# Local and global fundamental classes for multiquadratic extensions

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**Abstract**

Using a reformulation of an approach of Serre, we provide a description of the fundamental class of an arbitrary multiquadratic extension of local fields. For a multiquadratic extension of number fields $L/K$ such that 2 splits completely in $L$, we exhibit the global fundamental class. We obtain descriptions of the local and global reciprocity maps as consequences.

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## 1 Introduction

The fundamental class of a finite Galois extension of local or global fields is a central object in the cohomological approach to class field theory, as in Artin–Tate [1], Tate [10] and Serre [7]. In both the local and global theories, knowledge of the fundamental class is stronger than that of the reciprocity map (local or global) of the given extension. Indeed, one can determine the reciprocity map from the fundamental class, though not necessarily vice versa. In the local case, the fundamental class corresponds to a canonical class of central simple algebras in the Brauer group. The global setting is more intricate. The existence of the global fundamental class involves a subtle argument – see [10] – incorporating the local fundamental classes of all completions, and not just completions of the given global extension, but of possibly larger extensions: in general the global fundamental class of a given extension may not be determined from the local fundamental classes associated to *the same* extension.

The importance of fundamental classes to some of the current goals of algebraic number theory can be seen in, for example, the conjectures of Gruenberg–Ritter–Weiss and Burns–Flach extending Stark's Conjecture on special values of Artin $L$-functions. Indeed, the Equivariant Tamagawa Number Conjecture [2] for Tate motives, in its formulation as the Lifted Root Number Conjecture [4], has at its arithmetic core a certain four term sequence called a Tate sequence, which is defined purely in terms of local and global fundamental classes.

### 1.1 General multiquadratic extensions

Let $L/K$ be an extension of local or global fields of characteristic zero. In this article, we will be interested in the situation where $L/K$ is a multiquadratic extension, that is, $L/K$ is finite abelian with Galois group $G$ of exponent 2, so that $G$ is isomorphic to a product $\mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$. Equivalently, $L$ takes the form $K(\sqrt{a_1}, \ldots, \sqrt{a_n})$ for some $a_1, \ldots, a_n \in K^\times$. Our aim is to make the fundamental class (local or global as appropriate) of $L/K$ as explicit as possible, at least under some hypotheses.

### 1.2 Overview

The outline of the article is as follows: Section 2 recalls some group cohomological results that will be used later on, and may be ignored until required. Section 3 deals with the local theory. In particular, in Section 3.1 we describe the relevant result of Serre on local fundamental classes and then provide in Section 3.2 a general reformulation in language that is more amenable for our purposes. The main result on the local fundamental class for multiquadratic extensions is stated in Theorem 3.4, with a brief discussion on the reciprocity map and the odd residue characteristic case in Sections 3.3.5 and 3.3.6 respectively. After some remarks on obtaining global fundamental classes from local ones in Section 4.1, the main result on the global fundamental class – Theorem 4.4 – is stated and proven in Section 4.2. As a consequence of this theorem, we give a description of the global reciprocity map in Corollary 4.12.

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## 2 Group cohomological prerequisites

If $G$ is a finite group and $A$ a $\mathbb{Z}[G]$-module, $H^i(G, A)$ will denote Tate cohomology, as in [6, p.23]. A *normalized* 2-cocycle $f : G^2 \to A$ will mean a 2-cocycle such that $f(1, \sigma) = f(\sigma, 1) = 0$ for all $\sigma \in G$.

### 2.1 Dimension shifting

**Lemma 2.1** *(i) For any $\mathbb{Z}[G]$-module $A$ and any integer $i$, there is a canonical isomorphism*

$$H^i(G, \mathrm{Hom}_{\mathbb{Z}}(\Delta G, A)) \to H^{i+1}(G, A),$$

*where $\Delta G$ is the kernel of the augmentation map $\mathbb{Z}[G] \to \mathbb{Z}$ that sends every group element to 1.*

*(ii) In the case $i = 1$, the class of a 1-cocycle $c : G \to \mathrm{Hom}_{\mathbb{Z}}(\Delta G, A)$ corresponds to the class of the normalized 2-cocycle*

$$\begin{array}{rcl} G^2 & \to & A \\ (\sigma, \tau) & \mapsto & c(\sigma)(\sigma - 1) - c(\sigma\tau)(\sigma - 1), \end{array}$$

*and the class of a normalized 2-cocycle $f : G^2 \to A$ corresponds to the class of the 1-cocycle*

$$\begin{array}{rcl} G & \to & \mathrm{Hom}_{\mathbb{Z}}(\Delta G, A) \\ \sigma & \mapsto & (\tau - 1 \mapsto f(\tau, \tau^{-1}) - f(\tau, \tau^{-1}\sigma)). \end{array}$$

*Proof.* Part (i) follows from the exactness of the sequence

$$0 \to A \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \to \mathrm{Hom}_{\mathbb{Z}}(\Delta G, A) \to 0$$

together with the cohomological triviality of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$. Part (ii) is the result of a direct calculation, which is omitted. ■

## 2.2 Cup-product

Let $f : G^2 \to A$ be a 2-cocycle.

**Lemma 2.2** *Identifying $H^{-2}(G, \mathbb{Z})$ with $G^{\mathrm{ab}}$, the map $H^{-2}(G, \mathbb{Z}) \to H^0(G, A)$ obtained by cupping (on either side) with the class of $f$ sends the class of an element $\tau \in G$ to the class of the element $\sum_{\sigma \in G} f(\sigma, \tau)$ of $A^G$.*

*Proof.* Standard. ■

## 2.3 Inflation-restriction in dimension 2

For simplicity, assume now that $G$ is a finite abelian group that is the internal direct product of subgroups $H$ and $K$. Let $A$ be a $\mathbb{Z}[G]$-module, and assume that $H^1(H, A) = 0$. Suppose we are given a 2-cocycle $f : G^2 \to A$ whose restriction to $H$ is equal to the 2-coboundary associated to a map $g : H \to A$. For $\sigma \in G$ and $\tau \in H$, set

$$a_{\sigma,\tau} = f(\tau, \sigma) - f(\sigma, \tau) + (\sigma - 1)g(\tau) \in A.$$

It is straightforward to check that for fixed $\sigma \in G$, the map

$$\begin{array}{ccc} H & \to & A \\ \tau & \mapsto & a_{\sigma,\tau} \end{array}$$

is a 1-cocycle, and therefore by the assumption on $H^1(H, A)$, there is $a_\sigma \in A$ such that, for all $\tau \in H$, $a_{\sigma,\tau} = \tau a_\sigma - a_\sigma$.

**Lemma 2.3** *Let notation be as above.*
*(i) Given $\sigma, \rho \in K$ the element $f(\sigma, \rho) - (\sigma a_\rho - a_{\sigma\rho} + a_\sigma)$ is fixed by $H$.*
*(ii) The map*

$$\begin{array}{ccc} K^2 & \to & A^H \\ (\sigma, \rho) & \mapsto & f(\sigma, \rho) - (\sigma a_\rho - a_{\sigma\rho} + a_\sigma) \end{array}$$

*is a 2-cocycle whose inflation to $G$ (identifying $K$ with $G/H$) represents the class of $f$.*

*Proof.* (i) is a direct computation using only the definitions and the fact that $f$ is a 2-cocycle. (ii) follows immediately from (i). ■

# 3 The local fundamental class

In this section we fix a prime $p$, and a completion $\mathbb{C}_p$ of an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. All fields will be contained in $\mathbb{C}_p$. If $M$ is finite over $\mathbb{Q}_p$, then $\widehat{M}_{\mathrm{ur}}$ denotes the completion of the maximal unramified extension $M_{\mathrm{ur}}$ of $M$. We will denote by $\varphi_M$ the extension to $\widehat{M}_{\mathrm{ur}}$ of the Frobenius of $M$. In what follows, we will fix a base field $K$; for convenience, we will abbreviate $\varphi_K$ to just $\varphi$.

## 3.1 Serre's construction of local fundamental classes

Our strategy for computing local fundamental classes is to follow Serre's general construction in [8, p.202]. This construction is also described in more detail by Snaith in [9, pp.9–14], and is used in [5] in the proof of Chinburg's Second Conjecture for quaternion fields. Let us briefly recall the picture. $L/K$ can be any finite Galois extension of local fields with Galois group $G$. Let $\widehat{K}_{\mathrm{ur}}$, respectively $\widehat{L}_{\mathrm{ur}}$, be the completion of the maximal unramified extension of $K$, respectively $L$. Then there are short exact sequences

$$0 \to L^\times \to (\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times \to V \to 0 \tag{3.1}$$

and

$$0 \to V \to (\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times \overset{\omega}{\to} \mathbb{Z} \to 0, \tag{3.2}$$

where the map $\omega$ is the sum of the valuations on all components (see [9, p.10]) and $V = \mathrm{Ker}(\omega)$. To be more precise, there is a ring isomorphism

$$\begin{aligned} \widehat{K}_{\mathrm{ur}} \otimes_K L &\to \prod_{j=1}^{d} \widehat{L}_{\mathrm{ur}} \\ a \otimes b &\mapsto (\varphi^{d-j}(a)b)_j \end{aligned} \tag{3.3}$$

where $d$ is the residue degree of $L/K$, whose restriction to $(\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times$ is denoted $\Psi$. Then $\omega$ is equal to $\Psi$ followed by the sum of the valuations on the copies of $\widehat{L}_{\mathrm{ur}}$. The right-hand map in (3.1) is $\varphi - 1$, where $\varphi$ acts on $(\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times$ via $\widehat{K}_{\mathrm{ur}}$. Also, $G$ acts on $\widehat{K}_{\mathrm{ur}} \otimes_K L$ via its action on $L$, and this action restricts to an action on $(\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times$. Thus the above sequences become sequences of $\mathbb{Z}[G]$-modules, and in fact $V$ is cohomologically trivial. The upshot of [8, p.202] is that the local fundamental class for $L/K$ is the image of the class of $-1$ under the composition of connecting homomorphisms

$$H^0(G, \mathbb{Z}) \overset{\cong}{\to} H^1(G, V) \overset{\cong}{\to} H^2(G, L^\times) \tag{3.4}$$

associated to (3.1) and (3.2). These connecting homomorphisms will be examined in Section 3.3.2, where the proof of our main theorem in the local case is carried out.

