Homework 3

(due at 2:00 pm on May 25, 2009)

Problem 1. *Propagation of disturbances in a critical layer.*

Consider propagation of disturbances in the boundary layer with the speed $c \ll 1$ (in non-dimensional variables). Provide an estimate of the amplitude of such disturbances when nonlinear effects may become important based on consideration of the critical layer.

Solution. When $c \ll 1$, then the location of the critical layer is $y_c \sim c/U'(y_c)$ and the amplitude of a disturbance when nonlinear effects become important scales as

$$\epsilon \sim \frac{1}{c}\left[U'(y_c)\right]^{4/3}\left[\alpha\, Re_\delta\right]^{-2/3} \sim \frac{2\pi U_0}{\omega} L^{-2/3}\lambda^{-1/3}\left[U'(y_c)\right]^{4/3}.$$

Problem 2. *Kelvin-Helmholtz instability.*

Develop a theory of the Kelvin-Helmholtz instability in the presence of interfacial tension.

Solution. (for details, see Landau & Lifshitz "Fluid Mechanics") If the velocity of the upper layer of density ρ' relative to the lower layer of density ρ in the gravity field g is U, then the dispersion relation gives the following formula for the complex frequency of oscillations of perturbations:

$$\omega = k\frac{\rho' U}{\rho+\rho'} \pm \left[kg\frac{\rho-\rho'}{\rho+\rho'} - k^2U^2\frac{\rho\rho'}{(\rho+\rho')^2} + \frac{\sigma k^3}{\rho+\rho'}\right]^{1/2},$$

where k is the wavenumber and σ is the interfacial tension. The motion is marginally stable if ω is real, which holds if

$$U^4 \le \frac{4\sigma g(\rho-\rho')(\rho+\rho')^2}{\rho^2\rho'^2}.$$

Otherwise, the motion is unstable.

Problem 3. *Collapse of cavitation bubbles.*

Consider the problem of collapsing underwater bubbles and show that the Rayleigh-Taylor instability is reversed in this case. For simplicity, treat the case of negligible surface tension and when the density of the water phase is much larger than that of the collapsing bubbles.

Solution. The evolution equation for the amplitude $a(t)$ of a perturbation imposed on the collapsing bubble, $r = R(t) + a(t)Y_n$, where Y_n is a spherical harmonic of degree n, is

$$\ddot{\alpha} - G(t)\alpha = 0,\ a = (R_0/R)^{3/2}\,\alpha,$$

where

$$G(t) = \frac{3}{4}\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R}\left[n + \frac{1}{2}\right],$$

so that there is a possibility for instability in the case of collapsing bubbles, i.e. when $\ddot{R} < 0$. This is contrary to the Rayleigh-Taylor instability for flat interfaces.

Problem 4. *Stability of a gas jet.*

Consider a gas jet of radius a surrounded by a liquid phase. Deduce an appropriate dispersion relation and stability characteristics of this physical system.

Solution. The dispersion relation reads

$$a^3\rho\lambda^2/\sigma = -\frac{\alpha K_n'(\alpha)}{K_n(\alpha)}(1 - \alpha^2 - n^2),$$

where $\alpha = a\,k$ is the non-dimensional wavenumber, σ is the interfacial tension, and λ is the growth rate. Hence, instability occurs only for $n =$ and $-1 < \alpha < 1$.