Statistics 221 Midterm. Michael Kouritzin. University of Alberta.

No books or notes allowed. Non-Programmed Calculators allowed. SHOW YOUR WORK in the WORKBOOKS.

1. (11 marks) Put your name and identification number on your answer book.

2. (36 marks) Short description/expression answers.

(a) Supply the missing word. A random variable is neither random nor a variable but rather a ________.

(b) Expand $(x+2y+z)^3$. - By the multinomial theorem,

$$\begin{aligned}(x+2y+z)^3 &= \binom{3}{0,0,3}x^0(2y)^0z^3 + \binom{3}{0,3,0}x^0(2y)^3z^0 + \binom{3}{3,0,0}x^3(2y)^0z^0 \\ &+ \binom{3}{0,1,2}x^0(2y)^1z^2 + \binom{3}{1,0,2}x^1(2y)^0z^2 + \binom{3}{0,2,1}x^0(2y)^2z^1 \\ &+ \binom{3}{1,1,1}x^1(2y)^1z^1 + \binom{3}{1,2,0}x^1(2y)^2z^0 + \binom{3}{2,0,1}x^2(2y)^0z^1 \\ &+ \binom{3}{2,1,0}x^2(2y)^1z^0 \\ &= z^3 + 8y^3 + x^3 + 6yz^2 + 3xz^2 + 12y^2z + 12xyz + 12xy^2 + 3x^2z + 6x^2y\end{aligned}$$

(c) If we arrange the letters in ALBERTO, how many different arrangements are there with A adjacent to B? - We firstly determine the position of letters A and B bound together, there are 6 different ways; secondly, we determine the order between A and B, that is, AB or BA., there are 2 ways; finally, we need to determine the positions of the remaining 5 letters: since there is no restriction on them, this is just the full permutation of the 5 numbers and there are $P_5^5 = 5!$ different ways. Applying PC, the number of arrangements if A is adjacent to B is $6 \times 2 \times 5! = 1440$.

(d) A task needs two groups of at least 9 people to finish it on time. Each group starts with 10 workers and each person quits the job with probability of 0.1 (independent of other workers). What is the probability that at least 9 people stay until the end in one of the two groups, but not in both? - Let A be the event that there are at least 9 people stay until the end of the work in one group. Let B be the event that there are at least 9 people stay until the end of the work in the other group. Then,

$$P(A) = P(B) = \binom{10}{10}0.9^{10} + \binom{10}{9}0.9^9 0.1 = 0.7361.$$

Since A and B are independent, $P(AB) = P(A)P(B)$. The probability that at least 9 people stay until the end in one of the two groups, but not in both is:

$$P(A \cup B) - P(AB) = P(A) + P(B) - 2P(A)P(B) = 0.3885.$$

3. (20 marks) A random variable has cumulative distribution function:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 4 \\ 0.1 & \text{if } 4 \le x < 6 \\ 0.5 & \text{if } 6 \le x < 8 \\ 1 & \text{if } x \ge 8 \end{cases}.$$

Find

(a) $P(X = 5)$

(b) $P(X = 6)$

(c) $P(1 < X \le 6)$

(d) $P(X > 6)$ - We recall that $P(a < X \le b) = P(X \le b) - P(X \le a) = F_X(b) - F_X(a)$ for $a < b$ and let

$$F_X(5-) = \lim_{x \to 5-} F_X(x).$$

Then,

(a) $P(X = 5) = F_X(5) - F_X(5-) = 0$

(b) $P(X = 6) = F_X(6) - F_X(6-) = 0.5 - 0.1 = 0.4$

(c) $P(1 < X \le 6) = F_X(6) - F_X(1) = 0.5 - 0 = 0.5$

(d) $P(X > 6) = 1 - F_X(6) = 0.5.$

4. (20 marks) A fish store has an aquarium with r red, b blue and g green fish. Suppose that the fish are removed and sold randomly until there is only one type left.

(a) What is the probability that the blue fish will be the last remaining type?

(b) What is the probability that the red fish will be the first type extinct? - It will help to think of the experiment of drawing fish until there are none left. (a) If fish are drawn until none are left, then

$$P(\text{Last type remaining is blue}) = P(\text{Last fish drawn is blue})$$

However, there are b ways to choose a blue fish last and $r + b + g$ ways to choose a fish last. All are equally likely so

$$P(\text{Last type remaining is blue}) = P(\text{Last fish drawn is blue}) = \frac{b}{r + b + g}.$$