## 3.2 Alternative description of the short exact sequences

It is convenient to replace $(\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times$ by $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^\times$ in Section 3.1, where $H = \mathrm{Gal}(L/E)$ with $E$ the maximal unramified subextension of $L/K$. Let us explain why this is possible. We begin by observing that there is indeed a (canonical) $\mathbb{Z}[G]$-module isomorphism $\Theta : (\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^\times$. We introduce some notation first. Choose $\alpha \in L$ such that $L = K(\alpha)$, and fix a set $\mathcal{L}$ of representatives for $(G/H)_{\mathrm{left}}$. For each $\tau \in \mathcal{L}$, let $\alpha_\tau = \alpha^{\tau^{-1}}$. Also, let $f(x) \in K[x]$ be the minimal polynomial for $\alpha$ over $K$.

Given $g(x) = \sum_k a_k x^k \in \widehat{K}_{\mathrm{ur}}[x]$, define $[g(x)] = \sum_k a_k \otimes \alpha^k \in \widehat{K}_{\mathrm{ur}} \otimes_K L$. If $g(x)$ is coprime to $f(x)$, then $[g(x)] \in (\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times$ and

$$\Theta[g(x)] = \sum_{\tau \in \mathcal{L}} \tau \otimes g(\alpha_\tau).$$

This isomorphism is independent of the choice of $\alpha$ and $\mathcal{L}$.

We may also describe $\Theta$ as follows: If $\sum_i a_i \otimes b_i \in (\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times$, then

$$\Theta\left(\sum_i a_i \otimes b_i\right) = \sum_{\tau \in \mathcal{L}} \tau \otimes \sum_i a_i b_i^{\tau^{-1}}.$$

To describe the inverse, we let $f_\tau(x) \in \widehat{K}_{\mathrm{ur}}[x]$ be the minimal polynomial for $\alpha_\tau$ over $\widehat{K}_{\mathrm{ur}}$ for each $\tau \in \mathcal{L}$. Note that in fact $f_\tau(x) \in E[x]$. We then choose polynomials $\mathbf{1}_\tau(x) \in E[x]$ such that

$$\begin{aligned} \mathbf{1}_\tau(x) &\equiv 1 \bmod f_\tau(x) \\ \mathbf{1}_\tau(x) &\equiv 0 \bmod f_{\tau'}(x) \text{ for } \tau' \in \mathcal{L} \setminus \{\tau\}. \end{aligned}$$

Then if $g_\tau(x) \in \widehat{K}_{\mathrm{ur}}[x]$ is coprime to $f_\tau(x)$ for each $\tau \in \mathcal{L}$, we have

$$\Theta^{-1}\left(\sum_{\tau \in \mathcal{L}} \tau \otimes g_\tau(\alpha_\tau)\right) = \left[\sum_{\tau \in \mathcal{L}} g_\tau(x) \mathbf{1}_\tau(x)\right].$$

**Remark**. Even though $\widehat{L}_{\mathrm{ur}}^\times$ is written multiplicatively, we choose to write $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^\times$ additively.

### 3.2.1 Frobenius

We observe that since $\varphi$ acts on $(\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times$, by acting on $\widehat{K}_{\mathrm{ur}}$, $\varphi$ therefore also acts on $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^\times$ via $\Theta$. We will determine the precise effect of $\varphi$ on $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^\times$ in our setting in Section 3.3.2.

#### 3.2.2 Valuation

There is a natural valuation map $\omega' : \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^{\times} \to \mathbb{Z}$, namely $\omega' = \mathrm{aug} \otimes \nu$ where $\mathrm{aug} : Z[G] \to \mathbb{Z}$ is the augmentation map and $\nu : \widehat{L}_{\mathrm{ur}}^{\times} \to \mathbb{Z}$ is the normalized valuation on $L$ extended to $\widehat{L}_{\mathrm{ur}}$.

**Lemma 3.1** *The map $\omega' \circ \Theta$ agrees with the map $\omega$ of Section 3.1.*

*Proof.* For each $j \in \{1, \ldots, d\}$, let $\tau_j$ be the unique element of $\mathcal{L}$ such that $(\tau_j)|_E = \varphi|_E^j$. Let $\sigma_j$ be an automorphism of $\widehat{L}_{\mathrm{ur}}$ whose restrictions to $\widehat{K}_{\mathrm{ur}}$ and $L$ are $\varphi^j$ and $\tau_j$ respectively. Take $\sum_i a_i \otimes b_i \in (\widehat{K}_{\mathrm{ur}} \otimes_K L)^{\times}$. Then

$$\begin{aligned}
\omega' \circ \Theta \left( \sum_i a_i \otimes b_i \right) &= \sum_{j=1}^{d} \nu \left( \sum_i a_i b_i^{\tau_j^{-1}} \right) \\
&= \sum_{j=1}^{d} \nu \left( \left( \sum_i a_i^{\sigma_j} b_i \right)^{\sigma_j^{-1}} \right) \\
&= \sum_{j=1}^{d} \nu \left( \sum_i a_i^{\sigma_j} b_i \right) \\
&= \sum_{j=1}^{d} \nu \left( \sum_i a_i^{\varphi^j} b_i \right) \\
&= \omega \left( \sum_i a_i \otimes b_i \right),
\end{aligned}$$

the last equality holding because the order of the terms in the sum over $j$ is unimportant. ■

As a consequence of the above discussions, we may replace sequences (3.1) and (3.2) by

$$0 \to L^{\times} \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^{\times} \xrightarrow{\varphi-1} V' \to 0$$

and

$$0 \to V' \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^{\times} \xrightarrow{\omega'} \mathbb{Z} \to 0$$

where $V' = \mathrm{Ker}(\omega')$, and the map $L^{\times} \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^{\times}$ is given by

$$b \mapsto \sum_{\tau \in \mathcal{L}} \tau \otimes b^{\tau^{-1}}.$$

Let $\tau_0$ be the unique element of $\mathcal{L} \cap H$. The following lemma gives us a convenient way of recognizing elements of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^{\times}$ annihilated by $\varphi - 1$ as images of elements of $L^{\times}$.

**Lemma 3.2** *Let $\lambda = \sum_{\tau\in\mathcal{L}} \tau \otimes a_\tau \in \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^{\times}$. If $(\varphi - 1)\lambda = 0$, then $a_{\tau_0} \in L^\times$ and $\Theta^{-1}(\lambda)$ is the image of $a_{\tau_0}$ under $L^\times \to (\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times$.*

*Proof.* If $(\varphi - 1)\lambda = 0$, then in $\prod_{j=1}^{d} \widehat{L}_{\mathrm{ur}}^{\times}$ we have $(\Psi \circ \Theta^{-1}(\lambda))^{\varphi-1} = 1$. By the proof of [9, Lemma 1.2.7], all components of $\Psi \circ \Theta^{-1}(\lambda)$ are therefore equal and lie in $L^\times$, and further $\Theta^{-1}(\lambda)$ is the image under $L^\times \to (\widehat{K}_{\mathrm{ur}} \otimes_K L)^\times$ of any component. It thus remains to determine any component of $\Psi \circ \Theta^{-1}(\lambda)$; we choose the $d$-component.

For each $\tau \in \mathcal{L}$, write $a_\tau = g_\tau(\alpha_\tau)$ for some $g_\tau(x) \in \widehat{K}_{\mathrm{ur}}[x]$. Then $\Theta^{-1}(\lambda) = \left[\sum_{\tau\in\mathcal{L}} g_\tau(x)\mathbf{1}_\tau(x)\right]$, and one may verify that the image of this under $\Psi$ has $d$-component

$$\sum_{\tau\in\mathcal{L}} g_\tau(\alpha)\mathbf{1}_\tau(\alpha).$$

Now, since $\tau_0$ acts trivially on $\widehat{K}_{\mathrm{ur}}$,

$$\begin{aligned} f_{\tau_0}(\alpha) &= f_{\tau_0}^{\tau_0}(\alpha) \\ &= f_{\tau_0}^{\tau_0}(\alpha_{\tau_0}^{\tau_0}) \\ &= f_{\tau_0}(\alpha_{\tau_0})^{\tau_0} \\ &= 0. \end{aligned}$$

Thus, by the choice of the $\mathbf{1}_\tau(x)$,

$$\sum_{\tau\in\mathcal{L}} g_\tau(\alpha)\mathbf{1}_\tau(\alpha) = g_{\tau_0}(\alpha_{\tau_0}) = a_{\tau_0}.$$

■

## 3.3 Local multiquadratic extensions

We assume in this section that $L/K$ is a multiquadratic extension of local fields with Galois group $G$. Let $\mathfrak{r}$ be a minimal set of generators for $G$, so that $G$ is the internal direct product of distinct order 2 subgroups $\langle\rho\rangle$ with $\rho \in \mathfrak{r}$.

