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HIDDEN INVARIANCES IN PROBLEMS OF TWO-DIMENSIONAL AND THREE-DIMENSIONAL WALL JETS FOR NEWTONIAN AND NON-NEWTONIAN FLUIDS*

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Abstract. This work is devoted to the investigation of self-similar solutions for steady wall jets. The problem is considered in the context of two- and three-dimensional Prandtl boundary layer equations, and three-dimensional parabolized Navier–Stokes equations for Newtonian and non-Newtonian fluids. In contrast to dimensional analysis, which does not allow the determination of self-similar solutions in this case, a generating functions approach elaborated by Vinogradov [*Soviet Math. Dokl.*, 19 (1978), pp. 144–148] enables one to derive conservation laws for the above-mentioned problems and, as a consequence, to find new self-similarities of the Navier–Stokes equations.

Key words. invariants, self-similarity, conservation laws

AMS subject classifications. 76D10, 76D25, 76M55, 76M60

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1. Introduction.

Since Prandtl [39] introduced the concept of the laminar boundary layer and Blasius [9] reported the exact self-similar solution for the two-dimensional (2D) boundary layer equations, self-similar solutions of a wide variety of boundary problems have appeared in the literature. Moreover, a number of new and more general techniques have been devised to take into account different boundary conditions. In particular, laminar boundary layers with cross flow (see [58] and references therein), Falkner–Skan flow past stretching boundaries [40], submerged [44] and wall [1, 15] jets, and boundary layers in a stream with uniform shear [34] have been studied. Also, one may encounter many thin layer type of problems of practical importance involving convection mechanisms in Newtonian and non-Newtonian fluids [36, 20]. Most reductions are the result of the standard application of dimensional analysis and were determined long ago [44, 46, 31]. An interesting class is represented by problems of the second kind of self-similarity [7] and, in particular, by problems of hidden invariances [17], like submerged and wall jets. This class incorporates more than just dimensional analysis, since the appropriate conservation laws need to be applied for closure of the problem.

In this paper we are concerned with the investigation of self-similar solutions for three-dimensional (3D) wall jets, which fall into the paradox of hidden invariances as classified by [17]. This corresponds to the situation in which one would think under conditions of *rational statement* that there is insufficient data for determination of all parameters of the solution. The way out of this state of affairs is to find nontrivial hidden invariances, which entirely determine the leading asymptotic form of the solution. A well-known example of such a situation is the 2D submerged jet [43].

The importance of wall jet flows can be observed from the impressive number of

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references to one of the original works [15], well known in the West. The applications range from wall jet electrodes in electrochemistry [24, 47], boundary layer separation control [57] in aerodynamics, control of air contaminants [13, 18] in ecology and process hygiene, and coastal currents [48] in oceanology to processes of cooling and evaporation enhancement [27], just to name a few. This variety of applications necessitated further research on wall jet problems: stability and receptivity properties [6, 30, 22, 45, 50]; effects of compressibility [41]; effects of blowing, suction, and moving walls [32]; effects of ribbed walls [14]; and heat transfer [33]. However, all the above works are restricted to planar flow [1, 15], while the more important 3D case remains unexplored theoretically. Jet flows, in particular wall jets, not only play an important practical role, but also have nontrivial theoretical features. In contrast to the classical Prandtl boundary layer theory, the entrained velocity field outside the jet proper is described by the Navier–Stokes equations. Essentially, a wall jet is a thin layer of fluid directed tangential to a wall: the total flow field consists of an inner region, which resembles the conventional wall boundary layer, and an outer region more akin to a free shear layer. Such a flow may be produced either by a jet of liquid from a tap falling into a partially full sink and spreading out over the bottom, or by a downwards-directed jet from a vertical-take-off aircraft spreading out over the ground.

The following analysis is based on the results on higher local infinitesimal symmetries and conservation laws found by Vinogradov in 1975–1977 [51]. Other authors (Ibragimov [23], Olver [37], Tsujishita [49]) have developed similar ideas in the field of symmetry theory. The theory used here makes it possible to find all higher conservation laws for arbitrary nonlinear differential equations. In particular, it works effectively in situations when the Noether theorem, as well as other symmetry considerations, are not applicable. The adjective *higher* is used to stress that the symmetries and conservation laws under consideration are described by means of expressions containing arbitrary order derivatives of dependent variables that appear in the governing partial differential equations. The adjective *local* is used to point out that we deal with symmetries and conservation laws which admit localizations on arbitrary domains in the space of independent variables and which must be expressed by local, that is, differential, operators.

The paper proceeds as follows. In section 2 we provide the reader with an outline of the methods used throughout the paper. Then in section 3 we describe completely all results concerning 2D wall jets in order to illustrate the effectiveness of the generating functions approach for constructing conservation laws. In sections 4 and 5 we study 3D jets of a Newtonian fluid using Prandtl boundary layer and parabolized Navier–Stokes equations, respectively. Section 6 is devoted to an investigation of wall jets in a non-Newtonian fluid. A discussion of results and concluding remarks are given in section 7.

2. Outline of the method. One can encounter self-similar solutions in all branches of mathematical physics. For a discussion of the conceptual importance of self-similar solutions in the genesis of mathematically well-grounded theories, see [52]. The determination of self-similarities attracts great attention, especially since in complex nonlinear problems the discovery of these solutions is frequently the only way to overcome analytical difficulties and obtain a qualitative understanding of the underlying physics. Moreover, self-similar solutions are used as standards for the evaluation of various approximate methods, regardless of the actual importance of the problem. It is essential to stress that self-similar solutions represent basic value not only as exact solutions of individual, maybe actual concrete problems, but also as intermediate asymptotic representations of solutions of an immeasurably wider range of problems. Self-similar solutions always arise out of problems in which the parameters of dimensionality of independent variables are zero or infinite; that is, as a rule, self-similar solutions meet singular initial or boundary conditions. Therefore self-similar solutions usually represent intermediate asymptotics of solutions of nondegenerate problems.

