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Properties of traveling waves in an impulsive reaction–diffusion model with overcompensation

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Abstract. The phenomenon of overcompensation is widespread in ecology, and non-monotonic discrete-time map may admit complex dynamics. This paper focuses on the impact of overcompensation on the propagation of a spatially moving population with a birth pulse. We prove the upward convergence of the oscillating traveling wave when the birth function is a unimodal function. We establish the existence of monotone and non-monotone traveling wave solutions for the second iterative operator. Furthermore, we obtain the existence, uniqueness, and stability of the standing wave solutions for the second iterative operator. Numerical simulations are performed to illustrate and complement the theoretical results.

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1. Introduction

The overcompensatory phenomenon is universal in ecology, that is, over a certain threshold the population density at the next generation is described by a decreasing function with respect to the current density. Mathematically, this important phenomenon is represented by unimodal functions:

$$g(N) = Ne^{r(1-N/K)} \text{ (Ricker function)}, \tag{1.1}$$

$$g(N) = (1+r)N - rN^2/K \text{ (Logistic function)}. \tag{1.2}$$

Their positive fixed points may lose stability due to period-doubling bifurcations, resulting in stable two-point cycles. Many non-monotone discrete-time models exhibit complex dynamics [20]. If the reproduction map exhibits stable two-point cycles or more complicated behaviors, what are the dynamics in the presence of spatial movement?

Kot [12] studied the solutions of different waveforms appearing in some integral-difference equations. With the appearance of overcompensation, the scalar integral-difference equation suddenly presents a traveling wave that is much more complicated than the scalar reaction–diffusion equation. Li, Lewis, and Weinberger [14] showed that even in the overcompensated model, the spreading speed can still be represented as the slowest speed of a nonconstant traveling wave. They simulated a series of traveling waves and observed that, depending on the characteristics of the birth function, the tails of the waves may approach the carrying capacity monotonically, may approach the carrying capacity in an oscillatory manner, or may oscillate continually about the carrying capacity. Recently, Bourgeois, Leblanc, and Lutscher [2,3] conducted further studies on the integral-difference system with overcompensation, which explains the phenomenon of multiple propagation speeds and multilayer traveling waves in the system. In addition, they proved the existence of monostable and bistable traveling waves under the corresponding quadratic iterator and related the results to the existence of stacked waves.

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Many species give birth only at a particular time of each year. Such species have birth pulse, that is, reproduction takes place in a fixed short time period each year. It is more appropriate to use impulsive reaction–diffusion equations instead of reaction–diffusion equations to describe the propagation dynamics of population with birth pulse. In recent years, impulsive partial differential equations have been studied in [1,8,9,13,15,18,21,22,24–28,30–32]. Most efforts on the propagation dynamics of pulse models have focused on monotone traveling wave solutions; however, few studies have been conducted on non-monotone traveling wave solutions. Only some information can be obtained in [13,18]. Lewis and Li showed that oscillating traveling waves in the integral-difference systems also appear in the following impulsive reaction–diffusion equation (IRDE):

$$\begin{cases} u_t = du_{xx} - \alpha u - \gamma u^2, & (x,t) \in \mathbb{R} \times (0,1], \\ u(x,0) = g(N_m(x)), & x \in \mathbb{R}, \\ N_{m+1}(x) = u(x,1), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

Lewis and Li studied the spreading speed and the existence of traveling waves in the monostable case (i.e., the system has an unstable zero solution and a stable positive solution β). Lin and Wang [18] generalized the conclusions about traveling wave solutions to the model with general response terms and provided the results about the spreading speed and the existence of the traveling wave solution. Their results in [13,18] showed that if $g(N)$ is a non-monotone function, such as the Ricker function, the existence of traveling wave solution is still valid. The numerical simulation in [13] showed that if $g(N)$ is a Ricker function, then system (1.3) can possess oscillatory traveling waves. However, the upward convergence of the traveling wave solution (i.e., the asymptotic property of the wave profile at $+\infty$) and the dynamics of the system when the positive solution β loses stability have not been studied in previous work.

The purpose of this paper was to study the propagation phenomenon when the map $N \mapsto g(N)$ is unimodal. This paper is a further mathematical exploration beyond [13,18]. We rigorously study the propagation of the impulsive reaction–diffusion equation with a non-monotone growth function. We present the following impulsive reaction–diffusion system for any $m \in \mathbb{Z}^+$:

$$\begin{cases} u_t = du_{xx} + f(u), & (x,t) \in \mathbb{R} \times (0,1], \\ u(x,0) = g(N_m(x)), & x \in \mathbb{R}, \\ N_{m+1}(x) = u(x,1), & x \in \mathbb{R}. \end{cases} \tag{1.4}$$

Equation (1.4) defines a recurrence relation for $N_m(x)$ as

$$N_{m+1}(x) = Q[N_m(x)] \ \text{ for } \ x \in \mathbb{R}, \tag{1.5}$$

where $m \geq 0$ and Q is an operator that depends on d, f, g.

The main work of this paper includes two aspects: One is to study the upward convergence of non-monotone traveling wave solutions and the other is to study the properties of traveling wave solutions of quadratic iterative operators. The rest of this paper is organized as follows. In Sect. 2, we introduce some notations and assumptions that will be used later. In Sect. 3, we investigate the convergence of non-monotone traveling wave solutions. In Sect. 4, we study the properties of second iterative operators by the theory of monotone semiflows. In Sect. 5, we provide some numerical illustrations for our theoretical results. In Sect. 6, we summarize the paper with some concluding remarks.

2. Notations and assumptions

We start by introducing some notations. Let $C := BC(\mathbb{R}, \mathbb{R})$ be all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with the compact open topology. We equip C with the norm with respect to this topology $\|\phi\| = \sum_{k\geq 1} 2^{-k} \max_{|x|\leq k} |\phi(x)|$. For $\phi, \psi \in C$, we write $\phi \geq \psi$ if $\phi(x) - \psi(x) \geq 0$ for $x \in \mathbb{R}$. Denote $C_{[a,b]} := \{\phi \in C : b \geq \phi \geq a\}$, $C_+ := \{\phi \in C : \phi \geq 0\}$, and $C_r := C_{[0,r]}$.

Let Q_1 be the time-one solution map of the evolution system $u_t = du_{xx} + f(u), x \in \mathbb{R}$. Then, $N_m(x)$ satisfies the recursion system

$$N_{m+1}(x) = Q_1 \circ [g(N_m(\cdot))](x) = Q[N_m](x), x \in \mathbb{R}, \forall m \geq 0. \tag{2.1}$$

Note that $k(x,t)$ is Green's function of $\partial_t u = du_{xx}$, $k_1 * g(u) = \int\limits_R k(x-y,1)g(u(y))dy$, and $k_1 ** f(u) = \int\limits_0^1 \int\limits_R k(x-y,1-s)f(u(y,s))dyds$, where $k_1(x) = k(x,1) = \frac{1}{\sqrt{4\pi d}}\exp(-\frac{x^2}{4d})$. We are then able to derive an explicit relation between the initial value $g(N_m(x))$ and the time-one solution map $u(x,1,g(N_m(x)))$. It reads $Q[\cdot] = Q_1[g(\cdot)] = (k_1 * g + k_1 ** f)(\cdot)$. We say that $N_m(x)$ is a traveling wave solution of (1.5) if there exist a function W and a constant c such that $N_m(x) = W(x-cm)$ and $Q[W(\cdot - cm)](x) = W(x-(m+1)c)$ for all integers m.