### 3.3.1 Statement of the local theorem

For ease of notation, we break up the statement into two cases: (TR) is the case that $L/K$ is totally ramified, while (NTR) is the other case, namely that $L/K$ is not totally ramified.

In case (NTR), we can, and will, assume that there is $\tau \in \mathfrak{r}$ such that the subgroup of $G$ generated by $\mathfrak{r}' = \mathfrak{r} \smallsetminus \{\tau\}$ has fixed field equal to the quadratic unramified extension $E$ of $K$. Letting $H = \mathrm{Gal}(L/E)$, we thus have $H = \langle\mathfrak{r}'\rangle$. Note that $H$ acts on $\widehat{L}_{\mathrm{ur}}$ by extending to an automorphism fixing $\widehat{K}_{\mathrm{ur}}$. We also let $F$ be the fixed field of $\langle\tau\rangle$ in this case.

In general, if $\mathfrak{s}$ is a (possibly empty) subset of $\mathfrak{r}$, we write $\sigma_{\mathfrak{s}} = \prod_{\rho\in\mathfrak{s}} \rho$. If $\mathfrak{s}$ and $\mathfrak{t}$ are two such subsets, then define $\mathfrak{s} + \mathfrak{t} = (\mathfrak{s} \cup \mathfrak{t}) \smallsetminus (\mathfrak{s} \cap \mathfrak{t})$. In this way,

$\sigma_{\mathfrak{s}+\mathfrak{t}} = \sigma_{\mathfrak{s}}\sigma_{\mathfrak{t}}$. Observe that every element of $G$ is equal to $\sigma_{\mathfrak{s}}$ for a unique $\mathfrak{s}$. In case (NTR), we choose rather to write elements as $\sigma_{\mathfrak{s}}\tau^i$ with $\mathfrak{s} \subseteq \mathfrak{r}'$ and $i = 0, 1$.

The following is Proposition 15 in Section 5 of Chapter XIII of [8].

**Lemma 3.3** *If $L$ is a local field and $a$ is a unit in $\widehat{L}_{\mathrm{ur}}$, then there is a unit $b$ in $\widehat{L}_{\mathrm{ur}}$ such that $b^{\varphi_L - 1} = a$.*

Choose a uniformizer $\varpi$ of $L$. In case (NTR), we can, and will, assume that $\varpi$ lies in $F$. By Lemma 3.3, for each $\mathfrak{s} \subseteq \mathfrak{r}$ we may choose $\eta_{\mathfrak{s}} \in \widehat{L}_{\mathrm{ur}}^{\times}$ such that $\eta_{\mathfrak{s}}^{\varphi_L - 1} = \varpi^{\sigma_{\mathfrak{s}} - 1}$. In Section 3.3.3, we will explain how a choice of $\eta_{\mathfrak{s}}$ for each singleton $\mathfrak{s}$ leads to a choice of $\eta_{\mathfrak{s}}$ for all subsets $\mathfrak{s}$, but for now this does not matter.

**Theorem 3.4** *Let $\xi_{L/K}$ be the fundamental class of the extension $L/K$. There is a 2-cocycle $g : G^2 \to L^{\times}$ representing $-\xi_{L/K}$ given as follows:*

*(i) In case (TR),*

$$g(\sigma_{\mathfrak{s}}, \sigma_{\mathfrak{t}}) = \eta_{\mathfrak{t}}^{\sigma_{\mathfrak{s}}} \eta_{\mathfrak{s}} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1}$$

*for all subsets $\mathfrak{s}, \mathfrak{t} \subseteq \mathfrak{r}$.*

*(ii) In case (NTR),*

$$\begin{aligned} g(\sigma_{\mathfrak{s}}, \sigma_{\mathfrak{t}}) &= \eta_{\mathfrak{t}}^{\sigma_{\mathfrak{s}}} \eta_{\mathfrak{s}} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1} \\ g(\sigma_{\mathfrak{s}}, \sigma_{\mathfrak{t}}\tau) &= \eta_{\mathfrak{t}}^{\sigma_{\mathfrak{s}}} \eta_{\mathfrak{s}} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1} \varpi^{\sigma_{\mathfrak{s}} - 1} \\ g(\sigma_{\mathfrak{s}}\tau, \sigma_{\mathfrak{t}}) &= \eta_{\mathfrak{t}}^{\sigma_{\mathfrak{s}}\varphi_F} \eta_{\mathfrak{s}} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1} \\ g(\sigma_{\mathfrak{s}}\tau, \sigma_{\mathfrak{t}}\tau) &= \eta_{\mathfrak{t}}^{\sigma_{\mathfrak{s}}\varphi_F} \eta_{\mathfrak{s}} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1} \varpi^{\sigma_{\mathfrak{s}} - \sigma_{\mathfrak{s}+\mathfrak{t}} + 1} \end{aligned}$$

*for all subsets $\mathfrak{s}, \mathfrak{t} \subseteq \mathfrak{r}'$.*

**Remark**. A 2-cocycle for $\xi_{L/K}$ itself is obtained simply by taking the reciprocals of the elements in the theorem.

In Section 3.3.6, we will restrict to the case of odd residue characteristic. In that situation, there is essentially only one $\eta_{\mathfrak{s}}$ to choose (at most), and this itself can be made explicit as a certain root of unity. See Corollary 3.8.

### 3.3.2 Proof of Theorem 3.4

We now prove Theorem 3.4. We choose to do so only in the case (NTR). Indeed, the case (TR) is easier and contains nothing that is not dealt with in the other case. Further, we can avoid notational difficulties by restricting to just one case.

In the notation of Section 3.2, $\omega'(1 \otimes \varpi) = 1$, and so the image of the class of 1 under the connecting homomorphism $H^0(G, \mathbb{Z}) \to H^1(G, V')$ is represented by the 1-cocycle described by

$$\begin{aligned} \sigma_{\mathfrak{s}} &\mapsto (\sigma_{\mathfrak{s}} - 1)(1 \otimes \varpi) = 1 \otimes \varpi^{\sigma_{\mathfrak{s}} - 1} && (3.5) \\ \sigma_{\mathfrak{s}}\tau &\mapsto (\sigma_{\mathfrak{s}}\tau - 1)(1 \otimes \varpi) = 1 \otimes \varpi^{-1} + \tau \otimes \varpi^{\sigma_{\mathfrak{s}}} && (3.6) \end{aligned}$$

for $\mathfrak{s} \subseteq \mathfrak{r}'$.

Our next step is to lift the elements in the right-hand side of (3.5) and (3.6) under $\varphi - 1$.

**Lemma 3.5** *If $a_1, a_\tau \in \widehat{L}_{\mathrm{ur}}^\times$, then*

$$\varphi(1 \otimes a_1 + \tau \otimes a_\tau) = 1 \otimes a_\tau^{\varphi_F} + \tau \otimes a_1^{\varphi_F}.$$

*Proof.* In the notation of Section 3.2, we take $\mathcal{L}$ to be $\{1, \tau\}$, and we choose any $\alpha \in L$ such that $L = K(\alpha)$. Let $f_1(x), f_\tau(x)$ be as defined in the same section. We write $a_1 = g_1(\alpha_1)$ and $a_\tau = g_\tau(\alpha_\tau)$ for some $g_1(x), g_\tau(x) \in \widehat{K}_{\mathrm{ur}}[x]$. Then

$$\begin{aligned} \varphi(\Theta^{-1}(1 \otimes a_1 + \tau \otimes a_\tau)) &= \varphi[g_1(x)\mathbf{1}_1(x) + g_\tau(x)\mathbf{1}_\tau(x)] \\ &= [g_1^\varphi(x)\mathbf{1}_1^\varphi(x) + g_\tau^\varphi(x)\mathbf{1}_\tau^\varphi(x)], \end{aligned} \tag{3.7}$$

where for a polynomial $h(x) \in \widehat{K}_{\mathrm{ur}}[x]$, $h^\varphi(x)$ is the polynomial obtained by letting $\varphi$ act on the coefficients. Now, $\varphi$ permutes $f_1(x)$ and $f_\tau(x)$ but fixes neither (since neither lies in $K[x]$), and so $f_1^\varphi(x) = f_\tau(x)$ and $f_\tau^\varphi(x) = f_1(x)$. Consequently, $\mathbf{1}_1^\varphi(\alpha_\tau)$ and $\mathbf{1}_\tau^\varphi(\alpha_1)$ are both equal to 1 and $\mathbf{1}_1^\varphi(\alpha_1)$, and $\mathbf{1}_\tau^\varphi(\alpha_\tau)$ are both equal to 0. Therefore the element in (3.7) maps under $\Theta$ to

$$\begin{aligned} 1 \otimes g_\tau^\varphi(\alpha_1) + \tau \otimes g_1^\varphi(\alpha_\tau) &= 1 \otimes g_\tau(\alpha_\tau)^{\varphi_F} + \tau \otimes g_1(\alpha_1)^{\varphi_F} \\ &= 1 \otimes a_\tau^{\varphi_F} + \tau \otimes a_1^{\varphi_F}. \end{aligned}$$