There is a widespread opinion that similarity variables are always obtained through the application of dimensional analysis, that is, with similarity reasoning, and that this always leads to a form of solution in terms of those self-similar variables. After determining this solution it is easy to find a class of nondegenerate problems for which the considered self-similar solution is an intermediate asymptotic. It is clear that if a mathematical formulation is known, then, instead of dimensional analysis based on invariances relative to a subgroup of the similarity transformation group, one can reduce the number of function arguments by establishing the invariance of the problem relative to some group of continuous transformations. As a rule, the state of affairs is different—there are vast classes of problems for which, even though there exists a self-similar intermediate asymptotic, one cannot obtain the asymptotic solution form from the original problem statement by application of dimensional analysis. The form of self-similar variables can be obtained by solving a nonlinear eigenvalue problem and, in some cases, from some additional reasoning. In conclusion, note that consideration of self-similar solutions as intermediate asymptotic representations is aligned closely with singular perturbation methods [7] and renormalization groups [16]. That is to say, self-similar solutions are external and internal asymptotics of solutions of general problems, depending on the scale of independent variable used in the analysis of the intermediate asymptotics. And so the determination of constants incorporated into self-similar solutions of the second kind, according to Barenblatt's classification [7], can be carried out by matching an asymptotic solution with additional asymptotics.

The notion of a conservation law for a given differential equation is a concept "dual" with the concept of symmetry. A relation between them is established in some cases by the famous Noether theorem [35]. But implementation of the Noether theorem requires knowledge of a Lagrangian of the corresponding system of differential equations which, in the case of the Prandtl boundary layer and Navier–Stokes equations, does not exist, and one needs to find the differential consequences of the original equations, for which there is a weak Lagrangian. The determination of this weak Lagrangian is associated with specific computational difficulties. A substantial advancement in the calculation of conservation laws for nonlinear equations was achieved in connection with the development of higher symmetries theory, which can be distinguished from the classical theory, originating from the works of Lie [28, 29], in that symmetries are described in terms of derivatives of arbitrarily high order.

Here we provide a summary of the known methods for the determination of conservation laws; they may be obtained

1. by the reduction of equations for solving variational problems and acquisition of conservation laws on the basis of possible transformation groups [35]. Until recently this was the unique constructive method.

2. by the addition of expressions like

$$\sum_{i=1}^{n} \mathbf{J}_i(\mathbf{u}) = 0,$$

with unknown $\mathbf{J}_i$, to original system of equations $\mathbf{F}(\mathbf{u}) = 0$ and subsequent investigation of the question of compatibility of the overdetermined system that results. One can rarely perform all the necessary calculations using this method, as was noted by Ovsjannikov [38].

3. through use of the determination of *conservation currents* [55, 56].

4. through use of the *composite variational principle* [10]. At this moment, in view of "nonlocal trends" [2, 52], it becomes obvious that this approach cannot guarantee the complete set of conserved vectors.

5. by the *generating functions* method [26]. According to [53], for l-normal systems there is a unique conservation law for each generating function.

6. by direct construction of conservation laws from field equations [3]. Anco and Bluman [3, 4] have promised to prove in a forthcoming paper the ability of this algorithm to construct all local conservation laws.

7. using the *neutral action* method [21, 11], which is a systematic procedure for construction of conservation laws determined using the concept of the Gâteaux derivative.

It is interesting to note that the equivalence between the earlier method of *generating functions* and the two methods that follow it remains to be established, despite some *apparent* similarities, like an identity of *the linearization of system* by Anco and Bluman [2], the *Gâteaux derivative* by Honein, Chien, and Herrmann [21], and the *universal linearization operator* by Krasil'shchik and Vinogradov [26].

In addition there are some specific methods like *web geometry* used in the theory of nonlinear wave interaction [8]. However, as noted in the introduction, our analysis is based on generating functions. A short description of this method is provided below for the reader's convenience.

2.1. The generating functions method. It is known that the problem of determination of all symmetries and conservation laws for a given system of PDEs is equivalent to the consideration of a new nonlinear system of PDEs more complicated than the original one. However, an adaptation of the infinitesimal standpoint substantially simplifies analysis of the symmetry fields, whose flows transform the solutions of the system with n independent and m dependent variables,

$$\epsilon = \{F = 0\}, \qquad F = (F_1, \ldots, F_l), \tag{2.1}$$

into themselves. Such a field is a higher infinitesimal symmetry of (2.1) if it is tangent to the infinite prolongation ϵ^∞ of (2.1) defined by

$$D_\sigma(F_i) = 0 \quad \forall \sigma, i,$$

where

$$D_\sigma = \frac{\partial}{\partial x_\sigma} + u^i_\sigma \frac{\partial}{\partial u^i} + \cdots + u^i_{\sigma, j_1, \ldots, j_m} \frac{\partial}{\partial u^i_{j_1, \ldots, j_m}} + \cdots \tag{2.2}$$

is the full derivative operator.

Elaboration of this theory by Vinogradov [51] has led to a determination of the higher symmetries by finding the corresponding generating functions as solutions of

$$\bar{l}_F \overline{\varphi} = 0 \Longleftrightarrow \mathrm{Sym}\, \epsilon = \mathrm{Ker}\, \bar{l}_F, \tag{2.3}$$

where $\bar{l}_F$ is the *universal linearization* operator calculated in the case of n independent and m dependent variables according to

$$l_F^{\epsilon} = \sum_\sigma \begin{pmatrix} \frac{\partial F_1}{\partial p_\sigma^1} & \dots & \frac{\partial F_1}{\partial p_\sigma^m} \\ \dots & \dots & \dots \\ \frac{\partial F_l}{\partial p_\sigma^1} & \dots & \frac{\partial F_l}{\partial p_\sigma^m} \end{pmatrix} D_\sigma, \quad p_\sigma^j = \frac{\partial^{|\sigma|} u^j}{\partial x^\sigma}, \tag{2.4}$$

and restricted on ϵ^∞, that is, on the *internal coordinates* which are the maximal functionally independent part of coordinates x, u, p_σ^i on ϵ^∞.

As noted by Noether in [35] for Lagrangian systems, and by Vinogradov in [53] for general systems of PDEs, the notion of a conservation law for a given differential equation is a concept dual in the sense of the concept of symmetry. However, as was demonstrated in [53], not every conservation law is a reflection of some symmetry.

Let $S = (S_1, \dots, S_n)$ be a conservation current for (2.1). Then one can consider S on the infinite prolongation ϵ^∞. The conservation law is defined as

$$\sum_{i=1}^n D_i(S_i) = \sum_{j=1}^l A_j(F_j), \tag{2.5}$$

where l is the number of equations and

$$A_j = \sum_\sigma a_\sigma^j D_\sigma \tag{2.6}$$

are scalar $\mathcal{C}$-differential operators. Now, we introduce the definition of formally conjugated operators.

DEFINITION 2.1. *An operator formally conjugated to A is built by the rule*

$$A_j^* = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_\sigma^j. \tag{2.7}$$

If $A = \|A_{ij}\|$ is a matrix $\mathcal{C}$-differential operator, then $A^* = \|A_{ji}^*\|$.

