Final Exam

(April 15, 2014; 9:00-12:00)

Special instructions:

- This is a closed-book exam, i.e. books and lecture notes are not permitted.
- You are expected to solve all 3 problems; detailed solutions are required.

Problem 1. Consider a time varying point source $Q(t)$ which is located at $x = 0$ on a closed circular ring of circumferential length $2\,L$. The ring temperature $u(x,t)$ in the absence of heat exchange with the surroundings is governed by:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2},\ -L < x < L,\ x \neq 0,\ t > 0, \\ u(0+,t) = u(0-,t),\ -\frac{\partial u}{\partial x}\Big|_{x=0+} &+ \frac{\partial u}{\partial x}\Big|_{x=0-} = \frac{Q(t)}{\kappa}, \\ u(-L,t) = u(L,t),\ \frac{\partial u}{\partial x}\Big|_{x=-L} &= \frac{\partial u}{\partial x}\Big|_{x=L},\ t > 0, \\ u(x,0) &= 0,\ -L < x < L. \end{aligned} \tag{1}$$

(a) [5] Using symmetry considerations, prove that the solution expansion should be of the form

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos\frac{\pi n x}{L},\ -L < x < L.$$

(b) [15] Bearing in mind that $\partial u/\partial x$ is discontinuous at $x = 0$, find all $T_n(t)$ and the corresponding temperature distribution $u(x,t)$.

Problem 2.

(a) [20] Applying the Fourier transform and the convolution theorem, solve the Dirichlet problem for the Laplace equation in the upper half-plane

$$u_{xx}(x,y) + u_{yy}(x,y) = 0,\ x \in \mathbb{R}, y \in \mathbb{R}^+, \tag{2a}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R}, \tag{2b}$$

$$u(x,y) \to 0 \text{ for } x^2 + y^2 \to +\infty,\ y > 0. \tag{2c}$$

(b) [5] Give an interpretation of the obtained solution based on the potential theory.

(c) [5] Consider the case $u_0(x) = \text{const}$ and comment on where the maximum of the solution is achieved in the upper half-plane.

Problem 3.

(a) [5] Formulate the problem of oscillations of a string of unit length, $0 < x < 1$, with zero initial position and velocity and when one end, $x = 0$, is fixed, while the other one, $x = 1$, obeys the law $u_x|_{x=1} = \sin \omega t$. Assume unit sound speed in the string material.

(b) [5] Apply the Laplace transform in time and identify the singularities of the image solution.

(c) [10] Inverting the Laplace transform, find the solution in physical space.

(d) [5] Under which condition on the frequency of the driving force, ω, does the solution break down? Give possible physical reasons for that.