■

As a consequence of Lemma 3.5, and using the fact that $\varphi_F^2 = \varphi_L$, we find that if $\mathfrak{s} \subseteq \mathfrak{r}'$, then

$$\begin{aligned} (\varphi - 1)(1 \otimes \eta_\mathfrak{s} + \tau \otimes \eta_\mathfrak{s}^{\varphi_F}) &= 1 \otimes \eta_\mathfrak{s}^{\varphi_L} + \tau \otimes \eta_\mathfrak{s}^{\varphi_F} - 1 \otimes \eta_\mathfrak{s} - \tau \otimes \eta_\mathfrak{s}^{\varphi_F} \\ &= 1 \otimes \eta_\mathfrak{s}^{\varphi_L - 1} \\ &= 1 \otimes \varpi^{\sigma_\mathfrak{s} - 1}. \end{aligned}$$

Similarly,

$$(\varphi - 1)(1 \otimes \eta_\mathfrak{s} + \tau \otimes \eta_\mathfrak{s}^{\varphi_F} \varpi^{1 - \sigma_\mathfrak{s}}) = \tau \otimes \varpi^{\sigma_\mathfrak{s} - 1}.$$

This also uses the fact that $\varphi_F$ fixes $\varpi$. (Recall that $\varpi$ was chosen to lie in $F$.)

It is immediate that

$$(\varphi - 1)(1 \otimes \varpi) = 1 \otimes \varpi^{-1} + \tau \otimes \varpi$$

(again using that $\varphi_F$ fixes $\varpi$).

As a result of the preceding calculations, we see that we can lift the map $G \to V'$ described by (3.5) and (3.6) to the following map:

$$\begin{aligned} m : G &\to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^\times \\ \sigma_\mathfrak{s} &\mapsto 1 \otimes \eta_\mathfrak{s} + \tau \otimes \eta_\mathfrak{s}^{\varphi_F} \\ \sigma_\mathfrak{s}\tau &\mapsto 1 \otimes \eta_\mathfrak{s}\varpi + \tau \otimes \eta_\mathfrak{s}^{\varphi_F} \varpi^{1 - \sigma_\mathfrak{s}} \end{aligned}$$

for $\mathfrak{s} \subseteq \mathfrak{r}'$.

It remains to find $sm(t) - m(st) + m(s)$ for all $s, t \in G$, and recognize these elements as elements coming from $L^\times$. We may make the calculations easier as follows: Let $U_1 = \{1 \otimes b \mid b \in \widehat{L}_{\mathrm{ur}}^\times\}$ and $U_\tau = \{\tau \otimes b \mid b \in \widehat{L}_{\mathrm{ur}}^\times\}$, so that $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^\times = U_1 \oplus U_\tau$ as $\mathbb{Z}$-modules. Since, by construction, $(\varphi - 1)(sm(t) - m(st) + m(s)) = 0$ for all $s, t$, by Lemma 3.2 we need only look at the projection of $sm(t) - m(st) + m(s)$ onto $U_1$, and therefore we may work mod $U_\tau$ (remembering also that $\tau$ maps $U_1$ to $U_\tau$ and vice versa). Let $\sim$ denote the equivalence relation on $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \widehat{L}_{\mathrm{ur}}^\times$ given by equivalence mod $U_\tau$.

There are four cases:

$$\begin{array}{rll} \text{(i)} & s = \sigma_\mathfrak{s}, & t = \sigma_\mathfrak{t} \\ \text{(ii)} & s = \sigma_\mathfrak{s}, & t = \sigma_\mathfrak{t}\tau \\ \text{(iii)} & s = \sigma_\mathfrak{s}\tau, & t = \sigma_\mathfrak{t} \\ \text{(iv)} & s = \sigma_\mathfrak{s}\tau, & t = \sigma_\mathfrak{t}\tau \end{array}$$

with $\mathfrak{s}, \mathfrak{t} \subseteq \mathfrak{r}'$. In the first case,

$$\begin{aligned} sm(t) - m(st) + m(s) &\sim 1 \otimes \eta_\mathfrak{t}^{\sigma_\mathfrak{s}} - 1 \otimes \eta_{\mathfrak{s}+\mathfrak{t}} + 1 \otimes \eta_\mathfrak{s} \\ &= 1 \otimes \eta_\mathfrak{t}^{\sigma_\mathfrak{s}} \eta_\mathfrak{s} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1}. \end{aligned}$$

In the second case,

$$\begin{aligned} sm(t) - m(st) + m(s) &\sim 1 \otimes \eta_\mathfrak{t}^{\sigma_\mathfrak{s}} \varpi^{\sigma_\mathfrak{s}} - 1 \otimes \eta_{\mathfrak{s}+\mathfrak{t}} \varpi + 1 \otimes \eta_\mathfrak{s} \\ &= 1 \otimes \eta_\mathfrak{t}^{\sigma_\mathfrak{s}} \eta_\mathfrak{s} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1} \varpi^{\sigma_\mathfrak{s} - 1}. \end{aligned}$$

In the third case,

$$\begin{aligned} sm(t) - m(st) + m(s) &\sim 1 \otimes \eta_\mathfrak{t}^{\sigma_\mathfrak{s}\varphi_F} - 1 \otimes \eta_{\mathfrak{s}+\mathfrak{t}} \varpi + 1 \otimes \eta_\mathfrak{s} \varpi \\ &= 1 \otimes \eta_\mathfrak{t}^{\sigma_\mathfrak{s}\varphi_F} \eta_\mathfrak{s} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1}. \end{aligned}$$

Finally, in the fourth case,

$$\begin{aligned} sm(t) - m(st) + m(s) &\sim 1 \otimes \eta_\mathfrak{t}^{\sigma_\mathfrak{s}\varphi_F} \varpi^{\sigma_\mathfrak{s} - \sigma_\mathfrak{s}\sigma_\mathfrak{t}} - 1 \otimes \eta_{\mathfrak{s}+\mathfrak{t}} + 1 \otimes \eta_\mathfrak{s} \varpi \\ &= 1 \otimes \eta_\mathfrak{t}^{\sigma_\mathfrak{s}\varphi_F} \eta_\mathfrak{s} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1} \varpi^{\sigma_\mathfrak{s} - \sigma_{\mathfrak{s}+\mathfrak{t}} + 1}. \end{aligned}$$

This completes the proof of Theorem 3.4.

#### 3.3.3 Choosing the $\eta_\mathfrak{s}$

In this section, we show how a choice of $\eta_\mathfrak{s}$ for each singleton $\mathfrak{s} = \{\rho\} \subseteq \mathfrak{r}$ leads to a choice of $\eta_\mathfrak{s}$ for all $\mathfrak{s} \subseteq \mathfrak{r}$. Choose an ordering of $\mathfrak{r}$, and let $<$ denote strict inequality in that ordering. Then given $\mathfrak{s} \subseteq \mathfrak{r}$ and $\rho \in \mathfrak{s}$, define

$$\rho_\mathfrak{s} = \prod_{\substack{\delta \in \mathfrak{s} \\ \delta < \rho}} \delta.$$

Now, if $\rho \in \mathfrak{r}$, choose $\eta_\rho \in \widehat{L}_{\mathrm{ur}}^\times$ such that $\eta_\rho^{\varphi_L - 1} = \varpi^{\rho - 1}$. Then given any $\mathfrak{s} \subseteq \mathfrak{r}$, define

$$\eta_{\mathfrak{s}} = \prod_{\rho \in \mathfrak{s}} \eta_\rho^{\rho_{\mathfrak{s}}}.$$

**Lemma 3.6** *For any* $\mathfrak{s} \subseteq \mathfrak{r}$, $\eta_{\mathfrak{s}}^{\varphi_L - 1} = \varpi^{\sigma_{\mathfrak{s}} - 1}$.

*Proof.* We remark that the statement is vacuously true if $\mathfrak{s} = \emptyset$, both sides being equal to 1. We prove the lemma by induction on $\#\mathfrak{s} \geq 1$. If $\#\mathfrak{s} = 1$, then $\mathfrak{s} = \{\rho\}$ for some $\rho \in \mathfrak{r}$, and so $\eta_{\mathfrak{s}} = \eta_\rho$ and $\sigma_{\mathfrak{s}} = \rho$. Then the statement is clear by the choice of $\eta_\rho$.