One can then formulate the following theorem.

THEOREM 2.2. *Let $A_1, \dots, A_l$ be operators which satisfy* (2.5). *Then the restriction $\Omega = (A_1^*(1), \dots, A_l^*(1))|_{\epsilon^\infty}$ of the vector-function $(A_1^*(1), \dots, A_l^*(1))$ onto equation ϵ^∞ satisfies the equation*

$$\left(l_F^{\epsilon}\right)^* (\Omega) = 0. \tag{2.8}$$

Thus, on the strength of the equality $\Omega = (A_1^*(1), \dots, A_l^*(1))|_{\epsilon^\infty}$, one finds $A_j = \sum_\sigma a_\sigma^j D_\sigma$. The following result will be used in the course of our analysis.

THEOREM 2.3. *If the system of equations is l-normal (regular and definite, $l = m$), then the generating function Ω corresponds to each conservation law identically.*

Remark. Not all solutions of (2.8) are generating functions [26].

3. A 2D jet.

3.1. Mathematical formulation of the problem. In Cartesian coordinates the governing system for flow due to a jet spreading out over a planar surface in the Prandtl boundary layer approximation for an incompressible fluid is

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned} \tag{3.1}$$

with boundary conditions

$$y = 0 : u = v = 0, \tag{3.2}$$

$$y = \infty : u = 0. \tag{3.3}$$

In view of the fact that problem (3.1) is invariant under the dilation group, one can introduce the following change of variables:

$$\Psi = \nu x^{1-k} f(\eta), \quad \eta = \frac{y}{x^k},$$

where Ψ stands for the 2D stream-function introduced according to $\Psi_y = u$, $\Psi_x = -v$, and k is a parameter of the self-similarity which generates the nonlinear eigenvalue problem. Use of these variables yields the boundary eigenvalue problem:

$$\begin{aligned} f_{\eta\eta\eta} + (1-k) f f_{\eta\eta} + (2k-1) f_\eta^2 = 0, \\ \eta = 0 : f = f_\eta = 0, \\ \eta = \infty : f_\eta = 0. \end{aligned} \tag{3.4}$$

The hidden invariant in this problem has been found by Akatnov [1] and Glauert [15]. Modern methods of investigation of nonlinear equations allow one to find that conservation law and prove its uniqueness.

3.2. Calculation of conservation laws. The conjugated operator to the operator of universal linearization computed with the help of (2.4)–(2.7) is

$$(l_F^\epsilon)^* = \begin{pmatrix} \frac{\partial u}{\partial x} - L - \nu D_y^2 & -D_x \\ \frac{\partial u}{\partial y} & -D_y \end{pmatrix}, \quad L = uD_x + vD_y. \tag{3.5}$$

The final system is $(l_F^\epsilon)^* (\Omega) = 0$ with $\Omega = (\varphi_1, \varphi_2)^T$ or

$$\begin{aligned} u\frac{\partial \varphi_1}{\partial x} + v\frac{\partial \varphi_1}{\partial y} + \varphi_1 \frac{\partial v}{\partial y} + \frac{\partial \varphi_2}{\partial x} + \nu \frac{\partial^2 \varphi_1}{\partial y^2} = 0, \\ \varphi_1 \frac{\partial u}{\partial y} - \frac{\partial \varphi_2}{\partial y} = 0. \end{aligned} \tag{3.6}$$

Dropping the index for convenience, elimination of φ_2 yields

$$\nu\frac{\partial^3 \varphi}{\partial y^3} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \varphi}{\partial y^2} - 2\frac{\partial^2 \Psi}{\partial x \partial y}\frac{\partial \varphi}{\partial y} + 2\frac{\partial^2 \Psi}{\partial y^2}\frac{\partial \varphi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{\partial^2 \varphi}{\partial x \partial y} = 0,$$

which contains Ψ_x, Ψ_y, Ψ_{xy}, Ψ_{yy} and derivatives φ_x, φ_y, φ_{xy}, φ_{yy}, φ_{yyy}. So one can assume only the dependence $\varphi = \varphi(\Psi)$; the highest derivative Ψ_{yyy} must be excluded with the help of (3.1) . As a result we obtain $\varphi_{\Psi\Psi} = 0$, which gives the two solutions $\varphi = 1$ and $\varphi = \Psi$ to an approximation of a constant multiplier. The first one corresponds to a free submerged jet, and the second to a wall jet. Using Theorem 2.2 and expression (2.5), we obtain the conserved currents for the wall jet as

$$S = \begin{pmatrix} u^2 \Psi \\ uv\Psi + \frac{u^2}{2} - \Psi \frac{\partial u}{\partial y} \end{pmatrix}.$$

The integral form of this conservation law gives the well-known result

$$\frac{\partial}{\partial x}\left[\int_0^{+\infty} u^2 \Psi dy\right] = 0. \tag{3.7}$$

The form of this invariant in self-similar variables,

$$\frac{\partial}{\partial x} I = 0, \qquad I = \nu^2 x^{3-4k} \int_0^{+\infty} f f_\eta^2 d\eta, \tag{3.8}$$

gives rise to the value $k = 3/4$ for the self-similar parameter.

4. 3D jet described by the 3D Prandtl boundary layer equations. This problem corresponds to the flow of a jet spreading out over a plane surface from a slit, the vertical dimension of which is much less than its horizontal dimension, at a distance from the slit of the order of its horizontal dimension.

4.1. Mathematical formulation of the problem. In Cartesian coordinates the determinative system of 3D Prandtl boundary layer equations for an incompressible fluid is

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \nu \frac{\partial^2 u}{\partial y^2}, \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= \nu \frac{\partial^2 w}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \end{aligned} \tag{4.1}$$

to be solved with boundary conditions

$$\begin{aligned} &y = 0 : u = v = w = 0, \qquad && y = \infty : u = w = 0, \\ &z = 0 : \frac{\partial u}{\partial z} = \frac{\partial w}{\partial z} = 0, \qquad && z = \pm\infty : u = w = 0. \end{aligned}$$

In this 3D problem one can introduce the stream-function vector (Ψ^1, Ψ^2) according to

$$u = \frac{\partial \Psi^1}{\partial y}, \qquad w = \frac{\partial \Psi^2}{\partial y}, \qquad v = -\frac{\partial \Psi^1}{\partial x} - \frac{\partial \Psi^1}{\partial z}.$$

