Final (take-home) Exam

(due at 2:00 pm on June 8, 2009)

Problem 1. Consider the following two evolution equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a - bx^2, \tag{1a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bx^2, \tag{1b}$$

with $a, b \in \mathbb{R}$, and discuss local and global stability properties of their steady solutions. Connect your results to (a) bifurcation analysis, (b) exact solutions, and (c) finite-time singularity formation.

Problem 2. Consider the problem of deformations of a ferrofluid drop of permeability μ_2, placed in a fluid of permeability μ_1, in a magnetic field H. The interfacial tension is σ.

- Determine the total potential energy of the drop.
- Find the maxima and minima of the potential energy and conclude which of the steady states are stable and unstable.
- Prove the existence of hysteresis phenomena.

Problem 3. Consider the Rayleigh equation (from stability theory in the inviscid case). Prove that if the base state velocity profile $U(y)$ is symmetric in y in the domain $a < y < b$ and if $U''(y_s) = 0$ for $a < y_s < b$, then (1) there exists a neutral disturbance with the wavenumber α_N and $c_N = U(y_s)$ and (2) for α slightly less than α_N, there exist solutions with $c_i > 0$. Note that normal modes are of the standard form $\sim e^{i\alpha(x-ct)}$ with $c = c_r + ic_i$. Characterize the proved results in terms of sufficiency/necessity conditions for instability.

Problem 4. Consider the following nonlinear initial and boundary-value problem,

$$u_t - u^2 = u_{xx},$$

$$x = 0, 1: \ u_x = 0,$$

$$t = 0: \ u(x; 0) = u_0(x).$$

Show that the zero solution is nonlinearly unstable and determine the conditions for instability. Is energy stability analysis feasible in this case? If yes, develop one.

Problem 5. Prove, using as an example the following two-dimensional linear system

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = A\mathbf{u}, \;\; \mathbf{u} = (u_1, u_2)^T,$$
$$\mathbf{u}(0) = \mathbf{u}_0,$$

with A being a 2×2 matrix, that if the operator A (matrix in this case) is non-normal and its eigenvalues are located in the left-half of the complex plane, then there always exist initial conditions $\mathbf{u}_0$, which lead to transient growth phenomena.