Now suppose that the statement is true for all $\mathfrak{s} \subseteq \mathfrak{r}$ of a given cardinality $k$ with $1 \leq k < \#\mathfrak{r}$, and let $\mathfrak{t} \subseteq \mathfrak{r}$ have cardinality $k+1$. Let $\rho$ be the greatest element of $\mathfrak{t}$ and let $\mathfrak{s} = \mathfrak{t} \smallsetminus \{\rho\}$. Then

$$\begin{aligned}
\varpi^{\sigma_{\mathfrak{t}} - 1} &= \varpi^{\sigma_{\mathfrak{s}} \rho - 1} \\
&= \varpi^{\sigma_{\mathfrak{s}} \rho - \sigma_{\mathfrak{s}} + \sigma_{\mathfrak{s}} - 1} \\
&= (\varpi^{\rho - 1})^{\sigma_{\mathfrak{s}}} \varpi^{\sigma_{\mathfrak{s}} - 1} \\
&= (\eta_\rho^{\sigma_{\mathfrak{s}}} \eta_{\mathfrak{s}})^{\varphi_L - 1} \\
&= \left( \eta_\rho^{\rho_{\mathfrak{t}}} \prod_{\delta \in \mathfrak{s}} \eta_\delta^{\delta_{\mathfrak{t}}} \right)^{\varphi_L - 1} \\
&= \left( \prod_{\delta \in \mathfrak{t}} \eta_\delta^{\delta_{\mathfrak{t}}} \right)^{\varphi_L - 1} \\
&= \eta_{\mathfrak{t}}^{\varphi_L - 1}.
\end{aligned}$$

■

### 3.3.4 A simplification in the case $\mathfrak{s} < \mathfrak{t}$

For subsets $\mathfrak{s}, \mathfrak{t}$ of $\mathfrak{r}$, let us write $\mathfrak{s} < \mathfrak{t}$ if the greatest element of $\mathfrak{s}$ is less than the least element of $\mathfrak{t}$. We may use the choices of the $\eta_{\mathfrak{s}}$ made in Section 3.3.3 to simplify the expressions for the 2-cocycle in Theorem 3.4: If $\mathfrak{s} < \mathfrak{t}$, then $\eta_{\mathfrak{t}}^{\sigma_{\mathfrak{s}}} \eta_{\mathfrak{s}} \eta_{\mathfrak{s}+\mathfrak{t}}^{-1} = 1$. Indeed, in that case if $\rho \in \mathfrak{s} + \mathfrak{t}$ then

$$\rho_{\mathfrak{s}+\mathfrak{t}} = \begin{cases} \rho_{\mathfrak{s}} & \text{if } \rho \in \mathfrak{s} \\ \rho_{\mathfrak{t}} \sigma_{\mathfrak{s}} & \text{if } \rho \in \mathfrak{t}, \end{cases}$$

and so

$$\begin{aligned}
\eta_{\mathfrak{s}+\mathfrak{t}} &= \prod_{\rho \in \mathfrak{s}+\mathfrak{t}} \eta_\rho^{\rho_{\mathfrak{s}+\mathfrak{t}}} \\
&= \left( \prod_{\rho \in \mathfrak{t}} \eta_\rho^{\rho_{\mathfrak{t}} \sigma_{\mathfrak{s}}} \right) \left( \prod_{\rho \in \mathfrak{s}} \eta_\rho^{\rho_{\mathfrak{s}}} \right) \\
&= \eta_{\mathfrak{t}}^{\sigma_{\mathfrak{s}}} \eta_{\mathfrak{s}}.
\end{aligned}$$

Despite what the above may suggest, it is not possible to choose the $\eta_{\mathfrak{s}} \in \widehat{L}_{\mathrm{ur}}^{\times}$ such that $\sigma_{\mathfrak{s}} \mapsto \eta_{\mathfrak{s}}$ is a 1-cocycle. Indeed, if $r_{L/K} : H^0(G, L^{\times}) \to H^{-2}(G, \mathbb{Z})$ is the reciprocity map, then for $\sigma \in G$ we have

$$r_{L/K}^{-1}(\sigma) = (-\overline{g}) \cup \sigma,$$

identifying $H^{-2}(G, \mathbb{Z})$ with $G$ since $G$ is abelian, and remembering that $g$ represents the *negative* of the fundamental class. But then a simple application of Lemma 2.2 shows that if $\mathfrak{s}$ is a non-empty subset of $\mathfrak{r}'$ (working in the case (NTR) – the case (TR) is similar), then $r_{L/K}^{-1}(\sigma_{\mathfrak{s}})$ is represented by $N_{L/M}(g(\sigma_{\mathfrak{s}}, \sigma_{\mathfrak{s}})^{-1})$ where $M$ is any quadratic subextension of $L/K$ on which $\sigma_{\mathfrak{s}}$ acts non-trivially. Since $r_{L/K}$ is an isomorphism, this means that $N_{L/M}(g(\sigma_{\mathfrak{s}}, \sigma_{\mathfrak{s}})^{-1}) \neq 1$, and so in particular $g(\sigma_{\mathfrak{s}}, \sigma_{\mathfrak{s}}) \neq 1$. Thus $\eta_{\mathfrak{s}}^{\sigma_{\mathfrak{s}}} \eta_{\mathfrak{s}} \eta_{\mathfrak{s}+\mathfrak{s}}^{-1} \neq 1$, justifying the claim.

### 3.3.5 The reciprocity map

Let $r_{L/K} : H^0(G, L^{\times}) \to H^{-2}(G, \mathbb{Z})$ denote the (local) reciprocity map. Since $G$ is abelian, we may identify $H^{-2}(G, \mathbb{Z})$ with $G$. For each $\rho \in \mathfrak{r}$ choose $\eta_{\rho} \in \widehat{L}_{\mathrm{ur}}^{\times}$ such that $\eta_{\rho}^{\varphi_L - 1} = \varpi^{\rho - 1}$. We have the following corollary of Theorem 3.4:

**Corollary 3.7** *(i) In case (TR), $H^0(G, L^{\times})$ is generated by the classes of the elements in the set $\{N_{L/K}(\eta_{\rho}) \mid \rho \in \mathfrak{r}\}$, and $r_{L/K}$ maps the class of $N_{L/K}(\eta_{\rho})$ to $\rho$.*

*(ii) In case (NTR), $H^0(G, L^{\times})$ is generated by the classes of the elements in the set $\{N_{L/K}(\eta_{\rho}) \mid \rho \in \mathfrak{r}'\} \cup \{N_{F/K}(\varpi)\}$. Further, $r_{L/K}$ maps the class of $N_{L/K}(\eta_{\rho})$ to $\rho$ for $\rho \in \mathfrak{r}'$, and the class of $N_{F/K}(\varpi)$ to $\tau$.*

*Proof.* Let us restrict once again to the case (NTR), the case (TR) being similar. For each $\rho \in \mathfrak{s}$, let $H_{\rho}$ be the subgroup of $G$ generated by $\mathfrak{r} \smallsetminus \{\rho\}$ and let $N_{\rho}$ be the sum in $\mathbb{Z}[H_{\rho}]$ of the elements in $H_{\rho}$.

As discussed in Section 3.3.4, $r_{L/K}^{-1}(\rho)$ is represented by $N_{L/L^{H_{\rho}}}(g(\rho, \rho)^{-1}) = g(\rho, \rho)^{-N_{\rho}}$ for each $\rho \in \mathfrak{r}'$. In fact, since $H^0(G, L^{\times})$ has exponent 2, $r_{L/K}^{-1}(\rho)$ is equally well represented by $g(\rho, \rho)^{N_{\rho}}$. However, by Theorem 3.4,

$$\begin{aligned} g(\rho, \rho)^{N_{\rho}} &= \eta_{\rho}^{(\rho+1)N_{\rho}} \\ &= N_{L/K}(\eta_{\rho}). \end{aligned}$$

Similarly, from Theorem 3.4 we see that $r_{L/K}(\tau)$ is represented by

$$\begin{aligned} g(\tau, \tau)^{N_{\tau}} &= \varpi^{N_{\tau}} \\ &= N_{F/K}(\varpi) \end{aligned}$$

since restriction gives an isomorphism $H_{\tau} \to \mathrm{Gal}(F/K)$. ∎

### 3.3.6 Odd residue characteristic

We suppose in this section that the residue characteristic of $K$ is odd. In this case, the maximal abelian extension $L$ of $K$ of exponent 2 is biquadratic, and we may choose the $\eta_{\mathfrak{s}}$ of Section 3.3.1 explicitly. The only other multiquadratic extensions of $K$ are the three quadratic extensions; we omit these cases since their treatment is straightforward.

Let $q$ be the size of the residue field of $K$. Choose a uniformizer $\pi$ of $K$, a square root $\varpi$ of $-\pi$, and a square root $\varpi'$ of $-\zeta\pi$, where $\zeta$ is a generator for the $q-1$th roots of unity in $K$. Then $\varpi^{-1}\varpi'$ is a root of unity of order $2(q-1)$, and so there is a primitive $(q^2-1)$th root of unity $u$ such that $u^{(q+1)/2} = \varpi^{-1}\varpi'$. With this notation, the field $E = K(u)$ is the unramified quadratic extension of $K$, and the fields $F = K(\varpi)$ and $F' = K(\varpi')$ are the ramified quadratic extensions. We let $\sigma$, $\tau$ and $\rho$ be the non-trivial elements of $\mathrm{Gal}(L/E)$, $\mathrm{Gal}(L/F)$ and $\mathrm{Gal}(L/F')$ respectively, so that $G = \{1, \sigma, \tau, \rho\}$, and $\sigma\tau = \rho$.

In the notation of Section 3.3.1, we let $\mathfrak{r} = \{1, \tau\}$ and $\mathfrak{r}' = \{1\}$. Then the notation of the previous paragraph (i.e. $E$, $F$, $\varpi$, $\tau$) agrees with that of the general setup earlier in the case (NTR).

