Homework 1

(due at 2:00 pm on April 20, 2009)

Problem 1. Formulate *Lyapunov instability* using $(\varepsilon - \delta)$ language as a negation of the definition of *Lyapunov stability*. Give a physical/geometric interpretation.

Problem 2. *Rayleigh-Darcy convection in a porous medium.* You are given that two-dimensional convection in an infinite layer of a Boussinesq fluid in a porous medium is governed by the following non-dimensional initial-boundary value problem

$$\Delta\psi = -Ra\frac{\partial T}{\partial x}, \tag{1a}$$

$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial z} = \Delta T, \tag{1b}$$

with the boundary conditions

$$z = 0: \ \ \psi = 0, \ T = 0, \tag{2a}$$

$$z = 1: \ \ \psi = 0, \ T = -1, \tag{2b}$$

where ψ is the stream-function, T is the temperature, and Ra is the Rayleigh number.

- Give physical interpretations/assumptions behind derivation of the above equations and boundary conditions. Hint: start from Darcy's law.
- Study spectral stability of the base state $\psi_b = 0$, $T_b = -z$ and show that it is unstable for $Ra > 4\pi^2$.

Problem 3. *Derivation of the Lorenz equations*:

$$\frac{\mathrm{d}X}{\mathrm{d}\tau} = \sigma(Y - X), \tag{3a}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}\tau} = rX - Y - ZX, \tag{3b}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}\tau} = -bX + XY. \tag{3c}$$

- Start from the Rayleigh-Benard system considered in the class, but restrict it to a two-dimensional infinite layer with free perfectly conducting boundaries

$$z = 0, \pi: \ \frac{\partial u}{\partial z} = w = T = 0. \tag{4}$$

- You are given that there are roll cell of the (approximate) form

$$u(x,z,t) = \sqrt{2}(k^2+1)k^{-1}X(t)S_xC_z, \tag{5a}$$
$$w(x,z,t) = -\sqrt{2}(k^2+1)X(t)C_xS_z, \tag{5b}$$
$$T(x,z,t) = -(k^2+1)^3k^{-2}\left[\sqrt{2}Y(t)C_xS_z + Z(t)S_{2z}\right], \tag{5c}$$

  where $S_x = \sin kx,\ C_z = \cos z,\ C_x = \cos kx,\ S_z = \sin z,\ S_{2z} = \sin 2z$.

- Verify that the equation of continuity and the boundary conditions are satisfied.
- Show that the curl of the curl of the momentum equations fives (3a) if appropriate components may be truncated. Similarly, deduce (3b) and (3c) and provide the expressions for constants σ, r, and b.

Problem 4. Demonstrate that the principle of exchange of stabilities applies to the Rayleigh-Bernard problem.

Problem 5. Explain independence of the marginal stability curve on the Prandtl number in the Rayleigh-Bernard problem.