(1.7) The Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= \mu x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned} \qquad (L)$$

[→ C. Sparrow, The Lorenz Equations, Springer, 1982]

[→ Perko, p. 198–199 & p 373–380]

We see all relevant phenomena when we restrict to $\sigma = 10$, $\beta = \frac{8}{3}$ and let $\mu > 0$ vary.

A picture of the Lorenz-attractor is on Perko p. 199 for $\mu = 28$.

(a) System (L) is symmetric under the transformation $(x, y, z) \to (-x, -y, z)$. Hence if (L) has a periodic orbit $(x(t), y(t), z(t))$ then the mirror image $(-x(t), -y(t), z(t))$ is also a periodic orbit.

(b) There is a positively invariant ellipsoide E which all trajectories enter eventually:

Lyapunov-function:

$$V = \mu x^2 + \sigma y^2 + \sigma(z - 2\mu)^2$$

$$\dot V = 2\mu x\dot x + 2\sigma y\dot y + 2\sigma(z-2\mu)\dot z$$

$$= 2\mu x(\sigma(y-x)) + 2\sigma y(\mu x - y - xz) + 2\sigma(z-2\mu)(-\beta z + xy)$$

$$= 2\sigma\Big(\cancel{\mu xy} - \mu x^2 + \mu xy - y^2 - \cancel{yxz} - \beta z^2 + xyz + 2\mu\beta z - \cancel{2\mu xy}\Big)$$

$$= 2\sigma\left(2\mu\beta z - \mu x^2 - y^2 - \beta z^2\right)$$

$D = \{(x,y,z): \ \mu x^2 + y^2 + \beta z^2 \le 2\mu\beta z\}$

then $\dot V \geq 0$ on D.

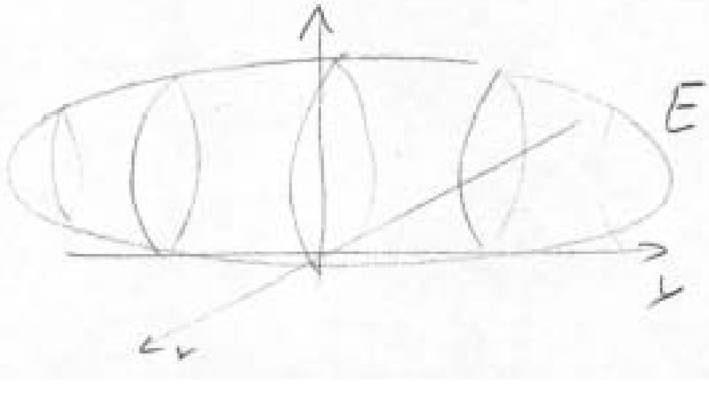
$C := \max\limits_{(x,y,z)\in D}\{V(x,y,z)\}$

For some small $\varepsilon > 0$:

$E := \{(x,y,z): \ V(x,y,z) \le C + \varepsilon\}$ ellipsoid.

If $\vec{x}_0 \notin E$, then $V(\vec{x}_0) > C + \varepsilon \Rightarrow \vec{x}_0 \notin D$.

$\Rightarrow \dot V(\vec{x}_0) < 0$. The orbit will decrease until it enters E.



(c) $0 < \mu < 1$: $(0,0,0)$ is globally stable
(Lyapunov function: $V_0(x,y,z) = x^2 + \sigma y^2 + \sigma z^2$).

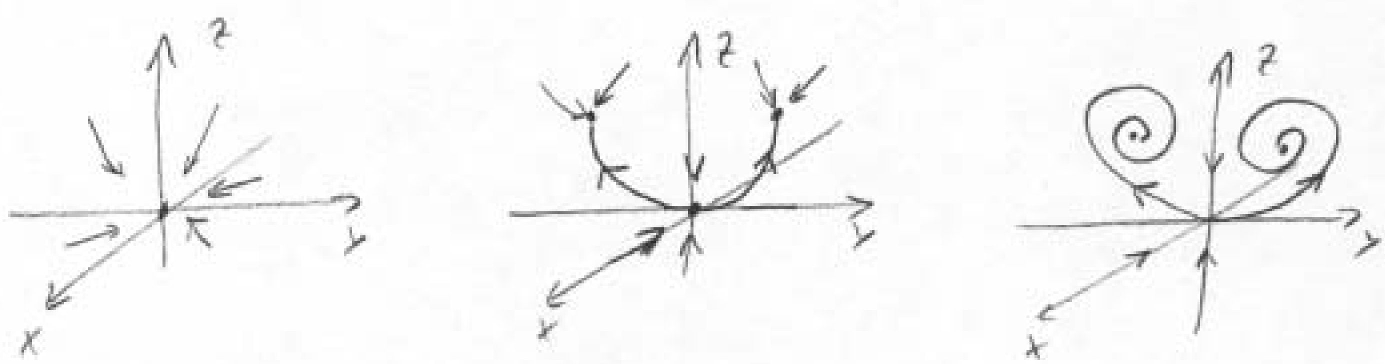
(d) at $\mu = 1$ there is a pitchfork bifurcation at $(0,0,0)$
and two new, stable critical points exist.