Consider the following impulsive model without spatial dispersal:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = f(u), 0 < t \leq 1, \\ u(0) = g(N_m), \\ N_{m+1} = u(1). \end{cases} \tag{2.2}$$

Let F denote the time-one solution map of the ordinary differential equation in model (2.2). It then follows that model (2.2) can be reduced to a discrete-time system

$$N_{m+1} = H[N_m] = F \circ g(N_m), \quad \forall m \geq 0. \tag{2.3}$$

The properties of the mapping H depend on the properties of f and g. Then, we make the following assumptions throughout the paper:

(A1) The function $f(u) \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following assumptions:
- $f(0) = 0$.
- $\frac{f(u)}{u}$ is non-increasing for $u > 0$.

(A2) The function $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following assumptions:
- $g(0) = 0, g'(0) > 0, g(N) > 0$ for $N > 0, g'(0)e^{f'(0)} > 1$.
- There exists a $\bar{N} > 0$ such that $g(\bar{N}) \leq \bar{N}$, and $\frac{g(N)}{N}$ is non-increasing for $N > 0$.

A commonly adopted function that satisfies assumption (A1) takes the following form:

$$f(u) = -au - bu^2$$

where $a > 0$ indicates the mortality rate in the dispersion stage and $b \geq 0$ indicates the competition coefficient. One can obtain the solution of $\dfrac{\mathrm{d}u}{\mathrm{d}t} = f(u)$ with initial state U is $F(U) = \dfrac{aU}{(e^a - 1)bU + ae^a}$. Note that the map F is always strictly monotonically increasing and satisfies $F(0) = 0, F(U) > 0$ if $U > 0$. In this paper, we focus on the propagation phenomenon when the map $N \mapsto g(N)$ is unimodal. For typical unimodal functions that satisfy the assumption (A2), see functions (1.1) and (1.2). Let $\hat{Q}$ be the restriction of Q to $\mathbb{R}$, that is, $\hat{Q} : \mathbb{R} \to \mathbb{R}$. Here, $\mathbb{R}$ is a subset of C and represents the set of constant functions. If system (1.4) starts to evolve with a constant positive profile $g(N_0) \in \mathbb{R}$, then the solution of (1.4) remains spatially constant and satisfies (2.2). Thus, Q_1 reduces to F, and $\hat{Q} : \mathbb{R} \to \mathbb{R}$ reduces to $H : \mathbb{R} \to \mathbb{R}$.

Assume that H satisfies the following conditions:

(H1) H is a continuously differentiable function on some right neighborhood of 0.
(H2) $H'(0) > 1$ and $H(N) \leq H'(0)N$ for all $N \in [0, \infty)$.
(H3) H has a unique positive fixed point β.

In the following, we always assume that (H1)–(H3) hold. By (H1)–(H3), we easily see that $H(N) > N$ for all $N \in (0, \beta)$ and $H(N) < N$ for all $N \in (\beta, \infty)$.

(H4) $|H'(\beta)| < 1$.

Conditions (H1–H4) mean that the second-iterate operator $H^2 := H \circ H$ has a unique stable positive fixed point. By [33, Lemma 5.3], the operator H satisfies the following properties:

(UM) For any interval $[a, b] \subseteq (0, \infty)$ with $a < b$, there exist $a', b' \in (0, \infty)$ such that $[a, b] \subseteq [a', b']$, $H([a', b']) \subseteq [a', b']$, and either $a < \min\{H(N) : N \in [a', b']\}$ or $b > \max\{H(N) : N \in [a', b']\}$.

In a later section, we will give examples of situations in which the system meets the enumerated assumptions.

3. Spreading speeds and traveling waves

We first provide a lemma to illustrate the relationship between two operators H and Q.

Lemma 3.1. *Assume that H satisfies (H1)–(H4). Then, the following statements are true:*

1. *If $H(a) = a$ for some $a \in \mathbb{R}$, then $Q^m[a] = a$ for all $m \geq 1$.*
2. *If $H([a, b]) \subseteq [a, b] \subseteq \mathbb{R}_+$, then $Q^m\left[C_{[a,b]}\right] \subseteq C_{[a,b]}$ for all $m \geq 1$.*
3. *If H is non-decreasing on $\mathbb{R}_+$ with $H(0) = 0$ and $H(\beta) = \beta$, where β is the only positive fixed point, then Q^m is non-decreasing on C_+, and for any $m \geq 1$, we have $Q^m[C_{[a,\beta]}] \gg a$ for all $a \in (0, \beta)$ and $Q^m[C_a] \ll a$ for all $a \in (\beta, \infty)$.*
4. *If $b > a > 0$, then there are three numbers $N(a, b) \in (0, \infty)$, $I(a, b) \in (a, \infty)$, and $S(a, b) \in (0, b)$ such that either $Q^m\left[C_{[a,b]}\right] \geq I(a, b)$ for all $m \geq N(a, b)$ or $Q^m\left[C_{[a,b]}\right] \leq S(a, b)$ for all $m \geq N(a, b)$.*

Proof. (1) $Q[a] = Q_1 \circ g[a] = F \circ g[a] = H[a] = a$, then $Q^m[a] = a$.
(2) Let $g_m := \min\{g(x) : x \in [a, b]\}$ and $g_M := \max\{g(x) : x \in [a, b]\}$. If $H([a, b]) \subseteq [a, b] \subseteq \mathbb{R}_+$, then $F \circ g_m \geq a, F \circ g_M \leq b$. Thus, for every $\varphi \in C_{[a,b]}$, by the comparison principle, we have $a \leq F \circ g_m = Q_1 \circ g_m \ll Q_1 \circ g(\varphi) \ll Q_1 \circ g_M = F \circ g_M \leq b$, and hence, $Q^m\left[C_{[a,b]}\right] \subseteq C_{[a,b]}$ for all $m \geq 1$. (3) Since the composite function $H = F \circ g$ is non-decreasing on $\mathbb{R}_+$ and F is strictly monotonically increasing, g and H have the same monotonicity and are non-decreasing on $\mathbb{R}_+$. By the monotonicity of g and the comparison principle for reaction–diffusion equations, we have Q is monotone in the sense $Q[u](x) \geq Q[v](x) \geq 0$ if $u(x) \geq v(x) \geq 0$. This result implies that Q is order preserving. Thus, Q^m is monotone on C_+. Next, we only consider the case where $a \in (0, \beta)$, and the case $a \in (\beta, \infty)$ can be proved similarly. For $a \in (0, \beta)$, $H(a) > a$ since $H'(0) > 1$ and β is the only positive fixed point. Note that $H([a, \beta]) \subseteq [a, \beta]$, by step (2), $Q^m\left[C_{[a,\beta]}\right] \subseteq C_{[a,\beta]}$ for all $m \geq 1$. Thus, for every $\varphi \in C_{[a,\beta]}$, by the comparison principle, we have $Q[\varphi] = Q_1 \circ g[\varphi] \gg Q_1 \circ g(a) = F \circ g(a) > a$. Thus, we have $Q^m[C_{[a,\beta]}] \gg a$ for all $a \in (0, \beta), m \geq 1$.
(4) Suppose that $b > a > 0$. By above (1)-(3) in proof and (UM), there exist $a', b' \in (0, \infty)$ such that $[a, b] \cup H([a', b']) \subseteq [a', b']$ and either $a < \min\{H(N) : N \in [a', b']\}$ or $b > \max\{H(N) : N \in [a', b']\}$. Without loss of generality, we assume that $I := \min\{H(N) : N \in [a', b']\} > a$. Since $H([a', b']) \subseteq [a', b']$, similar to step (2), one can obtain that $Q[\varphi] \geq I > a$ for every $\varphi \in C_{[a',b']}$, and $Q^m\left[C_{[a',b']}\right] \subseteq C_{[a',b']}$ for all $m \geq 1$. Obviously, $Q^m[\varphi] \geq I > a$ for every $\varphi \in C_{[a,b]} \subseteq C_{[a',b']}$. Thus, we have $\inf_{x\in\mathbb{R}} Q^m\left[C_{a,b}\right](x) > a$ or $\sup_{x\in\mathbb{R}} Q^m\left[C_{a,b}\right](x) < b$ for every $m \geq 1$. □

