Theorem 2:

(a) μ eigenvalue of A $\Rightarrow$ $\lambda = e^{\mu}$ eigenvalue of e^A.

(b) $\text{Re}\,\mu < 0$ $\Leftrightarrow$ $|\lambda| < 1$.

Proof: Homework!

Rule: " Lyapunov number $= e^{\text{Lyapunov exponent}}$ "

(1.4) Abel's formula, the Wronskian

Linear ODE in $\mathbb{R}^n$, nonautonomous:

$$\dot{x} = A(t)\,x \quad , \quad A(t) \in \mathbb{R}^{n\times n}$$

Let $X(t) \in \mathbb{R}^{n\times n}$ denote a fundamental system, i.e.

$$\dot{X} = A(t)\,X \quad \text{and} \quad X(t) = (x_1(t), \ldots, x_n(t)),$$

where $x_1(t), \ldots, x_n(t)$ are linearly independent solutions.

Consider $\det(X(t)) \triangleq$ n-dimensional Volume-element.

Leibniz-rule

$$\det(X) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, x_{\sigma(1)1} \cdots x_{\sigma(n)n}$$

group of permutations of $\{1, \dots, n\}$

$$\frac{d}{dt} \det(X) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \Big(\dot{x}_{\sigma(1)1}\, x_{\sigma(2)2} \cdots x_{\sigma(n)n} +$$

$$x_{\sigma(1)1}\, \dot{x}_{\sigma(2)2} \cdots x_{\sigma(n)n} +$$

$$\dots$$

$$+ x_{\sigma(1)1}\, x_{\sigma(2)2} \cdots \dot{x}_{\sigma(n)n} \Big)$$

$$= \det(\dot{x}_1, x_2, \dots, x_n) + \det(x_1, \dot{x}_2, x_3, \dots, x_n)$$

$$+ \cdots + \det(x_1, x_2, \dots, \dot{x}_n)$$

$$= \det(Ax_1, x_2, \dots, x_n) + \det(x_1, Ax_2, \dots, x_n)$$

$$+ \cdots + \det(x_1, x_2, \dots, Ax_n)$$

Now study the linear equations in z_i

$$X z_i = A x_i \qquad \text{for } i = 1, \dots, n \qquad (*)$$

The i-th component of the solution vector z_i is given by Kramers rule as

$$z_i^{(i)} = \frac{\det(x_1, \dots, Ax_i, \dots, x_n)}{\det(x_1, \dots, x_n)}$$

(*) can be written as $X Z = A X$ or

$Z = X^{-1} A X$ where $Z = (z_1 \dots z_n)$ has diagonal elements $z_i^{(i)} = \dfrac{\det(x_1, \dots, Ax_i, \dots, x_n)}{\det(x_1, \dots, x_n)}$

$$\Rightarrow \sum_{i=1}^{n} \frac{\det(x_1, \dots, Ax_i, \dots, x_n)}{\det(x_1, \dots, x_n)} = \operatorname{tr}(X^{-1} A X)$$

$$= \operatorname{tr} A$$

Hence we find a linear equation for $\det X$:

$$\frac{d}{dt} \det(X(t)) = \det(X(t)) \operatorname{tr}(A(t)).$$

Which is solved by

$$\boxed{\det(X(t)) = \det(X(0))\, e^{\int_0^t \operatorname{tr}(A(s))\,ds}}$$

WRONSKI'S FORMULA

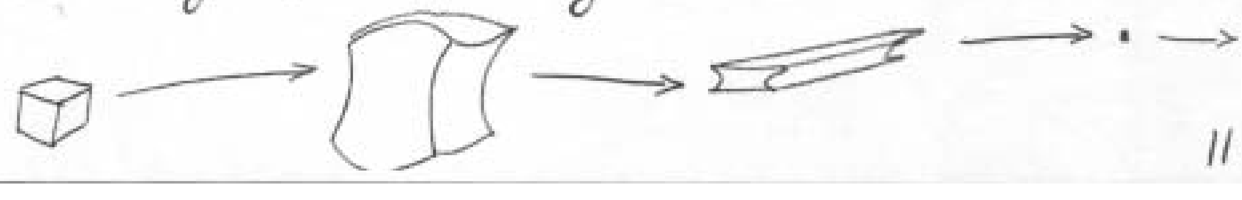
ABELS FORMULA

$\lambda_1 \cdots \lambda_n$

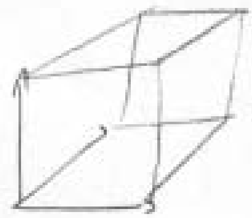
"Lyapunov mult."

"$\mu_1 + \cdots + \mu_n$ Lyapunov exponents"

The sign of $\int_0^t \operatorname{tr}(A(s))\,ds$ decides if an infinitessimal n-dimensional test volume is shrinking or expanding.



Example 1: Assume a test volume $V(0)$ in 3-D spanned by the three linear independent vectors $x_1(0)$, $x_2(0)$, $x_3(0)$.



Then

$V(0) = |\det(x_1, x_2, x_3)|$.

a) Study the time-evolution of this volume for the system

$$\dot{x} = \begin{pmatrix} -t^2+3 & 1 & t^3+t \\ e^t & -3+\sin(4t) & \sqrt{t} \\ \cos t & 0 & t^2 \end{pmatrix} x \quad , \; t \geq 0 .$$

b) What is the maximum and minimum volume?

a) Use Abel's formula: $\operatorname{tr} A(t) = -t^2+3-3+\sin(4t)+t^2$
$= \sin(4t)$

$$\int_0^t \sin(4t)\,dt = -\frac{1}{4}\left(\cos(4t)-1\right)$$

Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the solutions of $\dot{x} = A(t)x$ for initial data $x_1(0)$, $x_2(0)$, $x_3(0)$

Then

$$V(t) = |\det(x_1(t), x_2(t), x_3(t))|$$
$$= |\det(x_1(0), x_2(0), x_3(0))|\, e^{-\frac{1}{4}(\cos(4t)-1)}$$

$$= V(0)\, e^{\frac{1}{4}}\, e^{-\frac{1}{4}\cos(4t)}$$

$\Rightarrow$ $V(t)$ oscillates with period $T = \frac{\pi}{2}$.

max at $\cos(4t) = -1$, $4t_{max} = (2k+1)\pi \quad k \in \mathbb{N}$

$$t_{max} = \frac{2k+1}{4}\pi \quad k \in \mathbb{N}$$

$$V_{max} = V(t_{max}) = V(0)\, e^{\frac{1}{4}}\, e^{\frac{1}{4}} = V(0)\, e^{\frac{1}{2}}$$

min at $\cos(4t) = 1$, $4t_{min} = 2k\pi \quad k \in \mathbb{N}$

$$t_{min} = \frac{k}{2}\pi$$

$$V_{min} = V(t_{min}) = V(0)\, e^{\frac{1}{4}} e^{-\frac{1}{4}} = V(0) .$$