University of Alberta
Mathematical and Statistical Sciences
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Math 525, Differential Equations II

Winter 2015

Assignment 6, due April 10, 2015, 9 AM

Exercise 22: (Cantor set) (2)

Consider the middle-α Cantor set C_α. Instead of removing the middle 1/3, we now remove the middle fraction α with $\alpha \in (0,1)$. Compute the fractal dimension of the middle-α Cantor set.

Exercise 23: (Outer boxcounting dimension) (4)

We can define the *outer boxcounting dimension* as follows. Let $B(X,\varepsilon)$ denote the minimum number of boxes of side length ε that are needed to cover the set X and define

$$d_B(X) := \limsup_{\varepsilon\to 0} \frac{\ln B(X,\varepsilon)}{\ln(1/\varepsilon)}.$$

Show that if $X \subset \mathbb{R}^n$, then

$$d_f(X) = d_B(X).$$

Does your prove carry over to infinite dimensions? If not, why not?

Exercise 24: (Hausdorff measure) (2)

Show that

$$\mathcal{H}^d\left(\bigcup_{k=1}^\infty X_k\right) \le \sum_{k=1}^\infty \mathcal{H}^d(X_k).$$

Exercise 25: (Cattaneo system) (10)

The nonlinear Cattaneo system in one space dimension reads

$$\begin{aligned} u_t &= -\gamma v_x + f(u) \\ v_t &= -\gamma u_x - 2\mu v \end{aligned}$$

It is a model for correlated random walk on an interval $[0,l]$ of particles moving with speed γ and turning rate μ. The functions $u(x,t)$ and $v(x,t)$ are particle density and particle flux, respectively. We consider homogeneous Neumann boundary conditions, which have the form $v(0,t)=0$, and $v(l,t)=0$. For the nonlinearity f we assume

$$f \in C^2, \quad \|f'\|_\infty < 2\mu, \quad F(u) = \int_0^u f(s)ds, \quad \lim_{|u|\to\infty} F(u) = -\infty.$$

The solutions form a semigroup in $X = H^1([0,l]) \times H_0^1([0,l])$. We define

$$P(u,v) = \int_0^l F(u) + \mu v^2 + \gamma u_x v\, dx, \qquad Q(u,v) = \int_0^l u_t^2 + v_t^2\, dx.$$

1. Show that there exists a $\lambda < 0$ such that $L = \lambda P + Q$ is a strong Lyapunov function for the Cattaneo system.
2. With λ chosen as in part 1. show that

$$\lim_{(u,v)\to\infty \text{ in } X} L(u,v) = +\infty.$$

   You have to ensure that $L \to \infty$ for all of the following four limits: $u \to \infty$ in L^2, $u_x \to \infty$ in L^2, $v \to \infty$ in L^2, and $v_x \to \infty$ in L^2.
3. Show that the Cattaneo system has an attractor in L^2. If we assume that the set of all steady states $\mathcal{E}$ is finite, then show that

$$\mathcal{A} = \bigcup_{z\in\mathcal{E}} \overline{W^u(z)}.$$