$$C_{1/2} = \left(\pm 2\sqrt{\frac{2(\mu-1)}{3}},\ \pm 2\sqrt{\frac{2(\mu-1)}{3}},\ \mu - 1 \right)$$

(e) $1 < \mu < \mu_H := \frac{470}{19} \approx 24.74$.

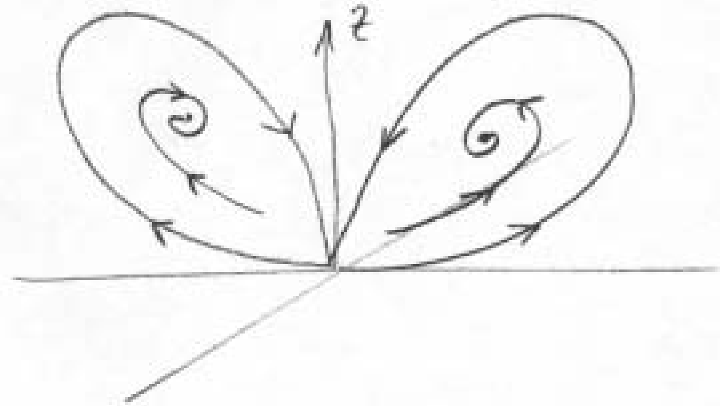
$(0,0,0)$ has a 1-dim. unstable mfd
and a 2-dim. stable mfd.

C_1 and C_2 are asymptotically stable:
nodes for $1 < \mu < 1.34$
spirals for $1.34 < \mu < 24.74$



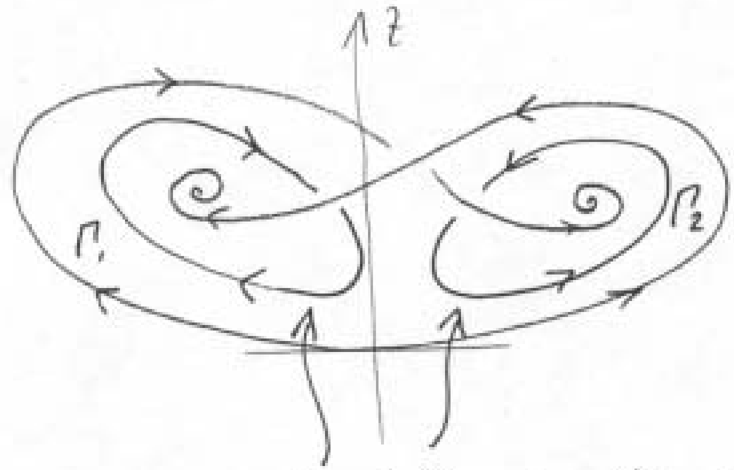
$\mu < 1$ $\quad$ $1 < \mu < 1.34$ $\quad$ $1.34 < \mu < 24.74$

(f) at $\mu = 13.926$: homoclinic loops:

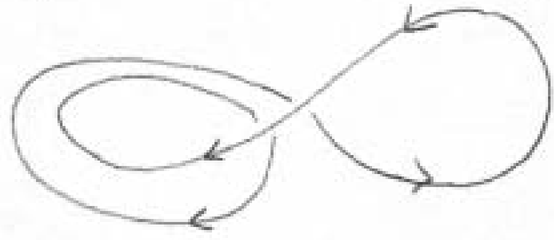


(g) homoclinic bifurcation at $\mu = 13.926$ leads to two unstable periodic orbits.

[→ Perko 4.8].



(h) as μ increases further we get more and more periodic orbits, e.g. "homoclinic explosion".



(i) At $\mu = \mu_H = 24.74$: Hopf bifurcation at both C_1 and C_2.

(j) $\mu > 24.74$: All three steady states are unstable. There exist ∞-many periodic orbits. For $\mu \gtrsim 24.74$ there exists a strange attractor