Motivated by [13,18], we introduce two monotone functions g^+ and g^-. We define

$$g^+(N) = \max_{0\leq V\leq N} g(V), \quad \forall N \geq 0.$$

It then follows that g^+ is non-decreasing, locally Lipschitz continuous, and $g^{+\prime}(0) = g'(0)$. In the case where $e^{f'(0)}g'(0) > 1$, system (1.4) with g replaced by g^+ has a positive fixed point $\beta^+ \in (0, \sigma]$. In such a case, we define g^- as

$$g^-(N) = \min_{N\leq V\leq\beta^+} g(V), \quad \forall 0 \leq N \leq \beta^+.$$

It is easy to see that g^- is also non-decreasing, locally Lipschitz continuous, and system (1.4) with g replaced by g^- admits a positive equilibrium β^-. Clearly, $0 < \beta^- \leq \beta \leq \beta^+$. It is easy to see $g^-(N) \leq g(N) \leq g^+(N), g^{\pm\prime}(0) = g'(0), g^{\pm}(N) \leq g'(0)N$, and there exists σ_0 such that $g^{\pm}(N) = g(N)$ for all $N \in (0, \sigma_0]$. With the above functions g^+ and g^-, we consider two auxiliary models:

$$\begin{cases} u_t = du_{xx} + f(u), & (x,t) \in \mathbb{R} \times (0,1], \\ u(x,0) = g^+(N_m(x)), & x \in \mathbb{R}, \\ N_{m+1}(x) = u(x,1), & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

and

$$\begin{cases} u_t = du_{xx} + f(u), & (x,t) \in \mathbb{R} \times (0,1], \\ u(x,0) = g^-(N_m(x)), & x \in \mathbb{R}, \\ N_{m+1}(x) = u(x,1), & x \in \mathbb{R}. \end{cases} \tag{3.2}$$

Similarly, systems (3.1) and (3.2) can be reduced to the following iterative systems:

$$N_{m+1}^+(x) = Q_1 \circ \left[g^+\left(N_m^+(\cdot)\right)\right](x) = Q^+\left[N_m^+\right](x), x \in \mathbb{R}, \forall m \geq 0, \tag{3.3}$$

and

$$N_{m+1}^-(x) = Q_1 \circ \left[g^-\left(N_m^-(\cdot)\right)\right](x) = Q^-\left[N_m^-\right](x), x \in \mathbb{R}, \forall m \geq 0. \tag{3.4}$$

Let $N_m^+(x)$ and $N_m^-(x)$ be solutions of systems (3.3) and (3.4), respectively. The comparison argument shows that if $0 < N_0^-(x) \leq N_0(x) \leq N_0^+(x) \leq \beta^+$, where $N_0^-, N_0, N_0^+ \in C_{\beta^+}$, then $0 \leq N_m^-(x) \leq N_m(x) \leq N_m^+(x) \leq \beta^+$ for all $m \geq 0$.

The main objective in this section is to establish the upward convergence and the existence of traveling waves when g is non-monotone. The following theorems are mostly from [13, Theorem 2.2] and [18, Theorem 4.1]. For completeness, the previous conclusions are listed here. However, we focus on upward convergence.

Theorem 3.2. *Assume that f satisfies (A1), g satisfies (A2), and H satisfies (UM). Then,*

$$c^* := 2\sqrt{d \ln(g'(0)e^{f'(0)})}$$

is the spreading speed of system (1.4) such that the following statements are valid:

1. *If $N_0(x) \in C_{\beta^+}$ has compact support, then $\lim_{m\to+\infty,|x|\geq cm} N_m(x) = 0$ for any $c > c^*$.*
2. *If $N_0(x) \in C_{\beta^+}\backslash\{0\}$, then $\beta^- \leq \liminf_{m\to+\infty,|x|\leq cm} N_m(x) \leq \limsup_{m\to+\infty,|x|\leq cm} N_m(x) \leq \beta^+$ for any $c \in (0, c^*)$.*
3. *If $|H'(\beta)| < 1$ holds, then for any $N_0(x) \in C_{\beta^+}\backslash\{0\}$, then $\lim_{m\to\infty,|x|\leq cm} N_m(x) = \beta$ for all $c \in (0, c^*)$.*

Proof. Define $Q[\cdot] = Q_1 \circ g[\cdot]$ and $Q^{\pm}[\cdot] = Q_1 \circ g^{\pm}[\cdot]$. Clearly, $Q^{\pm}$ is order preserving on C_{β^+} and $Q^-(\phi) \leq Q(\phi) \leq Q^+(\phi), \forall \phi \in C_+$. By [18, Theorem 3.1] and [18, Theorem 4.1], it follows that c^* is the spreading speed for the iterative systems $N_{m+1} = Q^{\pm}(N_m)$ on $C_{\beta^{\pm}}$.

(1) For a given $\phi \in C_{\beta^+}$ with compact support, let $N_m = Q^m(N_0), N_m^+ = (Q^+)^m(N_0), \forall m \geq 0$. By the comparison principle (see [26]), we have $0 \leq N_m(x) \leq N_m^+(x)$ for all $x \in \mathbb{R}$ and $m \geq 0$. For any $c > c^*$, [18, Theorem 3.1] implies $\lim_{m\to\infty,|x|\geq cm} N_m^+(x) = 0$, and hence $\lim_{m\to\infty,|x|\geq cm} N_m(x) = 0$.

(2) For a given $\phi \in C_{\beta^+}\backslash\{0\}$, define $\psi(x) = \min\{\phi(x), \beta^-\}$, then $\psi \in C_{\beta^+}\backslash\{0\}$. Let $N_m^- = (Q^-)^m(\psi)$, $\forall m \geq 0$. Since $\psi \leq \phi$, it follows from the comparison principle that

$$0 \leq N_m^-(x) \leq N_m(x) \leq N_m^+(x), \quad \forall x \in \mathbb{R}, m \geq 0.$$

For any $c \in (0, c^*)$, Theorem 3.1 in [18] implies $\lim_{m\to\infty,|x|\le cm} N_m^{\pm}(x) = \beta^{\pm}$. Thus, we have

$$\beta^- \le \liminf_{m\to\infty,|x|\le cm} N_m(x) \le \limsup_{m\to\infty,|x|\le cm} N_m(x) \le \beta^+. \tag{3.5}$$

(3) For any $c_1 < c^*$, by inequality (3.5), we have $\limsup_{m\to\infty} \max\{Q^m[\varphi](x) : |x| \le mc_1\} \le \beta^+$. Take a positive number $c_0 \in (c_1, c^*), \varphi \in C_{\beta^+}$ and let $\varepsilon_0 = c_0 - c_1$. For any $\varepsilon \ge 0$, define

$$V_-(\varepsilon) = \liminf_{m\to\infty} \min\left\{Q^m[\varphi](x) : |x| \le m(c_1+\varepsilon)\right\}$$

and

$$V_+(\varepsilon) = \limsup_{m\to\infty} \max\left\{Q^m[\varphi](x) : |x| \le m(c_1+\varepsilon)\right\}.$$

Clearly, $V_\pm(\varepsilon) \in [\beta^-, \beta^+]$ for any $\varepsilon \in [0, \varepsilon_0]$, $V_-(\varepsilon)$ is non-increasing and $V_+(\varepsilon)$ is non-decreasing in $\varepsilon \in [0, \varepsilon_0]$. Due to the monotonicity of $V_\pm$ and continuity of Q and φ, we see that $V_-(\varepsilon)$ and $V_+(\varepsilon)$ are continuous in $\varepsilon \in [0, \varepsilon_0]$ except possibly at a countable number of points in $[0, \varepsilon_0]$.