Group analysis suggests the following similarity variables:

$$\Psi^1 = \nu x^{1-k} f(\eta, \zeta), \qquad \Psi^2 = \nu x^{1-k} \varphi(\eta, \zeta), \qquad \eta = \frac{y}{x^k}, \qquad \zeta = \frac{z}{x^l},$$

in which k and l are parameters of the self-similarity which generates the nonlinear eigenvalue problem. Substitution in (4.1) furnishes the eigenvalue problem

$$\begin{aligned} f_{\eta\eta\eta} + [\varphi_\zeta f_{\eta\eta} - \varphi_\eta f_{\zeta\eta}] + l\zeta\, [f_\eta f_{\zeta\eta} - f_\zeta f_{\eta\eta}] + (1-k) f f_{\eta\eta} + (2k-1) f_\eta^2 = 0, \\ \varphi_{\eta\eta\eta} + [\varphi_\zeta \varphi_{\eta\eta} - \varphi_\eta \varphi_{\zeta\eta}] + l\zeta\, [f_\eta \varphi_{\zeta\eta} - f_\zeta \varphi_{\eta\eta}] + (1-k) f \varphi_{\eta\eta} + (2k-1) \varphi_\eta^2 = 0, \end{aligned}$$

with boundary conditions

$$\begin{aligned} &\eta = 0 : f_\eta = \varphi_\eta = (1-k) f - l\zeta f_\zeta + \varphi_\zeta = 0, \\ &\eta = \infty : f_\eta = \varphi_\eta = 0, \qquad \zeta = \pm\infty : f_\eta = \varphi_\eta = 0. \end{aligned}$$

4.2. Calculation of conservation laws. The operator of universal linearization is found as

$$l_F^\epsilon = \begin{pmatrix} \frac{\partial u}{\partial x} + L - \nu D_y^2 & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} + L - \nu D_y^2 \\ D_x & D_y & D_z \end{pmatrix}, \quad L \equiv u D_x + v D_y + w D_z, \tag{4.2}$$

and the conjugated operator is

$$(l_F^\epsilon)^* = \begin{pmatrix} \frac{\partial u}{\partial x} - L - \nu D_y^2 & \frac{\partial w}{\partial x} & -D_x \\ \frac{\partial u}{\partial y} & \frac{\partial w}{\partial y} & -D_y \\ \frac{\partial u}{\partial z} & \frac{\partial w}{\partial z} - L - \nu D_y^2 & -D_z \end{pmatrix}. \tag{4.3}$$

The final system is $(l_F^\epsilon)^*(\Omega) = 0$ with $\Omega = (\varphi_1, \varphi_2, \varphi_3)^T$ or

$$\begin{gathered} \left[\frac{\partial u}{\partial x} - L - \nu D_y^2\right]\varphi_1 + \frac{\partial w}{\partial x}\varphi_2 - D_x\varphi_3 = 0, \\ \frac{\partial u}{\partial y}\varphi_1 + \frac{\partial w}{\partial y}\varphi_2 - D_y\varphi_3 = 0, \\ \frac{\partial u}{\partial z}\varphi_1 + \left[\frac{\partial w}{\partial z} - L - \nu D_y^2\right]\varphi_2 - D_z\varphi_3 = 0. \end{gathered} \tag{4.4}$$

A local coordinate system on the infinite prolongation ϵ^∞ is given by $x_j, u^i_{j_1,\ldots,j_k}$, where if $i = 1$ ($u^1 = u$), etc., then $j_k \neq 1$ by virtue of the continuity equation. Using mathematical induction from the first and third equations of (4.4), one can show that φ_1, φ_2 are functions only of coordinates x_j, a consequence of which is the dependence $\varphi_3 = \varphi_3(x_j, u^i)$. A unique solution of (4.4) is then given by

$$\Omega = \begin{pmatrix} z \\ -x \\ uz - wx \end{pmatrix}, \tag{4.5}$$

and the corresponding conservation law is

$$S = \begin{pmatrix} u\,(uz - wx) \\ v\,(uz - wx) - \nu\big(z\frac{\partial u}{\partial y} - x\frac{\partial w}{\partial y}\big) \\ w\,(uz - wx) \end{pmatrix} = \begin{pmatrix} u\,(uz - wx) \\ S_y \\ S_z \end{pmatrix}. \tag{4.6}$$

Now the integral variant of the conservation law is given by

$$\frac{\partial}{\partial x}\left[\int_{-\infty}^{+\infty} dz \int_0^{+\infty} u\,(uz - wx)\,dy\right] + \int_{-\infty}^{+\infty} S_y\big|_{y=0}^{+\infty}\,dz + \int_0^{+\infty} S_z\big|_{z=-\infty}^{+\infty}\,dy = 0. \tag{4.7}$$

In terms of the similarity variables, the conservation law takes the form

$$\frac{\partial}{\partial x} I = 0, \qquad I = \nu^2 x^{2-3k+2l} \int_{-\infty}^{+\infty} d\zeta \int_0^{+\infty} f_\eta\,[\zeta f_\eta - \varphi_\eta]\,d\eta, \tag{4.8}$$

which requires that $2 - 3k + 2l = 0$.

4.3. Analysis in the plane of symmetry. Since we investigate self-similar solutions of the 3D Prandtl equations, the existence of the symmetry plane is a consequence of such a consideration. The leading order terms of the solution expansion near the symmetry plane are

$$u = u(x,y), \qquad v = v(x,y), \qquad w = zW(x,y),$$

so that the governing system of equations in this plane takes the form

$$\begin{gathered} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \\ u\frac{\partial W}{\partial x} + v\frac{\partial W}{\partial y} + W^2 = \nu\frac{\partial^2 W}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + W = 0, \qquad W = \frac{\partial w}{\partial z}, \end{gathered} \tag{4.9}$$

with associated boundary conditions

$$y = 0 : u = v = W = 0, \qquad y = \infty : u = W = 0.$$

Introduction of the self-similar variables $\eta = y/x^k$, $\Psi^1 = \nu x^{1-k} f(\eta)$, and $\Psi^2_z = \nu x^{-k}\Phi(\eta)$ furnishes the ODEs

$$\begin{gathered} f_{\eta\eta\eta} + (1-k)ff_{\eta\eta} + (2k-1)f_\eta^2 = -\Phi f_{\eta\eta}, \\ \Phi_{\eta\eta\eta} + \Phi\Phi_{\eta\eta} - \Phi_\eta^2 = -(1-k)f\Phi_{\eta\eta} - 2kf_\eta\Phi_\eta, \end{gathered} \tag{4.10}$$

with boundary conditions

$$\begin{aligned} &\eta = 0 : f = f_\eta = \Phi = \Phi_\eta = 0, \\ &\eta = \infty : f_\eta = \Phi_\eta = 0. \end{aligned}$$