**Corollary 3.8** *$-\xi_{L/K}$ is represented by the normalized 2-cocycle $g : G^2 \to L^\times$ given by*

| | $\sigma$ | $\tau$ | $\rho$ |
|---|---|---|---|
| $\sigma$ | $u$ | $-1$ | $-u$ |
| $\tau$ | $u^{-1}\varpi^{-1}\varpi'$ | $\varpi$ | $-u^{-1}\varpi'$ |
| $\rho$ | $\varpi^{-1}\varpi'$ | $\varpi$ | $-\varpi'$ |

*More precisely, the $(s,t)$-entry is $g(s,t)$ for $s, t \in \{\sigma, \tau, \rho\}$.*

*Proof.* According to Section 3.3.3, we only need to choose $\eta_1 \in \widehat{L}_{\mathrm{ur}}^\times$ such that $\eta_1^{\varphi_L - 1} = \varpi^{\sigma-1} = -1$. We may do this as follows: Let $\zeta_{q^4-1}$ be a primitive $(q^4-1)$th root of unity in $K_{\mathrm{ur}}$ such that $\zeta_{q^4-1}^{q^2+1} = u$ (such a $\zeta_{q^4-1}$ exists) and let $\eta = \zeta_{q^4-1}^{(q^2+1)/2}$. Then $\eta^{\varphi_L - 1} = \eta^{q^2-1} = -1$ and $\eta^{\sigma+1} = \eta^2 = u$. We then take $\eta_1 = \eta$. The rest is just repeated application of Theorem 3.4, noting that in that theorem, the only subsets $\mathfrak{s} \subseteq \mathfrak{r}'$ that appear are $\emptyset$ and $\{1\}$. ∎

**Remark**. The fact that the $\eta$ of the preceding proof is fixed by $\sigma$ is a result of the rather special fact that it lies not only in $\widehat{L}_{\mathrm{ur}}$, but also in $\widehat{K}_{\mathrm{ur}}$, on which $\sigma$ acts trivially. More precisely, $\eta$ lies in the quartic unramified extension of $K$.

# 4 The global fundamental class

In this section, $L/K$ will denote a Galois extension of number fields. All number fields will lie in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. For each prime $\mathfrak{p}$ of $K$, we fix once and for all a place of $\bar{\mathbb{Q}}$ above $\mathfrak{p}$. Given a number field $F$ containing $K$,

$\mathfrak{p}(F)$ will denote the place of $F$ below the chosen place of $\bar{\mathbb{Q}}$ above $\mathfrak{p}$. Then $F_{\mathfrak{p}}$ will denote the completion of $F$ at $\mathfrak{p}(F)$. For a number field $L$, $C_L$ will denote the idele class-group.

We forget all notation from Section 3.

## 4.1 Global fundamental classes from local ones

Let $L/K$ be a Galois 2-extension of number fields (not necessarily multiquadratic yet) with Galois group $G$. Our aim is to exhibit a Galois 2-extension $L'/K$ containing $L$ and an element $\xi'$ of $H^2(L'/K, J_{L'})$ such that the image of $\xi'$ in $H^2(L'/K, C_{L'})$ has global invariant $1/[L:K] \bmod \mathbb{Z}$. This being the case, the image of $\xi'$ in $H^2(L'/K, C_{L'})$ will be equal to the image of the fundamental class of $L/K$ under the injective map $H^2(L/K, C_L) \to H^2(L'/K, C_{L'})$.

**Lemma 4.1** *For each $n \geq 0$, let $\gamma_n = \omega_n + \omega_n^{-1}$, where $\omega_n$ is a dyadic primitive $2^{n+2}$th root of unity. Then $\mathbb{Q}_2(\gamma_n)/\mathbb{Q}_2$ is cyclic of degree $2^n$.*

*Proof.* Standard. ■

For $n \geq 0$, let $\alpha_n = \zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1}$, where $\zeta_{2^{n+2}}$ is a primitive $2^{n+2}$th root of unity in $\bar{\mathbb{Q}}$.

**Lemma 4.2** *Let $\mathfrak{p}$ be a dyadic prime of $K$. There is a non-negative integer $n$ such that $L(\alpha_n)/K$ is a Galois 2-extension, and such that $L(\alpha_n)_{\mathfrak{p}}/K_{\mathfrak{p}}$ has degree $[L:K]$.*

*Proof.* Let $[L:K] = 2^r$, $r \geq 1$. Then $[L_{\mathfrak{p}} : K_{\mathfrak{p}}] = 2^s$ for some non-negative integer $s \leq r$.

Let $\mathbb{Q}_{2,\infty} = \bigcup_n \mathbb{Q}_2(\gamma_n)$. Then $[L_{\mathfrak{p}} \cap \mathbb{Q}_{2,\infty} : \mathbb{Q}_2] = 2^t$ for some non-negative integer $t$. If $n \geq 0$, then

$$[L_{\mathfrak{p}}(\gamma_n) : L_{\mathfrak{p}}] = [\mathbb{Q}_2(\gamma_n) : \mathbb{Q}_2(\gamma_n) \cap L_{\mathfrak{p}}] = \begin{cases} 2^{n-t} & \text{if } n \geq t \\ 1 & \text{otherwise.} \end{cases}$$

If we take $n = t + r - s$, then since $r - s \geq 0$, $[L_{\mathfrak{p}}(\gamma_n) : L_{\mathfrak{p}}] = 2^{n-t} = 2^{r-s}$ and therefore

$$[L_{\mathfrak{p}}(\gamma_n) : K_{\mathfrak{p}}] = [L_{\mathfrak{p}}(\gamma_n) : L_{\mathfrak{p}}][L_{\mathfrak{p}} : K_{\mathfrak{p}}] = 2^{r-s} \cdot 2^s = 2^r = [L:K].$$

Finally, $L_{\mathfrak{p}}(\gamma_n) = L(\alpha_n)_{\mathfrak{p}}$, and $L(\alpha_n)/K$ is a Galois 2-extension because both $L/K$ and $K(\alpha_n)/K$ are. ■

**Proposition 4.3** *Let $L' = L(\alpha_n)$ as in Lemma 4.2. There is $\xi' \in H^2(L'/K, J_{L'})$ whose image in $H^2(L'/K, C_{L'})$ has global invariant $1/[L:K] \bmod \mathbb{Z}$.*

*Proof.* We use the canonical isomorphism

$$H^2(L'/K, J_{L'}) \simeq \bigoplus_{\mathfrak{q}} H^2(L'_{\mathfrak{q}}/K_{\mathfrak{q}}, (L'_{\mathfrak{q}})^{\times}),$$

where the direct sum runs over all primes of $K$. Taking this to be an identification, we let $\xi'$ be the element that has the local fundamental class for $L'_{\mathfrak{p}}/K_{\mathfrak{p}}$ in the $\mathfrak{p}$-component, where $\mathfrak{p}$ is the dyadic prime chosen in Lemma 4.2, and 0 elsewhere. The global invariant of the image in $H^2(L'/K, C_{L'})$ is then just the local invariant at $\mathfrak{p}$ of the local fundamental class of $L'_{\mathfrak{p}}/K_{\mathfrak{p}}$, i.e. $1/[L : K] \bmod \mathbb{Z}$. ■

## 4.2 Global multiquadratic extensions

Suppose now that $L/K$ is multiquadratic of degree $n = 2^r$, and assume further that 2 splits completely in $L/\mathbb{Q}$, so that $L_{\mathfrak{p}} = K_{\mathfrak{p}} = \mathbb{Q}_2$ where $\mathfrak{p}$ is our chosen dyadic prime, as in Section 4.1. In this case, if $\alpha = \alpha_r$, $K' = K(\alpha)$ and $L' = L(\alpha)$, then $L \cap K' = K$ and $LK' = L'$. Letting $G' = \mathrm{Gal}(L'/K)$ and $H = \mathrm{Gal}(L'/L)$, $G'$ is the internal direct product of $H$, cyclic of order $n$, and $\mathrm{Gal}(L'/K')$, of order $n$ and exponent 2. We observe that $H$ is the decomposition group of $\mathfrak{p}$ in $L'/K$. In fact, we freely identify $H$ with $G'_{\mathfrak{p}}$.

Fix a generator $\overline{a} \in (\mathbb{Z}/2^{r+2}\mathbb{Z})^{\times}/\{\pm\overline{1}\}$ – for example, $a = 5$ will do – and let $\tau$ be the unique element of $H$ satisfying $(\zeta + \zeta^{-1})^{\tau} = \zeta^{a} + \zeta^{-a}$, where $\zeta$ is a primitive $2^{r+2}$th root of unity in $\bar{\mathbb{Q}}$. Let $y$ be the idele in $J_{L'}$ with $a$ in the $\mathfrak{P}$-component for $\mathfrak{P}|\mathfrak{p}$ and 1 elsewhere, and let $\overline{y}$ be its image in $C_{L'}$.

We may now state our theorem on the global fundamental class.