Suppose that $V_-(\varepsilon) < V_+(\varepsilon)$ for any $\varepsilon \in [0, \varepsilon_0]$. By the continuity of $V_\pm$ and Lemma 3.1, we assume, without loss of generality, that for some $\varepsilon_1 \in (0, \varepsilon_0)$, V_- is continuous at ε_1 and

$$I_{V_-(\varepsilon_1),V_+(\varepsilon_1)} := \inf\left\{Q^{L_0}[\phi](x) : x \in \mathbb{R}, \phi \in C_{[V_-(\varepsilon_1),V_+(\varepsilon_1)]}\right\} > V_-(\varepsilon_1),$$

where $L_0 \ge 1$ is a positive integer greater. According to the definition of $V_-(\cdot)$, for any $\tau \in (0, \varepsilon_1)$, there exist sequences $m_k \to \infty$ as $k \to \infty$, $x_k \in [-m_k(c_1+\tau), m_k(c_1+\tau)]$ such that $\lim_{k\to\infty} N_{m_k}(x_k) = V_-(\tau)$. By $\tau < \varepsilon_1$, we know that for any bounded subset $\mathcal{B}$ of $\mathbb{R}$, $x_k + \mathcal{B} \subseteq [-(m_k - L_0)(c+\varepsilon_1), (m_k - L_0)(c+\varepsilon_1)]$ for all large k, which implies $\liminf_{k\to\infty} \min_{y\in\mathcal{B}} N_{m_k-L_0}(x_k+y) \in [V_-(\varepsilon_1), V_+(\varepsilon_1)]$ and $\limsup_{k\to\infty} \max_{y\in\mathcal{B}} N_{m_k-L_0}(x_k+y) \in [V_-(\varepsilon_1), V_+(\varepsilon_1)]$. Combined with (H1), we have

$$V_-(\tau) = \lim_{k\to\infty} N_{m_k}(x_k) = \lim_{k\to\infty} Q^{L_0}[N_{m_k-L_0}(x_k)] \ge I_{V_-(\varepsilon_1),V_+(\varepsilon_1)} > V_-(\varepsilon_1).$$

By the continuity of V_- at ε_1 and letting $\tau \to \varepsilon_1$, we have $V_-(\varepsilon_1) \ge I_{V_-(\varepsilon_1),V_+(\varepsilon_1)} > V_-(\varepsilon_1)$, a contradiction. Then, $V_-(\varepsilon) = V_+(\varepsilon)$ for some $\varepsilon \in [0, \varepsilon_0]$. By definitions of $V_\pm(\varepsilon)$ and the continuity of Q, if $\phi \in C_{[\epsilon,\beta^+]}$ with $0 < \epsilon \ll 1$, then we have $Q^m[\phi] \to V_+(\varepsilon)$ as $m \to \infty$. Thus, $Q[V_+(\varepsilon)] = V_+(\varepsilon) := \beta^*$. We claim that $\beta = \beta^*$; in fact, if $\beta \ne \beta^*$, we set $r = \min\{\beta, \beta^*\}$ and $s = \max\{\beta, \beta^*\}$, then $s > r$, $Q[r] = r$, and $Q[s] = s$. In particular, $r, s \in Q[C_{r,s}]$. However, the assumption (UM) and Lemma 3.1 imply $r \notin Q[C_{r,s}]$ or $s \notin Q[C_{r,s}]$, a contradiction. This shows $V_-(0) = V_+(0) = \beta$, and thus, the conclusion follows. □

Remark 3.3. In case $H'(\beta) \in [0, 1)$ holds, we see that $g'(N) \ge 0$ and $g^{\pm}(N) = g(N)$ for $N < \beta$, and hence, $\beta^- = \beta = \beta^+$. Therefore, upward convergence can be obtained from step (2) in the proof of Theorem 3.2.

The following theorem states the existence and nonexistence of traveling wave solutions and then obtains the result that the asymptotic spreading speed coincides with the slowest wave speed of traveling wave solutions for the corresponding linearized system (i.e., the spreading speed is linearly determined).

Theorem 3.4. *Assume that f satisfies (A1), g satisfies (A2), and H satisfies (UM). Then, the following statements are valid:*

1. *For any $c \in (0, c^*)$, system (1.5) has no traveling wave $W \in C_{\beta^+}\backslash\{0\}$ with $W(-\infty) = 0$.*
2. *For any $c \ge c^*$, system (1.5) has a traveling wave $W \in C_{\beta^+}\backslash\{0\}$ satisfying $W(-\infty) = 0$ and $W(+\infty) = \beta$. Moreover, if $H'(\beta) \in [0, 1)$, then for each $c \ge c^*$, the traveling wave W is non-decreasing and satisfies $\lim_{\xi\to+\infty} W(\xi) = \beta$. If $H'(\beta) \in (-1, 0)$, the traveling wave W also satisfies $\lim_{\xi\to+\infty} W(\xi) = \beta$.*

Proof. 1. Assume that for some $c_0 \in (0, c^*)$ has a traveling wave $N_m(x) = W(x + c_0 m)$ with $W \in C_{\beta^+}\backslash\{0\}$ and $W(-\infty) = 0$. By Theorem 3.2, we have $\liminf_{m\to\infty,|x|\leq cm} N_m(x) \geq \beta^- > 0, \quad \forall c \in (0, c^*)$. Choose $\tilde{c} \in (c_0, c^*)$ and let $x = -\tilde{c}m$. Then,

$$\liminf_{m\to\infty} N_m(-\tilde{c}m) = \liminf_{m\to\infty} W((c_0 - \tilde{c})m) > 0.$$

However, $\lim_{m\to\infty} W((c_0 - \tilde{c})m) = W(-\infty) = 0$, a contradiction. Therefore, we obtain the nonexistence of traveling wave solutions when $c \in (0, c^*)$.

2. For any $c > c^*$, the proof of the existence of traveling wave solutions can be accomplished by means of the Schauder–Tychonoff fixed point theorem. More details can be found in [18, Theorem 4.1]. Here, we only consider the asymptotic behavior of traveling wave solution $W(\xi)$ as $\xi \to +\infty$.

