<u>Note</u>: $\underline{\mathrm{Re}\ \mu_i = \text{Lyapunov exponents}}$

<u>(1.2) Discrete dynamical systems</u>

$$x_{n+1} = f(x_n), \quad f: \mathbb{R}^n \to \mathbb{R}^n$$

initial condition x_0, solution $\phi(n; x_0) = f^n(x_0)$

discrete dynamical system $(\mathbb{R}^n, \phi(n))$. <u>Ex</u>: Poincaré-map.

Fixpoints: $f(\bar{x}) = \bar{x}$.

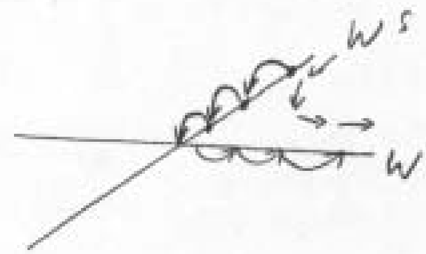
Stability of the fixpoint from the eigenvalues of the linearization

$$A = Df(\bar{x}), \quad \text{eigenvalues } \lambda_1, \dots, \lambda_n$$

<u>Theorem</u>: $|\lambda_i| < 1 \Rightarrow \bar{x}$ asymptotically stable

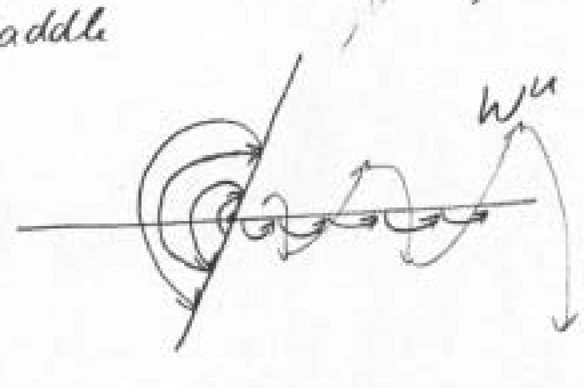
In <u>2-D</u>: zoo:

$0 < \lambda_1 < 1$, $\lambda_2 > 1$, saddle

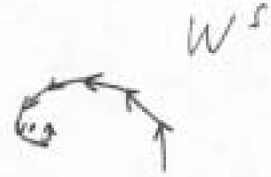


$-1 < \lambda_1 < 0$, $\lambda_2 > 1$

oscillating saddle



$\lambda_{1/2} = \pm i\beta, \quad \beta < 1$ stable "spiral"



<u>Note</u>: $|\lambda_i| =$ Lyapunov multipliers

<u>Trace - Determinant criterion in 2-D.</u>

$$\lambda_{1/2} = \frac{\operatorname{tr}A}{2} \pm \frac{1}{2}\sqrt{(\operatorname{tr}A)^2 - 4\det A}$$

$$\det A = \lambda_1 \cdot \lambda_2, \qquad \operatorname{tr}A = \lambda_1 + \lambda_2 \quad \in \mathbb{R}$$

Stability of $\bar{x} = 0 \Rightarrow \quad -1 < \det A < 1 \quad , -2 < \operatorname{tr}A < 2$

(i) $\lambda_1 = \bar{\lambda}_2$ complex, then $0 < \det A < 1$

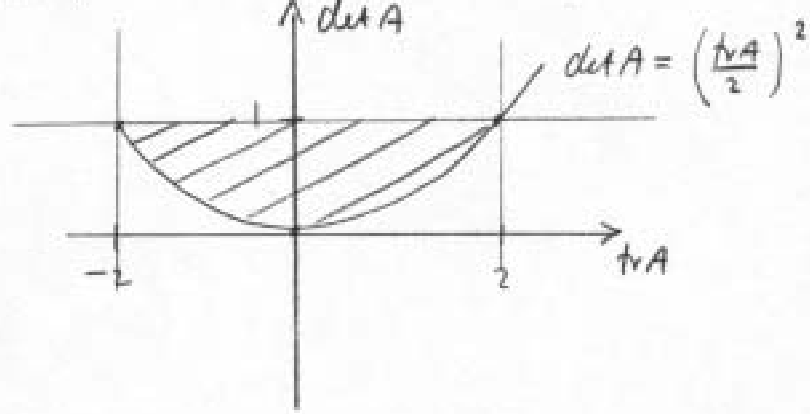
$$1 > \det A = \lambda_1 \cdot \lambda_2 = \lambda_1 \cdot \bar{\lambda}_1 = |\lambda_1|^2$$

