Homework 1

(due at 2:00 pm on April 20, 2009)

Problem 1. Formulate *Lyapunov instability* using $(\varepsilon - \delta)$ language as a negation of the definition of *Lyapunov stability*. Give a physical/geometric interpretation.

Solution. The Lyapunov definition of stability is given on p. 9 of D&R:

A basic state (flow) $\mathbf{U}(\mathbf{x}, t)$ *is Lyapunov stable if, for any* $\varepsilon > 0$*, there exists some positive number* $\delta(\varepsilon)$ *such that if*

$$\|\mathbf{u}(\mathbf{x}, 0) - \mathbf{U}(\mathbf{x}, 0)\| < \delta,$$

then

$$\|\mathbf{u}(\mathbf{x}, t) - \mathbf{U}(\mathbf{x}, t)\| < \varepsilon$$

for all $t \geq 0$*.*

The negation of this definition would be the notion of *Lyapunov instability* which takes place if either the solution $\mathbf{u}(\mathbf{x}, t)$ fails to exist (existence was implicitly assumed in the definition of stability) or if solutions arbitrary close to equilibrium escape a ball of some positive radius provided that they exist.

Problem 2. *Rayleigh-Darcy convection in a porous medium.* You are given that two-dimensional convection in an infinite layer of a Boussinesq fluid in a porous medium is governed by the following non-dimensional initial-boundary value problem

$$\Delta\psi = -Ra\frac{\partial T}{\partial x}, \tag{1a}$$

$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial z} = \Delta T, \tag{1b}$$

with the boundary conditions

$$z = 0: \ \psi = 0, \ T = 0, \tag{2a}$$

$$z = 1: \ \psi = 0, \ T = -1, \tag{2b}$$

where ψ is the stream-function, T is the temperature, and Ra is the Rayleigh number.

- Give physical interpretations/assumptions behind derivation of the above equations and boundary conditions. Hint: start from Darcy's law.

- Study spectral stability of the base state $\psi_b = 0$, $T_b = -z$ and show that it is unstable for $Ra > 4\pi^2$.

Solution. In order to derive (1a) start from Darcy's law

$$\mathbf{u} = -\frac{k}{\phi\mu}\left(\nabla p - \rho g \mathbf{k}\right), \tag{3}$$

where k is the permeability, ϕ is the porosity, μ is the viscosity, and $g\mathbf{k}$ is the gravity component in z-direction. Then, (1a) follows by eliminating the pressure and using the Boussinesq approximation for the density ρ. Equation (1b) is just the standard convection-diffusion equation for the temperature field.

Linearization around the base state $\psi_b = 0$, $T_b = -z$ leads to the following eigenvalue relation

$$\lambda = \frac{k^2 Ra}{k^2 + n^2\pi^2} - (k^2 + n^2\pi^2), \ \ n = 1, 2, \ldots. \tag{4}$$

Since the eigenvalues are real, the critical Rayleigh number is given by minimizing

$$Ra = \frac{(k^2 + \pi^2)^2}{k^2} \tag{5}$$

over $k \in \mathbb{R}$, which gives $Ra_c = 4\pi^2$.

Problem 3. *Derivation of the Lorenz equations*:

$$\frac{\mathrm{d}X}{\mathrm{d}\tau} = \sigma(Y - X), \tag{6a}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}\tau} = rX - Y - ZX, \tag{6b}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}\tau} = -bX + XY. \tag{6c}$$

- Start from the Rayleigh-Benard system considered in the class, but restrict it to a two-dimensional infinite layer with free perfectly conducting boundaries

$$z = 0, \pi : \ \frac{\partial u}{\partial z} = w = T = 0. \tag{7}$$

- You are given that there are roll cell of the (approximate) form

$$u(x, z, t) = \sqrt{2}(k^2 + 1)k^{-1}X(t)S_x C_z, \tag{8a}$$

$$w(x, z, t) = -\sqrt{2}(k^2 + 1)X(t)C_x S_z, \tag{8b}$$

$$T(x, z, t) = -(k^2 + 1)^3 k^{-2}\left[\sqrt{2}Y(t)C_x S_z + Z(t)S_{2z}\right], \tag{8c}$$

where $S_x = \sin kx$, $C_z = \cos z$, $C_x = \cos kx$, $S_z = \sin z$, $S_{2z} = \sin 2z$.

- Verify that the equation of continuity and the boundary conditions are satisfied.
- Show that the curl of the curl of the momentum equations gives (6a) if appropriate components may be truncated. Similarly, deduce (6b) and (6c) and provide the expressions for constants σ, r, and b.

Hints. The derivation is quite straightforward. When deriving (6a) show that the only component of vorticity is

$$\omega = \partial u/\partial z - \partial w/\partial x = -\sqrt{2}(k^2+1)^2 k^{-1} X S_x S_z. \tag{9}$$

When deducing (6b) and (6c), use the fact of linear independence of the functions $\cos z$ and $\sin z$, etc., and make sure that the convective term in the conduction equation is

$$\mathbf{u} \cdot \nabla T = (k^2+1)^4 k^{-2}(XYS_{2z} + 2^{3/2} ZXC_x S_z C_{2z}). \tag{10}$$

Problem 4. Demonstrate that the principle of exchange of stabilities applies to the Rayleigh-Bernard problem.

Solution. See the lectures or §9.1 of D&R. The 'cleanest' and more robust approach, though, would be to work explicitly with the notion of self-adjoint operators, which always have real eigenvalues. For example, in this particular problem one can rewrite the linear part of the system

$$Pr^{-1}\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p + R_T \theta \mathbf{k} + \nabla^2 \mathbf{u}, \tag{11a}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta - v = \nabla^2\theta, \tag{11b}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{11c}$$

with the boundary conditions at the top and bottom rigid boundaries

$$z = 0, 1: \ \ \theta = 0, \ \mathbf{u} = \mathbf{0}, \tag{12}$$

as

$$\frac{\partial}{\partial t}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & R_T \end{pmatrix}\begin{pmatrix} p \\ u \\ v \\ \theta \end{pmatrix} = L\begin{pmatrix} p \\ u \\ v \\ \theta \end{pmatrix}, \tag{13}$$

with

$$L = \begin{pmatrix} 0 & -\partial_x & -\partial_y & 0 \\ -\partial_x & \nabla^2 & 0 & 0 \\ -\partial_y & 0 & \nabla^2 & R_T \\ 0 & 0 & R_T & R_T\nabla^2 \end{pmatrix}, \tag{14}$$

where the symmetry of the linear operator is apparent. Then one has to demonstrate that it is self-adjoint in the inner product, i.e. $\langle L\mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{a}, L\mathbf{b}\rangle$ through the integration by parts.

Problem 5. Explain independence of the marginal stability curve on the Prandtl number in the Rayleigh-Bernard problem.

Solution. The Prandtl number enters the eigenvalue problem (or, equivalently, the dispersion relation) as a factor of the eigenvalue, i.e. λ/Pr. Since the principle of exchange of stabilities applies to the Rayleigh-Benard problem, the marginal stability curve, defined by the condition $\lambda = 0$, does not depend on the Prandtl number.