Motivated by the proof of [11, Theorem 3.1], we set $N_m(x) = W(x + cm)$ for all $m \geq 0$ and fix a number $\bar{c}_0 \in (0, c^*)$. By Theorem 3.2, it follows that

$$0 < \beta^- \leq \liminf_{m\to+\infty,|x|\leq\bar{c}_0 m} N_m(x) \leq \limsup_{m\to+\infty,|x|\leq\bar{c}_0 m} N_m(x) \leq \beta^+.$$

Thus, $\beta^- \leq \liminf_{m\to+\infty} N_m(-\gamma m) \leq \limsup_{m\to+\infty} N_m(-\gamma m) \leq \beta^+$ uniformly for $\gamma \in [0, \bar{c}_0]$. This implies that $\beta^- \leq \liminf_{m\to+\infty} W(sm) \leq \limsup_{m\to+\infty} W(sm) \leq \beta^+$ uniformly for $s \in [c - \bar{c}_0, c]$. Let $a_m = (c - \bar{c}_0)m, b_m = cm$ for all $m \geq 1$. Thus, there exists $j_0 > 0$ such that $a_{m+1} - b_m < 0$ for all $m \geq j_0$, and hence, $\cup_{m\geq j}[a_m, b_m] = [a_j, +\infty)$ for all $j \geq j_0$. It follows that $\beta^- \leq \liminf_{\xi\to+\infty} W(\xi) \leq \limsup_{\xi\to+\infty} W(\xi) \leq \beta^+$. If $|H'(\beta)| < 1$ holds, then Theorem 3.2 implies that $\lim_{m\to\infty,|x|\leq\tilde{c}m} N_m(x) = \beta$ for all $\tilde{c} \in (0, c^*)$. It follows that $\lim_{m\to\infty,|x|\leq\tilde{c}m} W(x + cm) = \beta$, and thus, we have $\lim_{m\to\infty} W(-\tilde{c}m + cm) = \lim_{m\to\infty} W((-\tilde{c} + c)m) = W(+\infty) = \beta$.

Suppose that $c = c^*$. There exist two sequences $\{c_m \in (c^*, \infty) : m \in \mathbb{N}\}$ and $\{\phi_m \in C_{\beta^+} : \phi_m(\infty) = \beta \text{ and } \phi_m(-\infty) = 0\}_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} c_m = c^*$ and $T_{-c_m}[Q[\phi_m]] = \phi_m$ for all $m \in \mathbb{N}$.

Let $B = \{\phi_m : m \in \mathbb{N}\}$, then $B \subseteq E_c$. By the compactness of $T_{-c}[Q[g(\cdot)]]$, we can show that B is precompact. Without loss of generality, we assume that the limiting of ϕ_m exists, and $\inf\{x \in \mathbb{R} : \phi_m(x) = \frac{\beta}{2}\} = 0$ due to translation invariance of Q. Hence, $\phi_m(0) = \frac{\beta}{2}$ and $\phi_m(x) \geq \frac{\beta}{2}$ for all $x \in [0, \infty)$. Let $\phi = \lim_{m\to\infty} \phi_m$. Then $\phi(0) = \frac{\beta}{2}$, $\phi(x) \geq \frac{\beta}{2}$ for all $x \in [0, \infty)$ and $T_{-c^*}Q[\phi] = \phi$. Similar to the previous discussion, we can set $N_m(x) := W(x + c^*m)$ for all $m \geq 0$ and fix a number $\bar{c}_0 \in (0, c^*)$. If $|H'(\beta)| < 1$ holds, then for any $N_0(x) \in C_{\beta^+}\backslash\{0, \beta\}$, then $\lim_{m\to\infty,|x|\leq c_0 m} N_m(x) = \beta$ for $c_0 \in (0, c^*)$. It follows that $\lim_{m\to\infty,|x|\leq c_0 m} W(x + c^*m) = \beta$, and hence, $\lim_{m\to\infty} W(-c_0 m + c^*m) = \lim_{m\to\infty} W((-c_0 + c^*)m) = W(+\infty) = \beta$, and thus W is a nonconstant function.

In particular, if $H'(\beta) \in [0, 1)$, then $H(N)$ is a monotonically increasing function on the interval $[0, \beta]$. Combining Lemma 3.1, it can be seen that Q is order preserving on $C_{[0,\beta]}$. According to the conclusion of [13, Theorem 2.1], for each $c \geq c^*$, the traveling wave W is non-decreasing. The proof is complete. □

4. Second-iterate operator

In Sect. 3, we always assume that (H4) is true, that is, $|H'(\beta)| < 1$. It is worth noting that some nonlinear difference equations can exhibit a remarkable spectrum of dynamical behaviors, from stable equilibrium points, to stable cyclic oscillations between two points, to stable cycles with four points, then eight, sixteen points,..., through to a chaotic regime. At $H'(\beta) = -1$, the positive state $N = \beta$ for H loses stability through a flip bifurcation, and there is a stable two-cycle for $H(N)$. We denote the values of the two-cycle as β_1 and β_2. They satisfy the relationships

$$\beta_1 = H(\beta_2) = H(H(\beta_1)), \ \beta_2 = H(\beta_1) = H(H(\beta_2)),$$

where $0 < \beta_1 < \beta < \beta_2$. It is well known that the two-point cycle is said to be stable if β_1 and β_2 are stable fixed points of second-iterate map H^2 and the two-point cycle is unstable if β_1 and β_2 are unstable fixed points of the second-iterate map H^2.

As is common when studying two cycles, we introduce the second-iterate operator

$$\mathcal{P}[N(x)] = Q \circ Q[N(x)] = [Q_1 \circ g] \circ [Q_1 \circ g][N(x)]. \tag{4.1}$$

When studying the properties of $\mathcal{P}$, it will be convenient to change indices and study $\tilde{N}_{m+1}(x) = \mathcal{P}[\tilde{N}_m](x)$. We will drop the tilde when no confusion can arise. The continuity and compactness of $\mathcal{P}$ is a direct result of the corresponding properties of Q. A traveling two-cycle for the operator Q corresponds to a pair of traveling wave profiles for the operator $\mathcal{P}$.

Let H be a function that satisfies the following condition:

(H5) β is the only positive fixed point of H, $H^{'}(\beta) < -1$, and $[H'(\beta)]^2 = \max_{N\in[\beta_1,\beta_2]}\{[H^2]'(N)\}$. H has exactly one stable two-cycle, i.e., there exist β_1 and β_2 such that $0 < \beta_1 < \beta < \beta_2, H(\beta_1) = \beta_2$ and $H(\beta_2) = \beta_1$, and all nonnegative initial conditions converge to this two-cycle under the map $N_{m+1} = H[N_m]$.

Lemma 4.1. *Assume that H satisfies (H1–H3) and (H5), and H is non-increasing on the interval $[\beta_1, \beta_2]$. Then the following statements are true:*

1. $\mathcal{P}[\beta_1] = \beta_1$, $\mathcal{P}[\beta_2] = \beta_2$, *and* $\mathcal{P}[\beta] = \beta$.
2. $\mathcal{P}[C_{[\beta,\beta_2]}] \subseteq C_{[\beta,\beta_2]}$ *and* $\mathcal{P}[C_{[\beta_1,\beta]}] \subseteq C_{[\beta_1,\beta]}$.
3. $\mathcal{P}[\alpha] \gg \alpha$ *for all* $\alpha \in (\beta, \beta_2)$ *and* $\mathcal{P}[\alpha] \ll \alpha$ *for all* $\alpha \in (\beta_1, \beta)$.
4. *If* $\beta_1 \leq n_1(x) \leq n_2(x) \leq \beta_2$, *then* $\beta_1 \leq \mathcal{P}[n_1(x)] \leq \mathcal{P}[n_2(x)] \leq \beta_2$.