**Theorem 4.4** *Let $\overline{y}$ be as above.*

*(i) There exists $\mu \in C_{L'}$ such that $\mu^{\sum_{k=0}^{n-1} \tau^k} = (\overline{y})^{-1}$.*

*(ii) For each $\sigma \in \mathrm{Gal}(L'/K')$ and each $i$ with $0 \leq i \leq n-1$, there exists $\nu_{\sigma\tau^i} \in C_{L'}$ such that $\nu_{\sigma\tau^i}^{\tau-1} = \mu^{(1-\sigma\tau^i)\tau}$.*

*(iii) For each $\sigma, \rho \in \mathrm{Gal}(L'/K')$, the element $\nu_{\rho}^{-\sigma}\nu_{\sigma\rho}\nu_{\sigma}^{-1}$ of $C_{L'}$ is fixed by $H$ and can therefore be considered as an element of $C_L$.*

*(iv) Identifying $G$ with $\mathrm{Gal}(L'/K')$, the map*

$$\begin{array}{rcl} G^2 & \to & C_L \\ (\sigma, \rho) & \mapsto & \nu_{\rho}^{-\sigma}\nu_{\sigma\rho}\nu_{\sigma}^{-1} \end{array}$$

*is a 2-cocycle representing the global fundamental class for $L/K$.*

**Remark**. The 2-cocycle in part (iv) of Theorem 4.4 appears to be a 2-coboundary. This would be true if all the elements $\nu_{\sigma}$ were themselves in $C_L$, but this is not the case.

The remainder of this section is devoted to proving Theorem 4.4. We begin with the 2-cocycle $f : (G'_{\mathfrak{p}})^2 \to (L'_{\mathfrak{p}})^\times$ defined, for $0 \le i, j \le n-1$, by

$$f(\tau^i, \tau^j) = \begin{cases} a & \text{if } i+j \ge n \\ 1 & \text{otherwise.} \end{cases}$$

**Lemma 4.5** *$f$ represents the local fundamental class of $L'_{\mathfrak{p}}/K_{\mathfrak{p}}$.*

*Proof.* Since $L'_{\mathfrak{p}}$ is a cyclic extension of $K_{\mathfrak{p}}$, the knowledge of the fundamental class is equivalent to the knowledge of the reciprocity map $H^0(G'_{\mathfrak{p}}, (L'_{\mathfrak{p}})^\times) \to G'_{\mathfrak{p}}$. As $K_{\mathfrak{p}} = \mathbb{Q}_2$ in our case, this map is described explicitly in, for example, [7, Section 3.1]. The details are left to the reader. ■

Let $x$ be the idele in $J_{L'}$ having $a$ in the $\mathfrak{p}(L')$-component and 1 elsewhere, so that the image of $\overline{f}$ in $H^2(G'_{\mathfrak{p}}, J_{L'})$ is represented by

$$f_1 : (\tau^i, \tau^j) \mapsto \begin{cases} x & \text{if } i+j \ge n \\ 1 & \text{otherwise.} \end{cases}$$

**Lemma 4.6** *The image of $\overline{f}_1$ under corestriction $H^2(G'_{\mathfrak{p}}, J_{L'}) \to H^2(G', J_{L'})$ is represented by*

$$g_1 : (\sigma\tau^i, \rho\tau^j) \mapsto \begin{cases} y & \text{if } i+j \ge n \\ 1 & \text{otherwise,} \end{cases}$$

*for $\sigma, \rho \in \mathrm{Gal}(L'/K')$ and $0 \le i, j \le n-1$.*

*Proof.* $\mathrm{Gal}(L'/K')$ is a set of representatives for $G'/H$. If $\sigma \in G'$, let $\overline{\sigma}$ be the unique element of $\mathrm{Gal}(L'/K')$ such that $\overline{\sigma}H = \sigma H$. By [3, Theorem 7], the image of $\overline{f}_1$ under corestriction is represented by the 2-cocycle $g_1$ given, for $\sigma, \rho \in \mathrm{Gal}(L'/K')$ and $0 \le i, j \le n-1$, by

$$\begin{aligned} g_1(\sigma\tau^i, \rho\tau^j) &= \prod_{\delta \in \mathrm{Gal}(L'/K')} f_1(\delta\rho\tau^i(\overline{\delta\sigma\tau^i})^{-1}, \overline{\delta\sigma\tau^i}\rho\tau^j(\overline{\delta\sigma\tau^i\rho\tau^j})^{-1})^{\delta^{-1}} \\ &= \prod_{\delta \in \mathrm{Gal}(L'/K')} f_1(\delta\sigma\tau^i(\delta\sigma)^{-1}, \delta\sigma\rho\tau^j(\delta\sigma\rho)^{-1})^{\delta^{-1}} \\ &= \prod_{\delta \in \mathrm{Gal}(L'/K')} f_1(\tau^i, \tau^j)^{\delta^{-1}} \\ &= f_1(\tau^i, \tau^j)^{\sum_{\delta \in \mathrm{Gal}(L'/K')} \delta} \\ &= \begin{cases} y & \text{if } i+j \ge n \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

■

Let $g : (G')^2 \to C_{L'}$ be the 2-cocycle satisfying, for $\sigma, \rho \in \mathrm{Gal}(L'/K')$ and $0 \le i, j \le n-1$,

$$g(\sigma\tau^i, \rho\tau^j) = \begin{cases} \overline{y} & \text{if } i+j \ge n \\ 1 & \text{otherwise.} \end{cases} \tag{4.1}$$

Then by construction, and by the proof of Proposition 4.3, the class of $g$ in $H^2(G', C_{L'})$ has invariant $2^{-r}$ mod $\mathbb{Z}$, and therefore must be the image of the fundamental class for $L/K$ under the inflation map $H^2(G, C_L) \to H^2(G', C_{L'})$. Let $c : G' \to \mathrm{Hom}_{\mathbb{Z}}(\Delta G', C_{L'})$ be the 1-cocycle corresponding to $g$, as in Lemma 2.1. One finds that for $\sigma, \rho \in \mathrm{Gal}(L'/K')$ and $0 \leq i, j \leq n-1$,

$$c(\sigma\tau^i)(\rho\tau^j - 1) = \begin{cases} \overline{y} & \text{if } 1 \leq j \leq i \\ 1 & \text{otherwise.} \end{cases}$$

Indeed,

$$\begin{aligned} c(\sigma\tau^i)(\rho\tau^j - 1) &= g(\rho\tau^j, \rho^{-1}\tau^{-j})g(\rho\tau^j, \rho^{-1}\tau^{-j}\sigma\tau^i)^{-1} \\ &= g(\tau^j, \tau^{-j})g(\tau^j, \tau^{i-j})^{-1}, \end{aligned}$$

and we observe that

$$g(\tau^j, \tau^{-j}) = \begin{cases} 1 & \text{if } j = 0 \\ \overline{y} & \text{otherwise} \end{cases}$$

while

$$g(\tau^j, \tau^{i-j}) = \begin{cases} 1 & \text{if } j \leq i \\ \overline{y} & \text{otherwise.} \end{cases}$$

**Definition 4.7** *Given $\lambda \in \mathrm{Hom}_{\mathbb{Z}}(\Delta H, C_{L'})$, write $\lambda_i = \lambda(\tau^i - 1)$ for each $i \in \mathbb{Z}$.*

**Proposition 4.8** *There is $\lambda \in \mathrm{Hom}_{\mathbb{Z}}(\Delta H, C_{L'})$ such that for $0 \leq i, j \leq n-1$,*

$$\lambda_{j-i}^{\tau^i}\lambda_{-i}^{-\tau^i}\lambda_j^{-1} = \begin{cases} \overline{y} & \text{if } 1 \leq j \leq i \\ 1 & \text{otherwise.} \end{cases} \tag{4.2}$$

*Proof.* Since $\overline{c}$ is in the image of the inflation map $H^1(G, \mathrm{Hom}_{\mathbb{Z}}(\Delta G, C_L)) \to H^1(G', \mathrm{Hom}_{\mathbb{Z}}(\Delta G', C_{L'}))$, it is therefore in the kernel of the restriction map $H^1(G', \mathrm{Hom}_{\mathbb{Z}}(\Delta G', C_{L'})) \to H^1(H, \mathrm{Hom}_{\mathbb{Z}}(\Delta H, C_{L'}))$. This says that there is $\lambda \in \mathrm{Hom}_{\mathbb{Z}}(\Delta H, C_{L'})$ such that for $i, j \in \mathbb{Z}$,

$$(\tau^i\lambda - \lambda)(\tau^j - 1) = c(\tau^i)(\tau^j - 1). \tag{4.3}$$

The left-hand side of (4.3) is $\lambda_{j-i}^{\tau^i}\lambda_{-i}^{-\tau^i}\lambda_j^{-1}$, while the right-hand side is $\overline{y}$ if $1 \leq j \leq i$ and 1 otherwise. ■

**Corollary 4.9** *There is $\mu \in C_{L'}$ such that $(\overline{y})^{-1} = \mu^{\sum_{k=0}^{n-1}\tau^k}$ in $C_{L'}$. In particular, part (i) of Theorem 4.4 holds.*

*Proof.* Choose $\lambda$ as in Proposition 4.8. Then $\overline{y} = \lambda_{-1}^{-\tau}\lambda_1^{-1}$, i.e.

$$(\overline{y})^{-1} = \lambda_1\lambda_{-1}^{\tau}. \tag{4.4}$$

Further, for $j = 2, \ldots, n-1$, $1 = \lambda_{j-1}^{\tau}\lambda_{-1}^{-\tau}\lambda_j^{-1}$, i.e.

$$\lambda_{j-1} = \lambda_j^{\tau^{-1}}\lambda_{-1}. \tag{4.5}$$

Using (4.5) we observe that, for $k = 0, \ldots, n-3$, we have $\lambda_{-1} = \lambda_{k+1}\lambda_{k+2}^{-\tau^{-1}}$, and therefore also

$$\lambda_{-1}^{\tau^{-k}} = \lambda_{k+1}^{\tau^{-k}}\lambda_{k+2}^{-\tau^{-(k+1)}}.$$