Numerical solutions of this system corresponding to profiles of the longitudinal velocity $u = f_\eta$ for several values of k are displayed in Figure 4.1. Integrating the first equation from η to ∞ yields

$$\begin{gathered} f_{\eta\eta} + (1-k)ff_\eta + (2-3k)F = -\Phi f_\eta - G, \\ F = \int_\eta^{+\infty} f_\eta^2 d\eta, \qquad G = \int_\eta^{+\infty} \Phi_\eta f_\eta d\eta. \end{gathered} \tag{4.11}$$

Multiplying this result by f_η and integrating from η to ∞ gives

$$\frac{f_\eta^2}{2} + (k-1)fF + (4k-3)\int_\eta^{+\infty} f_\eta F d\eta = \Phi F + \int_\eta^{+\infty} \left[\Phi_\eta F + f_\eta G\right] d\eta.$$

For $\eta = 0$ the above result gives

$$(4k-3)\int_0^{+\infty} f_\eta F d\eta = \int_0^{+\infty} \Phi_\eta F d\eta + \int_0^{+\infty} f_\eta G d\eta.$$

Assuming "nonreversed" velocity profiles f_η and Φ_η, we conclude that $k > \frac{3}{4}$ by virtue of the positiveness of all integrals in the equation. In the 2D case, $k = \frac{3}{4}$ because the

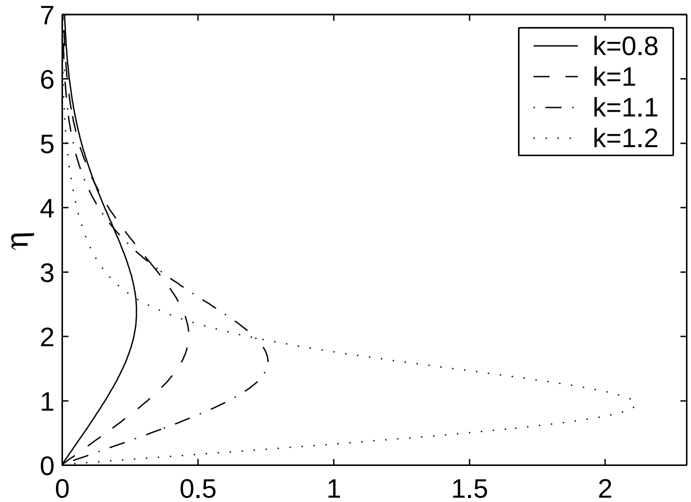

FIG. 4.1. *Profiles of the longitudinal velocity $u = f_\eta$ in the plane of symmetry.*

integrals on the right-hand side are equal to zero, which asserts the uniqueness of the solution of the appropriate eigenvalue problem. From the above expression one can determine the admissible spectrum of eigenvalues k, l:

$$k \in \left(\frac{3}{4}, +\infty\right), \qquad l \in \left(\frac{1}{8}, +\infty\right). \tag{4.12}$$

It should be noted that the set and structure of conservation laws changes drastically if the problem defined by (4.1) is oversimplified. The following example provides a demonstration of this fact.

4.4. Example of a 3D jet described by 2D equations. Let us represent an initial condition on the time-similar coordinate in the following form:

$$u = u_0(0, y, z), \qquad w = f_0(z)u_0(0, y, z). \tag{4.13}$$

This representation of w_0 entails an absence of secondary flows that enables one to search for a solution of the general problem in the form

$$u = u(x, y, z), \qquad w = f(x, z)u(x, y, z). \tag{4.14}$$

With that representation, the system of equations becomes

$$\begin{gathered} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + uf\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} = 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial uf}{\partial z} = 0, \end{gathered} \tag{4.15}$$

with associated boundary conditions

$$\begin{gathered} x = 0 : f = f_0(z), \\ y = 0 : u = v = 0, \qquad y = \infty : u = 0, \\ z = \pm\infty : u = 0. \end{gathered} \tag{4.16}$$

The equation for f is a nonlinear wave equation, solutions of which are both continuous and discontinuous functions. Let us confine ourselves to continuous functions, which have the implicit form

$$f(x,z) = f_0[z - xf(x,z)]. \tag{4.17}$$

For the sake of convenience let us represent this solution in the parametric form

$$f(x,z) = f_0(s), \qquad s = z - xf(x,z) = z - xf_0(s), \tag{4.18}$$

and we use the variables (x,y,s) instead of (x,y,z). As a result, we arrive at the system

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + u\frac{f_0'}{1+xf_0'} &= 0. \end{aligned}$$

One can convert this system to the 2D boundary layer equations with the help of the substitution $(x,y,s) \Rightarrow (\xi(x,s), \eta(x,y,s), s)$. An appearance of new variables is determined by zeroing appropriate terms in the system. From the continuity equation, we find that $\eta = y[1+xf']$, and from the momentum equation, $\xi = [1+xf']^3/3f'$. As a result,

$$\begin{aligned} u\frac{\partial u}{\partial \xi} + \widetilde{v}\frac{\partial u}{\partial \eta} &= \nu\frac{\partial^2 u}{\partial \eta^2}, \\ \frac{\partial u}{\partial \xi} + \frac{\partial \widetilde{v}}{\partial \eta} &= 0, \end{aligned} \tag{4.19}$$

where

$$\widetilde{v} = \frac{v + auy}{\eta_y}, \qquad a = \frac{f_0'}{1+xf_0'}.$$

An invariant for this problem was already obtained in section 3, viz.,

$$\frac{\partial}{\partial \xi}\int_0^{+\infty} \Psi\Psi_\eta^2 d\eta = 0, \qquad u = \Psi_\eta, \qquad \widetilde{v} = -\Psi_\xi. \tag{4.20}$$

5. 3D jet described by the 3D parabolized Navier–Stokes equations. These Navier–Stokes equations must be parabolized when the horizontal dimension of the slit is of the same order as its vertical dimension (circular pipe, for example). Assuming the existence of a local streamwise flow direction, designated herein as x, such that the effects of viscous diffusion in this direction are of higher order in the parameter Re^{-n}, for $n > 0$ and $Re \gg 1$, for large Re, these streamwise diffusion terms can be neglected at lowest order. In view of the absence of an outer flow, the pressure gradient in the x direction is asymptotically small, thus making the resulting governing system mathematically parabolic in x. Subsequently, marching or initial value methods can be applied [12, 42].