Proof. Since $H^2(\beta) = \beta$ and $H^2(\beta_i) = \beta_i, i = 1, 2$, we have $\mathcal{P}(\beta) = \beta$ and $\mathcal{P}(\beta_i) = \beta_i, i = 1, 2$ in the sense of constant functions. The function H maps the interval $[\beta, \beta_2]$ into $[\beta_1, \beta]$ and vice versa. Hence, if $N \in [\beta, \beta_2]$, then $Q[N] \in [\beta_1, \beta]$ and $\mathcal{P}[N] = Q(Q[N]) \in [\beta, \beta_2]$. Therefore, $C_{[\beta,\beta_2]}$ is invariant under $\mathcal{P}$. Similarly, we have $\mathcal{P}[C_{[\beta_1,\beta]}] \subseteq C_{[\beta_1,\beta]}$. From (H5), we have $(H^2)^{'}(\beta) > 1$; thus, $H^2(\alpha) > \alpha$ for some $\alpha > \beta$. Since there is no fixed point between β and β_2, we must have $H^2(\alpha) > \alpha$ for $\alpha \in (\beta, \beta_2)$. The same relation holds for constant functions under $\mathcal{P}$. To show monotonicity, assume that $\beta \leq n_1(x) \leq n_2(x) \leq \beta_2$. Then, we have $\beta \geq Q(n_1(x)) \geq Q(n_2(x)) \geq \beta_1$ and $\beta \leq Q(Q[n_1(x)]) \leq Q(Q[n_2(x)]) \leq \beta_2$. □

The translation invariance, continuity, and compactness of $\mathcal{P} : C_{[\beta,\beta_2]} \mapsto C_{[\beta,\beta_2]}$ follow from the corresponding properties of Q. More mathematical details can be found in [26]. The monostability and monotonicity of $\mathcal{P} : C_{[\beta,\beta_2]} \mapsto C_{[\beta,\beta_2]}$ can be obtained by Lemma 4.1. According to the conclusions in [16,17,29], we can directly give the following theorem.

Theorem 4.2. *There exists a spreading speed $c^*_{[\beta,\beta_2]}$ for the operator $\mathcal{P}$ in the following sense:*

1. *For any $N_0 \in C_{[\beta,\beta_2]}$ such that $N_0 - \beta$ has compact support, the solution of (4.1) satisfies*

$$\lim_{m\to\infty} \sup_{|x|\geq cm} N_m(x) = \beta \text{ for all } c > c^*_{[\beta,\beta_2]}.$$

2. *For any $N_0 \in C_{[\beta,\beta_2]}\backslash\{\beta\} := \{N \in C_{[\beta,\beta_2]} : N - \beta \neq 0\}$, the solution of (4.1) satisfies*

$$\lim_{m\to\infty} \sup_{|x|\leq cm} (\beta_2 - N_m)(x) = 0 \text{ for all } c \in \left(0, c^*_{[\beta,\beta_2]}\right).$$

Furthermore, for every $c \geq c^*_{[\beta,\beta_2]}$, there exists a monotone traveling wave $W(x + cm)$ with $W(-\infty) = \beta$ and $W(+\infty) = \beta_2$.

Theorem 4.3. *There exists a unique pair of stable standing waves $W_1(x), W_2(x) \in \mathcal{C}_+$ for $\mathcal{P} : C_{[\beta_1,\beta_2]} \mapsto C_{[\beta_1,\beta_2]}$ such that $\mathcal{P}^m[W_{1,2}](x) = W_{1,2}(x), \forall x \in \mathbb{R},\ m \geq 0$, with $W_1(-\infty) = \beta_1, W_1(+\infty) = \beta_2$ and $W_2(-\infty) = \beta_2, W_2(+\infty) = \beta_1$, where $W_1(x)$ is an increasing function and $W_2(x)$ is a decreasing function.*

Proof. The translation invariance, continuity, and compactness of $\mathcal{P} : C_{[\beta_1,\beta_2]} \mapsto C_{[\beta_1,\beta_2]}$ follow from Theorem 4.2. According to Lemma 4.1, the bistability and monotonicity of $\mathcal{P} : C_{[\beta_1,\beta_2]} \mapsto C_{[\beta_1,\beta_2]}$ can be obtained.

Let $\bar{P}$ be the operator defined by $\bar{P}[u] = \mathcal{P}[u + \beta_1] - \beta_1$. $\bar{P}$ is an operator on $C_{\beta_2-\beta_1}$ that defines the recursion $u_{m+1} = \bar{P}[u_m]$ for $u_0 \in C_{\beta_2-\beta_1}$. Translation invariance, monotonicity, continuity, and compactness of $\bar{P} : C_{\beta_2-\beta_1} \to C_{\beta_2-\beta_1}$ follow from the corresponding properties of $\mathcal{P}$. We then verify that $\bar{P} : C_{\beta_2-\beta_1} \to C_{\beta_2-\beta_1}$ satisfies the bistability and counter-propagation. Let $\hat{P}$ be the restriction of $\bar{P}$ to $\mathbb{R}$, that is, $\hat{P} : \mathbb{R} \to \mathbb{R}$. Here, $\mathbb{R}$ is a subset of C and represents the set of constant functions. For any $N \in [0, \beta_2 - \beta_1]$, $\hat{P}[N] = \mathcal{P}[N + \beta_1] - \beta_1 = H^2[N + \beta_1] - \beta_1$. $\hat{P}$ admits exactly three fixed points $\beta_2 - \beta_1 > \beta - \beta_1 > 0$, and $\hat{P}[N]$ is non-decreasing in $N \in [0, \beta_2 - \beta_1]$. Moreover, fixed points 0 and $\beta_2 - \beta_1$ are stable and $\beta - \beta_1$ is unstable, that is, $\hat{P}'[0] \in [0, 1)$, $\hat{P}'[\beta_2 - \beta_1] \in [0, 1)$, and $\hat{P}'[\beta - \beta_1] \in (1, +\infty)$.

By the theory developed in [16,17], $\bar{P} : C_{[\beta-\beta_1,\beta_2-\beta_1]} \mapsto C_{[\beta-\beta_1,\beta_2-\beta_1]}$ admits leftward and rightward spreading speeds $c^*_-(\beta - \beta_1, \beta_2 - \beta_1)$ and $c^*_+(\beta - \beta_1, \beta_2 - \beta_1)$. Since $\bar{P}$ is reflectively invariant, that is, $\bar{P}[\phi(-x)] = \bar{P}[\phi](-x)$, we have $c^*(\beta - \beta_1, \beta_2 - \beta_1) := c^*_-(\beta - \beta_1, \beta_2 - \beta_1) = c^*_+(\beta - \beta_1, \beta_2 - \beta_1)$, which is called the spreading speed of this monostable subsystem. Similarly, this monostable subsystem $\bar{P} : C_{[0,\beta-\beta_1]} \mapsto C_{[0,\beta-\beta_1]}$ also admits a spreading speed $c^*(0, \beta - \beta_1)$.

From [13, Theorem 2.1] and [16, Theorem 3.10], one can obtain that if $\hat{P}'[\beta-\beta_1] > 1, 0 \leq \hat{P}'[\beta_2-\beta_1] < 1$ is satisfied (i.e., $\beta - \beta_1$ is unstable, and $\beta_2 - \beta_1$ is stable), then $\bar{P} : C_{[\beta-\beta_1,\beta_2-\beta_1]} \to C_{[\beta-\beta_1,\beta_2-\beta_1]}$ has a spreading speed $c^\star(\beta - \beta_1, \beta_2 - \beta_1) = \hat{c} \geq \inf_{\mu>0} \ln[\hat{P}'(\beta - \beta_1)K(\mu)]/\mu > 0$, where $K(\mu)$ is the moment generating function of Gaussian Kernel $k_1(x) = \frac{1}{\sqrt{4\pi d}} \exp\left(-\frac{x^2}{4d}\right)$. Similarly, we can obtain $c^*_{[0,\beta-\beta_1]} > 0$, and hence, $\bar{P} : C_{\beta_2-\beta_1} \to C_{\beta_2-\beta_1}$ satisfies the following condition:

$$c^*_-(\beta - \beta_1, \beta_2 - \beta_1) + c^*_+(0, \beta - \beta_1) > 0,$$

where $c^*_-(\alpha, \beta)$ and $c^*_+(0, \alpha)$ represent the leftward and rightward spreading speeds of monostable subsystem $\bar{P}$ restricted on $C_{[\beta-\beta_1,\beta_2-\beta_1]}$ and $C_{[0,\beta-\beta_1]}$, respectively. By the theory developed in [7], there exists c_0 such that $\bar{P} : C_{[0,\beta_2-\beta_1]}$ admits a bistable traveling wave $\bar{W}(x - c_0m)$ connecting 0 to $\beta_2 - \beta_1$.