(b) We will use part (a) repeatedly to solve this problem. Let R_i, G_i, and B_i be the events that the red, blue and green fish become extinct i^{th} for $i = 1, 2, 3$. Then, we want

$$\begin{aligned} P(R_1) &= P(R_1 B_2 G_3 \cup R_1 G_2 B_3) \\ &= P(R_1 B_2 | G_3) P(G_3) + P(R_1 G_2 | B_3) P(B_3) \end{aligned}$$

by mutual exclusivity. However, we find by analogy to the previous part that

$$P(G_3) = \frac{g}{r + b + g} \text{ and } P(B_3) = \frac{b}{r + b + g}.$$

Moreover, restricting to the 'new subspace' G_3 and using the argument of previous part, we have that

$$\begin{aligned} P(R_1B_2|G_3) &= P(\text{The blue fish out last the red}|G_3) \\ &= P(\text{Last blue or red chosen is blue}\}|G_3) \\ &= \frac{b}{r+b}. \end{aligned}$$

Putting everything together, we have that

$$\begin{aligned} P(R_1) &= P(R_1B_2|\,G_3)P(G_3) + P(R_1G_2|\,B_3)P(B_3) \\ &= \frac{b}{r+b}\frac{g}{r+b+g} + \frac{g}{r+g}\frac{b}{r+b+g} \\ &= \frac{bg}{r+b+g}\left[\frac{1}{r+b} + \frac{1}{r+g}\right]. \end{aligned}$$

5. (15 marks) Suppose that you are dealt 7 cards from a deck of 52, and you choose the best possible 5-card poker hand. What is the probability that the chosen hand is a: i) flush including straight flush? ii) 4 of a kind? iii) full house? (Use the following definitions of poker hands, listed in descending order of value. Straight flush: contains five cards in rank sequence, all of the same suit; ace can be high or low, but wrap-around is not allowed. Four of a kind: contains four cards of one rank, and an unmatched card. Full house: contains three cards of one rank, and two cards of another rank. Flush: contains five cards of the same suit, but not all in rank sequence.) -

For P(flush including straight flush), we count the number of ways to get at least 5 cards with the same suit. We don't exclude forming a straight because we are intentionally including straight flushes. So, we first choose a suit for the flush, then using classification we choose either 5 cards of the same suit and another 2 cards from a different suit, 6 cards of the same suit and another card from a different suit, or 7 cards of the same suit:

$$\begin{aligned} \#(\geq 5 \text{ same suit}) &= \underbrace{\begin{array}{l} \#(\text{suits for the flush}) \\ \times\ \ [\#(5 \text{ cards of suit}) \times \#(2 \text{ cards not of suit}) \\ \quad\ +\#(6 \text{ cards of suit}) \times \#(1 \text{ card not of suit}) \\ \quad\ + \#(7 \text{ cards have same suit})] \end{array}}_{\text{using classification}} \\ &= \binom{4}{1}\left[\binom{13}{5}\binom{39}{2} + \binom{13}{6}\binom{39}{1} + \binom{13}{7}\right] \\ &= 4,089,228 \\ P(\geq 5 \text{ same suit}) &= \frac{4,089,228}{\binom{52}{7}} = 3.0566 \times 10^{-2}. \end{aligned}$$

For P(4 of a kind), we first choose a rank for the 4 of a kind, then we choose all 4 cards of that rank, then we choose any other 3 cards:

$$\#(4 \text{ of a kind}) = \#(\text{choose a rank})$$

$$\begin{aligned}
&\times\#(\text{choose 4 of that rank})\\
&\times\#(\text{choose any other 3 cards})\\
&= \binom{13}{1}\binom{4}{4}\binom{48}{3}\\
&= 224,848\\
P(\text{4 of a kind}) &= \frac{224,848}{\binom{52}{7}} = 1.6807 \times 10^{-3}.
\end{aligned}$$

For $P(\text{full house})$, we use classification to break it into 3 disjoint cases: 1 triple, 1 pair, and 2 kickers; 1 triple and 2 pairs; and 2 triples and 1 kicker.

$$\begin{aligned}
\#(\text{1 triple, 1 pair, and 2 kickers}) &= \#(\text{1 triple}) \times \#(\text{1 pair}) \times \#(\text{2 kickers})\\
&= \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}\binom{11}{2}\binom{4}{1}^2\\
&= 3,294,720.\\
\#(\text{1 triple and 2 pairs}) &= \#(\text{1 triple}) \times \#(\text{2 pairs})\\
&= \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{2}^2\\
&= 123,552.\\
\#(\text{2 triples and 1 kicker}) &= \#(\text{2 triples}) \times \#(\text{1 kicker})\\
&= \binom{13}{2}\binom{4}{3}^2\binom{11}{1}\binom{4}{1}\\
&= 54,912.
\end{aligned}$$

So, we have

$$P(\text{full house}) = \frac{3,294,720 + 123,552 + 54,912}{\binom{52}{7}} = 0.02596102.$$