5.1. Formulation of the problem. In Cartesian coordinates the system of parabolized Navier–Stokes equations for an incompressible fluid is

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \nu\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right],\\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right],\\
u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right],\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}
\tag{5.1}
$$

to be solved with boundary conditions

$$
\begin{aligned}
y = 0 : u = v = w = 0, &\qquad y = \infty : u = w = 0,\\
z = 0 : \frac{\partial u}{\partial z} = \frac{\partial w}{\partial z} = 0, &\qquad z = \pm\infty : u = w = 0.
\end{aligned}
$$

Also, it is necessary to impose specified boundary conditions on the pressure function p, which will be determined later. Once again, group analysis brings about the following variable transformation:

$$
\Psi^1 = \nu x^{1-k} f(\eta,\zeta), \qquad \Psi^2 = \nu\varphi(\eta,\zeta), \qquad p_1 = \rho\nu^2 x^{-2k} g(\eta,\zeta); \quad \eta = \frac{y}{x^k}, \quad \zeta = \frac{z}{x^k},
$$

where k is a parameter of the self-similarity which generates the nonlinear eigenvalue problem. Note the difference of this problem from that of a Prandtl boundary layer, wherein the eigenvalue problem contains two parameters, k and l.

5.2. Calculation of conservation laws by the generating functions method. The operator of universal linearization,

$$
\begin{aligned}
l_F^{\epsilon} &= \begin{pmatrix}
\frac{\partial u}{\partial x} + L - \nu\triangle & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} & 0\\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} + L - \nu\triangle & \frac{\partial v}{\partial z} & \frac{1}{\rho}D_y\\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} + L - \nu\triangle & \frac{1}{\rho}D_z\\
D_x & D_y & D_z & 0
\end{pmatrix},\\
\triangle &= D_y^2 + D_z^2,
\end{aligned}
\tag{5.2}
$$

has for its conjugated operator

$$
\begin{aligned}
(l_F^{\epsilon})^* &= \begin{pmatrix}
\frac{\partial u}{\partial x} + L^* - \nu\triangle & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} & -D_x\\
\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + L^* - \nu\triangle & \frac{\partial w}{\partial y} & -D_y\\
\frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} + L^* - \nu\triangle & -D_z\\
0 & -\frac{1}{\rho}D_y & -\frac{1}{\rho}D_z & 0
\end{pmatrix},\\
L^* &\equiv -[uD_x + vD_y + wD_z].
\end{aligned}
\tag{5.3}
$$

The final system is $(l_F^\epsilon)^*(\Omega) = 0$ for $\Omega = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T$. Solutions of this system are

$$\Omega_1 = \begin{pmatrix} y \\ -x \\ 0 \\ uy - vx \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} z \\ 0 \\ -x \\ uz - wx \end{pmatrix}, \qquad \Omega_3 = \begin{pmatrix} 0 \\ z \\ -y \\ vz - wy \end{pmatrix},$$

(5.4)

$$\Omega_4 = a \begin{pmatrix} 1 \\ 0 \\ 0 \\ u \end{pmatrix}, \qquad \Omega_5 = b \begin{pmatrix} 0 \\ 1 \\ 0 \\ v \end{pmatrix}, \qquad \Omega_6 = c \begin{pmatrix} 0 \\ 0 \\ 1 \\ w \end{pmatrix}, \qquad \Omega_7 = d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where a, b, c, and d are constants.

Let us consider the first three conservation laws, appropriate to Ω_1, Ω_2, and Ω_3. For Ω_1 the operators A_j take the form

$$A_1 = y, \qquad A_2 = -x, \qquad A_3 = 0, \qquad A_4 = uy - vx. \tag{5.5}$$

Then the differential form of conservation law (2.5) is

$$\begin{aligned} &\frac{\partial}{\partial x}\left[u\left(yu - xv\right)\right] + \frac{\partial}{\partial y}\left[v\left(yu - xv\right) - x\frac{p}{\rho} - \nu\left(y\frac{\partial u}{\partial y} - u - x\frac{\partial v}{\partial x}\right)\right] \\ &\qquad + \frac{\partial}{\partial z}\left[w\left(yu - xv\right) - \nu\left(y\frac{\partial u}{\partial z} + x\frac{\partial v}{\partial z}\right)\right] = 0, \end{aligned} \tag{5.6}$$

which has the corresponding conservation current

$$S = \begin{pmatrix} u\left(uy - vx\right) \\ v\left(uy - vx\right) - x\frac{p}{\rho} - \nu\left(y\frac{\partial u}{\partial y} - u - x\frac{\partial v}{\partial y}\right) \\ w\left(uy - vx\right) - \nu\left(y\frac{\partial u}{\partial z} + x\frac{\partial v}{\partial z}\right) \end{pmatrix} = \begin{pmatrix} u\left(uy - vx\right) \\ S_y \\ S_z \end{pmatrix}. \tag{5.7}$$

The integral variant of this conservation law is

$$\frac{\partial}{\partial x}\left[\int_{-\infty}^{+\infty} dz \int_0^{+\infty} u\left(uy - vx\right) dy\right] + \int_{-\infty}^{+\infty} S_y|_{y=0}^{+\infty}\, dz + \int_0^{+\infty} S_z|_{z=-\infty}^{+\infty}\, dy = 0.$$

Hence one can find the boundary condition to impose on the pressure, namely,

$$p(x,y,z)|_{y=0} = p(x,y,z)|_{y=+\infty}. \tag{5.8}$$

The form of the invariant in self-similar variables is

$$\frac{\partial}{\partial x} I = 0, \qquad I = \nu^2 x^{2-k} \int_{-\infty}^{+\infty} d\zeta \int_0^{+\infty} f_\eta\left[(1-k)\eta f_\eta + (1-k)f - k\zeta f_\zeta + \varphi_\zeta\right] d\eta, \tag{5.9}$$

which produces $k = 2$.

