(15) Floquet Theory

Consider a T-periodic linear system

(1) $\dot{x} = A(t)x$, $A(t) \in \mathbb{R}^{n \times n}$, $A(t+T) = A(t)$
A continuous.

Let $\mathcal{X}(t)$ denote a fundamental system (fundamental matrix) of (1). Then

$$\dot{\mathcal{X}}(t) = A(t)\mathcal{X}(t). \quad \text{and}$$

$\mathcal{X}(t) = (x_1(t), \ldots, x_n(t))$ are n linear independent solutions to (1).

<u>Theorem 1:</u> (Floquet's Theorem)
There exists a T-periodic, non-singular, differentiable matrix $Q(t)$ and a constant matrix B such that

$$\mathcal{X}(t) = Q(t)e^{Bt}.$$

If $\mathcal{X}(0) = I$ then $Q(0) = I$.

<u>Proof:</u> In class presentation I,
$\rightarrow$ Hartman, ODE's, 1964, Wiley & Son's p. 61, 62.
requires: Linear Algebra, Jordan Forms.

Corollary: The linear periodic system (1) transforms under the linear change of coordinates

$$y := Q^{-1}(t)\, x$$

to an autonomous linear system

$$\dot{y} = By,$$

(where B is a constant matrix)

Proof. From Theorem 1: $Q(t) = X^t(t) e^{-Bt}$, hence

$$\dot{Q}(t) = \dot{X}^t e^{-Bt} - X^t e^{-Bt} B$$

$$= AX^t e^{-Bt} - X^t e^{-Bt} B$$

$$= AQ - QB$$

Now $y := Q^{-1}x$, then $x = Qy$ and

$$\dot{x}(t) = \dot{Q} \cdot y + Q\dot{y}$$

$$= AQy - QBy + Q\dot{y}$$

$$= Ax + Q[-By + \dot{y}]$$

$\Longrightarrow \quad \dot{y} = By.$

qed

Note, that from Theorem 1, $X(t) = Q(t)e^{Bt}$, it follows that the stability of the steady state $\bar{x} = 0$ is determined by the term e^{Bt}. Hence by the stability of $\dot{y} = By$.

We call the eigenvalues of B,

$$\mu_1, \ldots, \mu_n \quad \text{the } \underline{\text{Floquet exponents}}$$

And for period T we call the eigenvalues of e^{BT}, $\lambda_j = e^{\mu_j T}$ the <u>Floquet multipliers</u>.

These are used for the stability of periodic orbits.