Let $\bar{W}$ be a non-decreasing traveling wave. According to Proposition 1 and Lemma 5 in [19], we obtain that $\bar{W}(\xi) \in C^1(\mathbb{R}, \mathbb{R})$, $\bar{W}(\xi)$, and $\bar{W}^{'}(\xi)$ are uniformly continuous, and $\hat{P}^{'}(N)$ is uniformly continuous. Slightly modifying the proof in [26, Lemma 4.2], we can show that the function $\bar{W}(\xi)$ is strictly increasing on $\mathbb{R}$ and $\lim_{|\xi|\to\infty} \bar{W}'(\xi) = 0$. All the assumptions in [35, Theorem 2.7] are satisfied, which indicates the uniqueness and stability of the bistable traveling wave solution. Since $\mathcal{P}$ has the same dynamics on $C_{[\beta_1,\beta_2]}$ as $\bar{P}$ on $C_{[0,\beta_2-\beta_1]}$, and two semiflows share the spreading speed, that is, $\mathcal{P} : C_{[\beta_1,\beta_2]} \to C_{[\beta_1,\beta_2]}$ admits a unique (up to translation) bistable traveling wave $W_1(x - c_0m)$ connecting β_1 to β_2.

Let $W_1(x - c_0m)$ be an increasing traveling wave of (4.1) with $W_1(-\infty) = \beta_1$ and $W_2(+\infty) = \beta_2$. By the spatial symmetry of system (4.1), then $W_2(x + c_0m) := W_1(-x - c_0m) = W_1(-(x + c_0m))$ is a decreasing traveling wave of (4.1) with $W_2(-\infty) = \beta_2$ and $W_2(+\infty) = \beta_1$. Applying the operator Q to $W_1(x - c_0m)$, we obtain a traveling wave $W_3(x - c_0m) = Q[W_1(x - c_0m)]$ with speed c_0 and asymptotic behavior $W_3(-\infty) = \beta_2$ and $W_3(+\infty) = \beta_1$ for the operator $\mathcal{P}$. By the uniqueness of the bistable traveling wave, we have $W_3(x - c_0m) = W_2(x + c_0m + s_0)$ for some $s_0 \in \mathbb{R}$. If $c_0 > 0$, then $W_3(-\infty) = \beta_2 = W_2(+\infty) = \beta_1$, and this is a contradiction. Similarly, there is a contradiction when $c_0 < 0$, so $c_0 = 0$, that is, W_1 and W_2 are standing waves. □

An overcompensatory growth function that has a stable two-cycle, given by β_1, β_2, is not necessarily non-increasing on $[\beta_1, \beta_2]$. Assuming that H is non-monotonic on $[\beta_1, \beta_2]$ and H is unimodal, then there exists a unique $l_0 \in (\beta_1, \beta)$ such that $H(l_0) = H(\beta_1) = \beta_2$. In addition, there is a unique $m \in (\beta_1, l_0)$ such that H is increasing on $[0, m)$ and decreasing on $(m, H(m))$, where $H(m) := \max\{H(N)\}$ for all $N \geq 0$. Obviously, we can obtain $H^2(m) < \beta_1$, and there exists a unique $l^* \in (m, \beta)$ such that $H(l^*) = H(H^2(m)) = H^3(m) > \beta$. Since $H^2(m) < \beta_1 < m < l^* < \beta$, one can obtain $\beta = H(\beta) < H(l^*) =$

$H^3(m) < H(\beta_1) = \beta_2 < H(m)$. By assumption (H5), the fixed points of $H \circ H$ are exactly β_1, β and β_2. Furthermore, we have $(H\circ H)^{'}(\beta) > 1, 0 \leq| (H\circ H)^{'}(\beta_i) |< 1, i = 1, 2$. Thus, $(H\circ H)(N) < N$ on (β_1, β), and $(H \circ H)(N) > N$ on (β, β_2). Thus, we have $H^2(l^*) < l^*$, and hence, $H^4(m) < l^*$. Considering the abovementioned factors, we can conclude that $H^2(m) < \beta_1 < H^4(m) < l^* < \beta < H^3(m) < \beta_2 < H(m)$.

Inspired by [2], we construct the following two non-increasing functions:

$$H^+(N) := \begin{cases} H(m), & 0 \leq N \leq m, \\ H(N), & N > m, \end{cases} \quad \text{and} \quad H^-(N) := \begin{cases} H^3(m), & 0 \leq N \leq l^*, \\ H(N), & N > l^*. \end{cases}$$

It is easy to verify that the fixed points of $[H^+]^2(N)$ are $N = H^2(m), N = \beta$, and $N = H(m)$, and the fixed points of $[H^-]^2(N)$ are $N = H^4(m), N = \beta$, and $N = H^3(m)$. Then, Lemma 4.1 still holds when H is replaced by $H^\pm$ and the interval $[\beta_1, \beta_2]$ is replaced by $[H^2(m), H(m)]$ and$[H^4(m), H^3(m)]$, respectively.

Since the operator F is strictly increasing for $N \geq 0$, we can define its inverse operator F^{-1} and set $\tilde{g}^\pm = F^{-1}[H^\pm]$. We then define the second-iterate operators $\mathcal{P}^\pm$ as in (4.1) with g replaced by $\tilde{g}^\pm$, that is,

$$\mathcal{P}^+ = [Q_1 \circ \tilde{g}^+] \circ [Q_1 \circ \tilde{g}^+], \;\; \mathcal{P}^- = [Q_1 \circ \tilde{g}^-] \circ [Q_1 \circ \tilde{g}^-].$$

By construction of $\tilde{g}^+$ and $\tilde{g}^-$, we have the inequalities $\mathcal{P}^-[N](x) \leq \mathcal{P}[N](x) \leq \mathcal{P}^+[N](x)$ for $\beta \leq N(x) \leq \beta_2$. Since $| (H\circ H)'(\beta_2) |\in [0, 1)$, similar to the discussion in Theorems 3.2 and 3.4, we give the following theorem and omit the proof.

Theorem 4.4. *Assume that the spreading speeds of $\mathcal{P}^\pm$ as above are linearly determined. There exists a spreading speed $c^*_{[\beta,\beta_2]}$ for the operator $\mathcal{P}$ in the following sense:*

1. *For any $N_0 \in C_{[\beta,H(m)]}$ such that $N_0 - \beta$ has compact support, the solution of (4.1) satisfies*

$$\lim_{m\to\infty} \sup_{|x|\geq cm} N_m(x) = \beta \text{ for all } c > c^*_{[\beta,\beta_2]}.$$

2. *For any $N_0 \in C_{[\beta,H(m)]}\backslash\{\beta\} := \left\{N \in C_{[\beta,H(m)]} : N - \beta \neq 0\right\}$, the solution of (4.1) satisfies*

$$\lim_{m\to\infty} \sup_{|x|\leq cm} (\beta_2 - N_m)(x) = 0 \text{ for all } c \in \left(0, c^*_{[\beta,\beta_2]}\right).$$

Furthermore, for every $c \geq c^*_{[\beta,\beta_2]}$, there exists a traveling wave $W(x + cm)$ with $W(-\infty) = \beta$ and $W(+\infty) = \beta_2$.