For Ω_2 we have

$$S = \begin{pmatrix} u\left(uz - wx\right) \\ v\left(uz - wx\right) - \nu\left(z\frac{\partial u}{\partial y} - x\frac{\partial w}{\partial y}\right) \\ w\left(uz - wx\right) - x\frac{p}{\rho} - \nu\left(z\frac{\partial u}{\partial z} - u - x\frac{\partial w}{\partial z}\right) \end{pmatrix} = \begin{pmatrix} u\left(uz - wx\right) \\ S_y \\ S_z \end{pmatrix}, \tag{5.10}$$

which has for its integral variant

(5.11)
$$\frac{\partial}{\partial x}\left[\int_{-\infty}^{+\infty} dz \int_{0}^{+\infty} u\,(uz - wx)\,dy\right] + \int_{-\infty}^{+\infty} S_y|_{y=0}^{+\infty}\,dz + \int_{0}^{+\infty} S_z|_{z=-\infty}^{+\infty}\,dy = 0.$$

Hence another boundary condition to be imposed on the pressure is

$$p(x,y,z)|_{z=-\infty} = p(x,y,z)|_{z=+\infty}. \tag{5.12}$$

Here the parameter of self-similarity is also $k = 2$.

For Ω_3, the conservation law takes the form

$$S = \begin{pmatrix} u\,(vz - wy) \\ v\,(vz - wy) + z\frac{p}{\rho} - \nu\big(y\frac{\partial w}{\partial y} - w - z\frac{\partial v}{\partial y}\big) \\ w\,(vz - wy) - y\frac{p}{\rho} + \nu\left(y\frac{\partial w}{\partial z} - v - z\frac{\partial v}{\partial z}\right) \end{pmatrix} = \begin{pmatrix} u\,(vz - wy) \\ S_y \\ S_z \end{pmatrix}, \tag{5.13}$$

with integral variant

$$\frac{\partial}{\partial x}\left[\int_{-\infty}^{+\infty} dz \int_{0}^{+\infty} u\,(vz - wy)\,dy\right] + \int_{-\infty}^{+\infty} S_y|_{y=0}^{+\infty}\,dz + \int_{0}^{+\infty} S_z|_{z=-\infty}^{+\infty}\,dy = 0.$$

Again, the parameter of self-similarity is $k = 2$. It is obvious that conservation laws appropriate to Ω_1, Ω_2, and Ω_3 give the same similarity exponent $k = 2$. A search of all solutions has been fulfilled similarly to that of section 4. As for conservation laws appropriate to $\Omega_4 - \Omega_7$, the integral forms under specified conditions are not physically relevant. An analogous analysis conducted for the incompressible Navier–Stokes equations has shown that there are only seven conservation laws: three conservation laws of the impulse components, three conservation laws of the impulse moment components, and mass conservation [10, 19].

6. Wall jets for non-Newtonian fluids. For simplicity we have chosen one class of time-independent fluids for which the shear rate at any point is a function of only the local shear stress; these are called dilatant fluids, the behavior of which can be described by the empirical functional relation known as the Ostwald–de Waele power law model with $n > 1$. In the case of rheological dilatancy, an increase in apparent viscosity with increasing shear rate occurs. The case $n < 1$ corresponding to pseudoplastic fluids is not considered here. The momentum equation takes the following appearance in tensor notation:

$$\begin{gathered} \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_k}{\partial x_k} = -\frac{\partial p \delta_{ik}}{\partial x_k} + \frac{\partial \sigma'_{ik}}{\partial x_k}, \\ \sigma'_{ik} = \kappa \left|\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right|^{n-1} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right). \end{gathered} \tag{6.1}$$

6.1. The 2D case. In Cartesian coordinates the Prandtl boundary layer equations for an incompressible power-law liquid, after nondimensionalization of the variables, become

$$\begin{gathered} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \end{gathered} \tag{6.2}$$

which must satisfy the boundary conditions

$$y = 0 : u = v = 0, \qquad y = \infty : u = 0.$$

Let us try to determine the conservation laws using the generating functions method, treating derivatives in a generalized sense. One thereby obtains the operator of universal linearization

$$l_F^{\epsilon} = \begin{pmatrix} \frac{\partial u}{\partial x} + L - n(n-1)\mathrm{sign}\left(\frac{\partial u}{\partial y}\right)\left|\frac{\partial u}{\partial y}\right|^{n-2}\frac{\partial^2 u}{\partial y^2}D_y - n\left|\frac{\partial u}{\partial y}\right|^{n-1}D_y^2 & \frac{\partial u}{\partial y} \\ D_x & D_y \end{pmatrix}, \tag{6.3}$$

which furnishes the conjugated operator

$$(l_F^{\epsilon})^* = \begin{pmatrix} \frac{\partial u}{\partial x} - L - nD_y\left|\frac{\partial u}{\partial y}\right|^{n-1}D_y & -D_x \\ \frac{\partial u}{\partial y} & -D_y \end{pmatrix}. \tag{6.4}$$

The final system determined from $(l_F^{\epsilon})^*(\Omega) = 0$ with $\Omega = (\varphi_1, \varphi_2)^T$ is

$$\frac{\partial u}{\partial x}\varphi_1 - u\frac{\partial \varphi_1}{\partial x} - v\frac{\partial \varphi_1}{\partial y} - n\frac{\partial}{\partial y}\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x} = 0,$$
$$\varphi_1\frac{\partial u}{\partial y} - \frac{\partial \varphi_2}{\partial y} = 0.$$

Eliminating φ_2, introducing the stream-function $u = \Psi_y$, $v = -\Psi_x$, and dropping the subscripts, one obtains

$$2\Psi_{xy}\varphi_y - 2\Psi_{yy}\varphi_x + \Psi_x\varphi_{yy} - \Psi_y\varphi_{xy} - n\frac{\partial^2}{\partial y^2}\left[|\Psi_{yy}|^{n-1}\varphi_y\right] = 0. \tag{6.5}$$

Using analogous reasoning to that of section 3.2, we assume a solution only in the form $\varphi = \varphi(\Psi, \Psi_y)$. Eliminating Ψ_{yyy}, Ψ_{yyyy} with the help of (6.2) and its differential consequences, we find $\varphi_\Psi = \varphi_{\Psi_y} = 0$, which leads to a conclusion of triviality of the proposed solution φ and implies the absence of local conservation laws in this case. One can generalize the problem statement (6.2) by considering an apparent viscosity in a form $f(\frac{\partial u}{\partial y})$ or some more complicated dependence. Investigation of that problem may give the proper limitations on the form of functional dependence that is indispensable for the existence of invariants.