Hence

$$\begin{aligned}
\lambda_{-1}^{\sum_{k=0}^{n-1}\tau^k} &= \lambda_{-1}^{\sum_{k=0}^{n-1}\tau^{-k}} \\
&= \left(\prod_{k=0}^{n-3}\lambda_{k+1}^{\tau^{-k}}\lambda_{k+2}^{-\tau^{-(k+1)}}\right)\lambda_{-1}^{\tau^2+\tau} \\
&= \lambda_1\lambda_{-1}^{-\tau^2}\lambda_{-1}^{\tau^2+\tau} \\
&= \lambda_1\lambda_{-1}^{\tau} \\
&= (\overline{y})^{-1},
\end{aligned}$$

the last equation by (4.4). ∎

**Lemma 4.10** *Suppose $\mu \in C_{L'}$ is an idele class satisfying $\mu^{\sum_{k=0}^{n-1}\tau^k} = (\overline{y})^{-1}$. (Such a $\mu$ exists by Corollary 4.9.) If $\lambda$ is the element of $\operatorname{Hom}_{\mathbb{Z}}(\Delta H, C_{L'})$ defined by*

$$\lambda(\tau^j - 1) = (\overline{y})^{-1}\mu^{-\sum_{k=1}^{j}\tau^k}$$

*for $j = 1, \ldots, n-1$, then $c(\tau^i) = \tau^i\lambda - \lambda$ for all $i \in \mathbb{Z}$.*

*Proof.* This is equivalent to showing that the equation in (4.2) holds for $0 \le i, j \le n-1$. Further, we may assume that $i, j \ge 1$. We split the verification up into three cases: (i) $i = j$, (ii) $j < i$, and (iii) $i < j$.

Case (i):

$$\begin{aligned}
\lambda_{-i}^{-\tau^i}\lambda_i^{-1} &= \lambda_{n-i}^{-\tau^i}\lambda_i^{-1} \\
&= \overline{y}^2\mu^{\sum_{k=i+1}^{n}\tau^k}\cdot\mu^{\sum_{k=1}^{i}\tau^k} \\
&= \overline{y}^2(\overline{y})^{-1} \\
&= \overline{y}.
\end{aligned}$$

Case (ii):

$$\begin{aligned}
\lambda_{j-i}^{\tau^i}\lambda_{-i}^{-\tau^i}\lambda_j^{-1} &= \lambda_{n+j-i}^{\tau^i}\lambda_{n-i}^{-\tau^i}\lambda_j^{-1} \\
&= \overline{y}\mu^{-\sum_{k=i+1}^{n+j}\tau^k}\cdot\mu^{\sum_{k=i+1}^{n}\tau^k}\cdot\mu^{\sum_{k=1}^{j}\tau^k} \\
&= \overline{y}\mu^{-\sum_{k=n+1}^{n+j}\tau^k}\cdot\mu^{\sum_{k=1}^{j}\tau^k} \\
&= \overline{y}.
\end{aligned}$$

Case (iii):

$$\begin{aligned}
\lambda_{j-i}^{\tau^i}\lambda_{-i}^{-\tau^i}\lambda_j^{-1} &= \lambda_{n+j-i}^{\tau^i}\lambda_{n-i}^{-\tau^i}\lambda_j^{-1} \\
&= \overline{y}\mu^{-\sum_{k=i+1}^{j}\tau^k}\cdot\mu^{\sum_{k=i+1}^{n}\tau^k}\cdot\mu^{\sum_{k=1}^{j}\tau^k} \\
&= \overline{y}\mu^{\sum_{k=1}^{n}\tau^k} \\
&= 1.
\end{aligned}$$

■

Lemma 4.10 exhibits the 1-cocycle $c$ as an explicit 1-coboundary. Translating into 2-cocycles as in Lemma 2.1, we find that the restriction of the 2-cocycle $g$ to $H$ is equal to the 2-coboundary associated to the map

$$\begin{aligned}
h : H &\to C_{L'} \\
\tau^i &\mapsto \lambda(\tau^{-i}-1)^{-\tau^i}.
\end{aligned}$$

**Lemma 4.11** *Let $\mu$ and $\lambda$ be chosen as in Lemma 4.10. If $j \in \mathbb{Z}$, then*

$$\lambda(\tau^{-j}-1)^{-\tau^j} = \mu^{-\sum_{k=1}^{j}\tau^k} = \overline{y}\cdot\lambda(\tau^j-1).$$

*Proof.* The second equality follows immediately from the definition of $\lambda$. As for the first, if $1 \le j \le n-1$,

$$\begin{aligned}
\lambda(\tau^{-j}-1)^{-\tau^j} &= \left((\overline{y})^{-1}\mu^{-\sum_{k=1}^{n-j}\tau^k}\right)^{-\tau^j} \\
&= \overline{y}\mu^{\sum_{k=1}^{n-j}\tau^{j+k}} \\
&= \overline{y}\mu^{\sum_{k=j+1}^{n}\tau^k} \\
&= \mu^{-\sum_{k=1}^{n}\tau^k+\sum_{k=j+1}^{n}\tau^k} \\
&= \mu^{-\sum_{k=1}^{j}\tau^k}.
\end{aligned}$$

■

Following Section 2.3, given $\sigma \in \mathrm{Gal}(L'/K')$ and $0 \le i, j \le n-1$, we let

$$a_{\sigma\tau^i,\tau^j} = g(\tau^j,\sigma\tau^i)g(\sigma\tau^i,\tau^j)^{-1}h(\tau^j)^{\sigma\tau^i-1}.$$

Observe that, as a consequence of the definition of $g$ and of Lemma 4.11,

$$a_{\sigma\tau^i,\tau^j} = h(\tau^j)^{\sigma\tau^i-1} = \mu^{-(\sigma\tau^i-1)\sum_{k=1}^{j}\tau^k}.$$

By the discussion in Section 2.3, the map

$$\begin{aligned}
H &\to C_{L'} \\
\tau^j &\mapsto a_{\sigma\tau^i,\tau^j}
\end{aligned} \tag{4.6}$$

is a 1-cocycle for each $\sigma\tau^i \in G'$. Since $H^{-1}(H, C_{L'}) = 0$, there is consequently $\nu_{\sigma\tau^i} \in C_{L'}$ such that

$$\nu_{\sigma\tau^i}^{\tau-1} = \mu^{(1-\sigma\tau^i)\tau},$$

remembering that $a_{\sigma\tau^i,\tau} = \mu^{(1-\sigma\tau^i)\tau}$. Thus part (ii) of Theorem 4.4 holds. Furthemore, for all $\sigma\tau^i \in G'$, the 1-cocycle in (4.6) is equal to the 1-coboundary associated to the element $\nu_{\sigma\tau^i}$. Using part (i) of Lemma 2.3 together with the fact that $g(\sigma, \rho) = 1$ for $\sigma, \rho \in \mathrm{Gal}(L'/K')$, we thus obtain part (iii) of Theorem 4.4. Finally, part (ii) of Lemma 2.3 gives part (iv) of Theorem 4.4.

## 4.3 The global reciprocity map

We keep the notation and assumptions of Section 4.2. Let $r_{L/K} : H^0(G, C_L) \to H^{-2}(G, \mathbb{Z})$ be the global reciprocity map, which we view as a map into $G$ since $G$ is abelian.

**Corollary 4.12** *Let $N \in \mathbb{Z}[\mathrm{Gal}(L'/K')]$ be the sum of the elements of* $\mathrm{Gal}(L'/K')$. *Then for $\rho \in G$, $\nu_\rho^N$ lies in $C_K$ and represents the class of $r_{L/K}^{-1}(\rho)$.*

**Remark**. At first sight, Corollary 4.12 seems to say that $r_{L/K}^{-1}$ is the trivial map, all elements in the image being represented by the norm of an idele. Just as in the remark following Theorem 4.4, this would be true if each $\nu_\rho$ was in $C_L$. We emphasize again, however, that this is not the case.

*Proof.* We let $G$ act on $C_{L'}$ via the canonical isomorphism $G \simeq \mathrm{Gal}(L'/K')$. That $\nu_\rho^N$ lies in $C_K$ for each $\rho \in G$ will be a consequence of the proof of the rest of the corollary. However, we may justify it directly: It is immediate that $\nu_\rho^N$ is fixed by $\mathrm{Gal}(L'/K')$. That it is fixed by $H$ follows from the way the elements were chosen in part (ii) of Theorem 4.4.

Now, let $w : G^2 \to C_L$ be the 2-cocycle from part (iv) of Theorem 4.4. The map $r_{L/K}^{-1}$ is induced by cup-product with the global fundamental class of $L/K$, i.e. with the class of the normalized 2-cocycle $w$. Therefore by Lemma 2.2, $r_{L/K}^{-1}(\rho)$ is represented by

$$\begin{aligned}\prod_{\sigma\in G} w(\sigma,\rho) &= \prod_{\sigma\in G} \nu_\rho^{-\sigma}\nu_{\sigma\rho}\nu_\sigma^{-1} \\ &= \prod_{\sigma\in G} \nu_\rho^{-\sigma} \\ &= \nu_\rho^{-N}.\end{aligned}$$

Since $H^0(G, C_L)$ has exponent 2, $\nu_\rho^{-N}$ and $\nu_\rho^N$ represent the same class. ■

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