$$= \frac{1}{4}(\operatorname{tr}A)^2 + \frac{1}{4}\left((\operatorname{tr}A)^2 - 4\det A\right)$$

$$\Rightarrow \quad 4 > 2(\operatorname{tr}A)^2 - 4\det A$$

$$(\operatorname{tr}A)^2 < 2 + 2\det A \quad , \quad \det A > \frac{1}{2}(\operatorname{tr}A)^2 - 2$$

since λ_1 complex we have $(\operatorname{tr}A)^2 < 4\det A$



Since $(\mathrm{tr}A)^2 < 4\det A$ and $\det A < 1$ we have automatically

$$(\mathrm{tr}A)^2 < 4\det A < 2 + 2\det A$$

(ii) λ_1, λ_2 real: $\det A < \frac{\mathrm{tr}A^2}{4}$

$$-1 < \frac{\mathrm{tr}A}{2} \pm \frac{1}{2}\sqrt{(\mathrm{tr}A)^2 - 4\det A} < 1 \quad \text{for stability}$$

$$-2 < \mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A} < 2$$

$$-2 - \mathrm{tr}A < \pm\sqrt{\phantom{(\mathrm{tr}A)^2 - 4\det A}} < 2 - \mathrm{tr}A$$

$\ominus$

$$4 + 4\,\mathrm{tr}A + \cancel{(\mathrm{tr}A)^2} > \cancel{(\mathrm{tr}A)^2} - 4\det A$$

$$1 + \mathrm{tr}A > -\det A$$

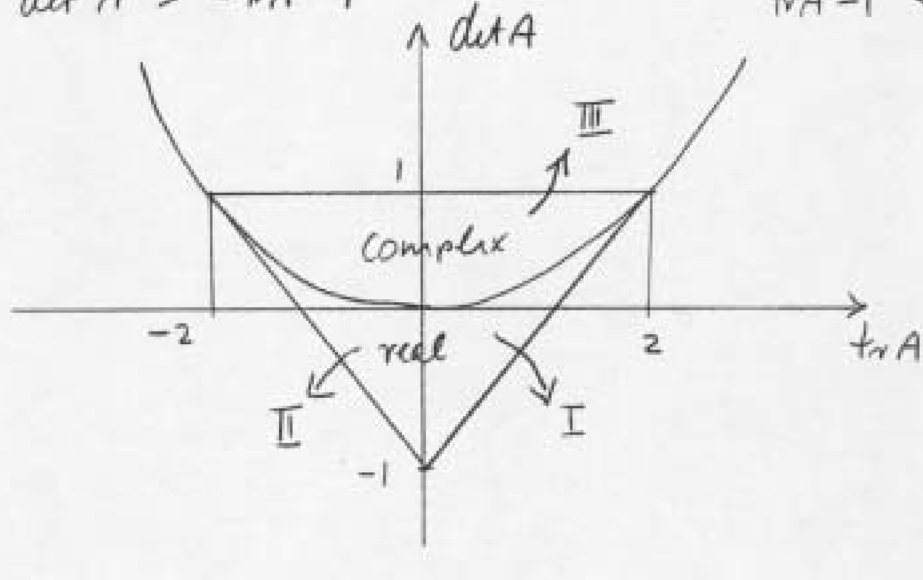
$$\det A > -\mathrm{tr}A - 1$$

$\oplus$

$$\cancel{(\mathrm{tr}A)^2} - 4\det A < 4 - 4\,\mathrm{tr}A + \cancel{(\mathrm{tr}A)^2}$$

$$-\det A < 1 - \mathrm{tr}A$$

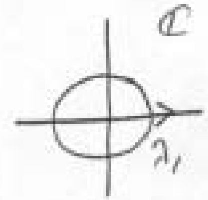
$$\mathrm{tr}A - 1 < \det A$$



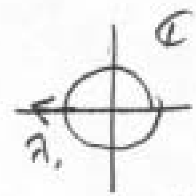
<u>Stability triangle</u>

Possible Bifurcations

I: One e-value leaves the unit ball at $\lambda_1 = +1$
(transcritical bifurcation)



II: One real e-value leaves the unit ball at $\lambda_1 = -1$
(period doubling bifurcation)



III: Two complex conjugate e-values leave the unit ball together
(discrete Hopf bifurcation)