6.2. The 3D case. Introducing the asymptotic solution representation ($\epsilon \ll 1$)

$$\begin{aligned} u &\Rightarrow u_0[u + \epsilon u' + \cdots], \\ v &\Rightarrow \epsilon u_0[v + \epsilon v' + \cdots], \\ w &\Rightarrow \epsilon u_0[w + \epsilon w' + \cdots], \\ p &\Rightarrow p_0[1 + \epsilon^2 p' + \cdots], \end{aligned}$$

and rescaling the coordinates according to

$$x \Rightarrow Lx, \qquad y \Rightarrow \epsilon Ly, \qquad z \Rightarrow \epsilon Lz, \tag{6.6}$$

after substitution into the Navier–Stokes equations, we obtain for the condition $(n > 1,\ L^n u_0^{2-n} \rho \epsilon^{1+n}/\kappa \sim 1)$ the system of PDEs:

$$\begin{aligned}
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial}{\partial y}\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z},\\
&u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p'}{\partial y} + \frac{\partial}{\partial x}\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y},\\
&u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p'}{\partial z} + \frac{\partial}{\partial x}\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z},\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{aligned} \tag{6.7}$$

Following the same procedure as in previous sections, we need to solve the system of equations $(l_F^{\epsilon})^*(\Omega) = 0$, $\Omega = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T$. Direct substitution of Ω_1, obtained in the previous section for $n = 1$, shows its validity. The appropriate differential form of the conservation law is

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[u\,(yu - xv) + x\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial y}\left[v\,(yu - xv) - xp' - y\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right]\\
&\qquad + \frac{\partial}{\partial z}\left[w\,(yu - xv) - y\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z}\right] = 0.
\end{aligned} \tag{6.8}$$

Group analysis provides the appropriate similarity reduction form

$$\begin{aligned}
&u = x^{\frac{k(1+n)-1}{n-2}} f_u(\eta, \zeta), && v = x^{\frac{k(2n-1)-(n-1)}{n-2}} f_v(\eta, \zeta),\\
&w = x^{\frac{k(2n-1)-(n-1)}{n-2}} f_w(\eta, \zeta), && p' = x^{2\frac{k(2n-1)-(n-1)}{n-2}} f_p(\eta, \zeta),\\
&\eta = \frac{y}{x^k}, && \zeta = \frac{z}{x^k}.
\end{aligned}$$

As a result, we have the integral constraint

$$\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} dz \int_0^{+\infty}\left[u\,(uy - vx) + x\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right] dy = 0, \tag{6.9}$$

or, in self-similar variables,

$$\frac{\partial}{\partial x} I = 0, \qquad I = x^{2k + \frac{3kn-2}{n-2}} \int_{-\infty}^{+\infty} d\zeta \int_0^{+\infty}\left[\eta f_u^2 - f_u f_v + \left|\frac{\partial f_u}{\partial \eta}\right|^{n-1}\frac{\partial f_u}{\partial \eta}\right] d\eta, \tag{6.10}$$

which requires the similarity exponent $k = \frac{2}{5n-4}$. Obviously, $n \to 1$ corresponds to $k \to 2$, as was already found in section 5.

7. Discussion and conclusions. The notion of conservation laws used throughout this paper is associated with equations describing the physical phenomenon but not with the phenomenon itself. For this reason, it may happen that different equations describing the same physical situation have different groups of conservation laws. For example, Euler and Lagrange approaches to the same continuum medium may lead to different sets of conservation laws, because the transition from Euler to Lagrange coordinates is a nonlocal transformation. Once again it should be noted that the results obtained above were found in the frame of local conservation law theory. However, the consideration of various types of nonlocality may, in principle, lead to new conservation currents according to the hypothesis [53, 54] that there exists a complete set of nonlocal conservation laws in a sufficiently small vicinity of any regular point of the PDEs. We can give the following informal justification of this suggestion. Consider an evolution system of PDEs,

$$\frac{\partial \mathbf{U}}{\partial t} = L(\mathbf{x}, \mathbf{U})\mathbf{U}, \quad \mathbf{x} \in \Omega.$$

If the domain Ω is finite and the nonlinear operator $L(\mathbf{x}, \mathbf{U})$ contains no singularities, one can apply Galerkin's method, the convergence of which was proved by Keldysh [25] and later by other authors, by representing the solution in a form

$$\mathbf{U} = \sum_{n=1}^{\infty} u_n(t)\varphi_n(\mathbf{x}),$$

which leads to an infinite-dimensional system of ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}), \quad \mathbf{u} = (u_1, u_2, \dots)^T.$$

Consider a truncation of this system such that $\mathbf{u}$ belongs to some domain $\mathcal{U}$ in m-dimensional Euclidean space. It is known [5] that there exists a neighborhood of a nonsingular point $\mathbf{u}$ such that the truncated system has $m-1$ functionally independent first integrals, which are referred to as *local first integrals*. Now recalling that in the case of orthogonal basis functions $\varphi_n(\mathbf{x})$,

$$u_n(t) = \frac{\langle \mathbf{U}, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle},$$

one concludes that the corresponding conservation laws for the original solution vector $\mathbf{U}$ are *nonlocal*. The convergence of Galerkin's method admits the limit $m \to \infty$. In this case the set of conservation laws is countable. One can expect in the case of extended domain Ω a "continuous" spectrum of conservation laws, or both "discrete" and "continuous" spectra.

In addition, one should mention that there is another question still pending about the correspondence between a manifold of solutions obtained by determining hidden invariances and a manifold of solutions for the associated nonlinear eigenvalue problem.

This study provides, for the first time, an overall picture for hidden invariances in wall jet problems which are obtained with the help of constructive mathematical tools for the determination of the local conservation laws of a given system of PDEs. First, the classical result of Akatnov [1] and Glauert [15] was reproduced, and uniqueness of the self-similarity exponent was demonstrated. Consideration of the 3D jet described by the Prandtl boundary layer equations, along with an analysis in the plane of symmetry, leads to a two-parameter self-similar solution with the admissible spectrum of eigenvalues k, l:

$$k \in \left(\frac{3}{4}, +\infty\right), \qquad l \in \left(\frac{1}{8}, +\infty\right).$$

Analogously, the 3D wall jet described by the 3D parabolized Navier–Stokes equations has been investigated, producing the unique self-similarity exponent $k = 2$.

Finally, time-independent dilitant non-Newtonian power-law liquids, for which the consistency index satisfies $n > 1$, were considered in both 2D and 3D situations.

For the 2D case an absence of local conservation laws was demonstrated. Analysis of the 3D case, on the other hand, yielded a local conservation law under the requirement that the similarity exponent k satisfy $k = \frac{2}{5n-4}$, consistent with the planar wall jet of a Newtonian fluid for which $n = 1$ and $k = 3/4$.

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