University of Alberta
Mathematical and Statistical Sciences
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Math 525, Ordinary Differential Equations II
Winter 2015

Assignment 1, due January 16, 2015, 9 AM in class

Exercise 1: (Linearization of discrete dynamical systems) (4)

We study the discrete dynamical system in $\mathbb{R}^n$ with differentiable function $f(x)$:

$$x_{n+1} = f(x_n).$$

1. Assume that $\bar{x}$ is a fixed point and consider small perturbations around $\bar{x}$ and define $x_n := \bar{x} + y_n$ where y_n is small. Derive the linearization of the above equation for y_n.
2. Prove the linear stability theorem which says that
If λ_j are the eigenvalues of $Df(\bar{x})$, and if $|\lambda_j| < 1$, then $\bar{x}$ is asymptotically stable.

Exercise 2: (Spectral Theorem) (2)

Proof Theorem 1 in (1.3), which reads
(a) If μ is an eigenvalue of a real matrix A, then $\lambda = e^\mu$ is an eigenvalue of e^A.
(b) $Re\mu < 0$ if and only if $|\lambda| < 1$.

Exercise 3: (Abel's formula) (2)

Study the time evolution of a test volume $V(t)$ for the three dimensional system

$$\dot{x}(t) = \begin{pmatrix} 1 & 100 & \ln(1+t) \\ \frac{1}{(2-t)^2} & \frac{1}{1-t} & \frac{1}{3-t} \\ 0 & \sin 2t & \frac{1}{1+t} \end{pmatrix} x(t).$$

Show that volumes blow-up in finite time and find the blow-up time. Does the blow-up time depend on the initial volume?

Exercise 4: (Perko, p. 231, Problem Set 5, No. 2:) (7)

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= x - 4y - \frac{x^3}{4} - xy^2 \\ \dot{y} &= x + y - \frac{x^2 y}{4} - y^3 \\ \dot{z} &= z \end{aligned}$$

1. Show that $\gamma(t) = (2\cos 2t, \sin 2t, 0)$ is a π-periodic solution.
2. Determine the linearization at $\gamma(t) : A(t)$.
3. Show that
$$\Phi(t) = \begin{pmatrix} e^{-2t}\cos 2t & -2\sin 2t & 0 \\ \frac{1}{2}e^{-2t}\sin 2t & \cos 2t & 0 \\ 0 & 0 & e^t \end{pmatrix}$$
is a fundamental matrix of the linearized system.
4. Write $\Phi(t)$ as $Q(t)e^{Bt}$ with a π-periodic matrix $Q(t)$. Find the Floquet exponents and multipliers of $\gamma(t)$.
5. Sketch $\gamma(t), W^s, W^u, W^c$ and a few typical trajectories.

Exercise 5: (All those functions) (5)

1. Find a function $f \in C_c^\infty(\mathbb{R})$ with supp$f \subset [a, b]$, where $a < b \in \mathbb{R}$.
2. Let $\Omega \subset \mathbb{R}^n$ be bounded. Show that if $f \in L^2(\Omega)$ then it follows that $f \in L^1(\Omega)$.

3. If Ω is unbounded the above statement is not true. Show that $\rho(x) = \frac{1}{1+x}$ is contained in $L^2([0,\infty))$ but not in $L^1([0,\infty))$.
4. Show that $\rho(x) = e^{-x}x^{-\frac{2}{3}}$ is contained in $L^1([0,\infty))$ but not in $L^2([0,\infty))$.
5. Find a value $\gamma^* \in [0,1]$ such that the function $f(x) = x^{\frac{3}{2}}$ is element of the Hölder space $C^{1,\gamma}([0,1])$ for $\gamma \leq \gamma^*$ and $f(x)$ is not contained in $C^{1,\gamma}([0,1])$ for $\gamma > \gamma^*$.