Proof. By construction of $\tilde{g}^+$ and $\tilde{g}^-$, we have the inequalities $\mathcal{P}^-[N](x) \leq \mathcal{P}[N](x) \leq \mathcal{P}^+[N](x)$ for $\beta \leq N(x) \leq \beta_2$.

By construction, we have $H^-(N) = H(N) = H^+(N)$ near $N = \beta$, so that the derivatives of these three functions at $N = \beta$ agree. By the assumption that the spreading speeds $c^*_\pm$ of $\mathcal{P}^\pm$ are linearly determined, we have $c^*_+ = c^*_- = c^*_{(\beta,\beta_2)}$.

Let $N_0 \in C_{[\beta,H(m)]}$ such that $N_0 - \beta$ has compact support. If $N_m = \mathcal{P}^m(N_0)$ and $N^+_m = (\mathcal{P}^+)^m(N_0)$, by the comparison principle, we have $\beta \leq N_m(x) \leq N^+_m(x)$. For any $c > c^*$, Theorem 4.2 implies that $\lim_{m\to\infty,|x|\geq cm} N^+_m(x) = 0$, and hence, $\lim_{m\to\infty,|x|\geq cm} N_m(x) = 0$.

Let $N_0 \in C_{[\beta,H(m)]}, N \not\equiv \beta$ and $M_0 = \min\left\{N_0, H^3(m)\right\}$. Then, $M_0 \leq N_0$ and $M_0 \in C_{[\beta,H^3(m)]}, M_0 \not\equiv 1$. Since $\tilde{g}^-$ is non-increasing, $(\mathcal{P}^-)^m(M_0) \leq (\mathcal{P}^-)^m(N_0)$. If $M^-_m = (\mathcal{P}^-)^m(M_0)$, $N_m = \mathcal{P}^m(N_0)$, and $N^+_m = (\mathcal{P}^+)^m(N_0)$, then by the comparison principle, $\beta \leq M^-_m \leq N_m \leq N^+_m$. For any $c \in (0, c^*)$, Theorem 4.2 implies that $\lim_{m\to\infty,|x|\leq cm} N^+_m(x) = H(m)$ and $\lim_{m\to\infty,|x|\leq cm} N^-_m(x) = H^3(m)$. Thus, we have

$$H^3(m) \le \liminf_{m\to\infty,|x|\le cm} N_m(x) \le \limsup_{m\to\infty,|x|\le cm} N_m(x) \le H(m). \tag{4.2}$$

Since $|(H\circ H)'(\beta_2)| \in [0,1)$, similar to the discussion in Theorem 3.2 and Theorem 3.4, the conclusion in the theorem can be obtained. □

5. Numerical examples

In this section, we illustrate our theoretical results from the previous section with numerical simulations. Consider $f(u) = -\alpha u - \gamma u^2$, and $g(N) = Ne^{r-N}$ is the unimodal function. According to the conclusion of Shang et al. in [23], we know that an important parameter affecting the dynamic behavior of the system is r, and e^r is the growth rate. An elementary analysis shows that $H(N) = F\circ g(N) = \frac{N}{cN+e^{N-r+\alpha}}$, where $c = \frac{(e^\alpha-1)\gamma}{\alpha}$. It is easy to see that $H(N)$ satisfies hypotheses (H1–H3) in Sect. 2, and $H(N) = N$ has a unique positive solution on $[0,+\infty)$, denoted as β. Clearly, $c\beta = 1 - e^{\beta-r+\alpha} < 1$. Then, $H'(\beta) = (c\beta-1)(c\beta-1) < 1$. For differential system $N_{m+1} = H(N_m)$, if $H'(\beta)\in(-1,1)$, then the fixed point β is asymptotically stable. However, if $H'(\beta) < -1$, then the system undergoes period-doubling bifurcation, β loses its stability, and a pair of stable period-two fixed points β_1, β_2 emerge with $H(\beta_1) = \beta_2$ and $H(\beta_2) = \beta_1$. For the composition function $(H\circ H)(N) = \frac{H(N)}{cH(N)+e^{H(N)-r+\alpha}}$. It is easy to get that $0 < \beta_1 < \beta < \beta_2$ are the roots of the equation $(H\circ H)(N) = N$, and $(H\circ H)'(\beta_1) = (H\circ H)'(\beta_2) = H'(\beta_1)H'(\beta_2) = (c\beta_1-1)(\beta_1-1)(c\beta_2-1)(\beta_2-1)$.

We choose parameters $\alpha = 0.1$ and $\gamma = 0.01$, which have been used in the [13]. It then follows that $H(N)$ satisfies (H4) in Sect. 2, $H'(\beta)\in(0,1)$ if $r\in(0.100, 1.111)$, and $H'(\beta)\in(-1,0]$ if $r\in[1.111, 2.143)$. However, $H(N)$ does not satisfy property (H4) if $r > 2.143$. Next, we further discuss the case where $H(N)$ satisfies (H5). By calculation, we can obtain that $(H\circ H)'(\beta_2)\in[0,1)$ if $r\in(2.143, 2.413]$, and $(H\circ H)'(\beta_2)\in(-1,0)$ if $r\in(2.413, 2.701)$. Furthermore, if $r\ge 2.701$, then $(H\circ H)'(\beta_2)\le -1$, and $H(N)$ no longer satisfies hypothesis (H5).

To describe the influence of parameter r on population propagation dynamics, four different parameter values of r are selected: $r_1 = 0.8, r_2 = 1.8, r_3 = 2.4$, $r_4 = 2.6$. By calculating the four cases of r, we obtain that if $r = 0.8$, then $\beta = 0.693$ and $H'(\beta) = 0.305 \in (0,1)$; if $r = 1.8$, then $\beta = 1.682$ and $H'(\beta) = -0.670 \in (-1,0)$; if $r = 2.4$, then $\beta = 2.276$, $\beta_1 = 1.019$, $\beta_2 = 3.532$, and $H'(\beta) = -1.245 < -1$, $(H\circ H)'(\beta_2) = 0.046 \in (0,1)$; and if $r = 2.6$, then $\beta = 2.474$, $\beta_1 = 0.781$, $\beta_2 = 4.166$, and $H'(\beta) = -1.435 < -1$, $(H\circ H)'(\beta_2) = -0.658 \in (-1,0)$. Now, we truncate the infinite domain $\mathbb{R}$ to finite domain $[-L, L]$, where L is sufficiently large. We select parameters $d = 4$, $L = 150$, and the following front-like function $f_0(x)$ as the initial function $N_0(x)$ to satisfy $\liminf_{x\to+\infty} N_0(x) > 0$ and $N_0(x) = 0$ for $x \le 0$,

$$f_0(x) = \begin{cases} 0, & -150 \le x \le 0, \\ 0.12x, & 0 < x < 10, \\ 1.2, & 10 \le x \le 150. \end{cases}$$

According to Fig. 1, we can observe that the original iterative operator Q and the second iterative operator $Q\circ Q$ when r increases generate monotone traveling waves and non-monotone traveling waves. There is an increasing platform in Fig. 1c, d to separate the initial propagation state with the final two cyclic states. From an ecological perspective, the result means that a species that has been introduced in a certain area a long time ago, and population densities may suddenly turn to periodic fluctuations from long-term stable states.

As r further increases, the stability of the two-point cycle will be destroyed, and a four-point cycle will appear according to the period-doubling bifurcation theorem. To verify this, we choose $L = 200, N_0(x) = f_0(x)$, and $r = 2.8$. Figure 2 shows that the system structure presents a higher degree of complexity with the increase of parameter r. Our results demonstrate that the invasion process may occur in two or more

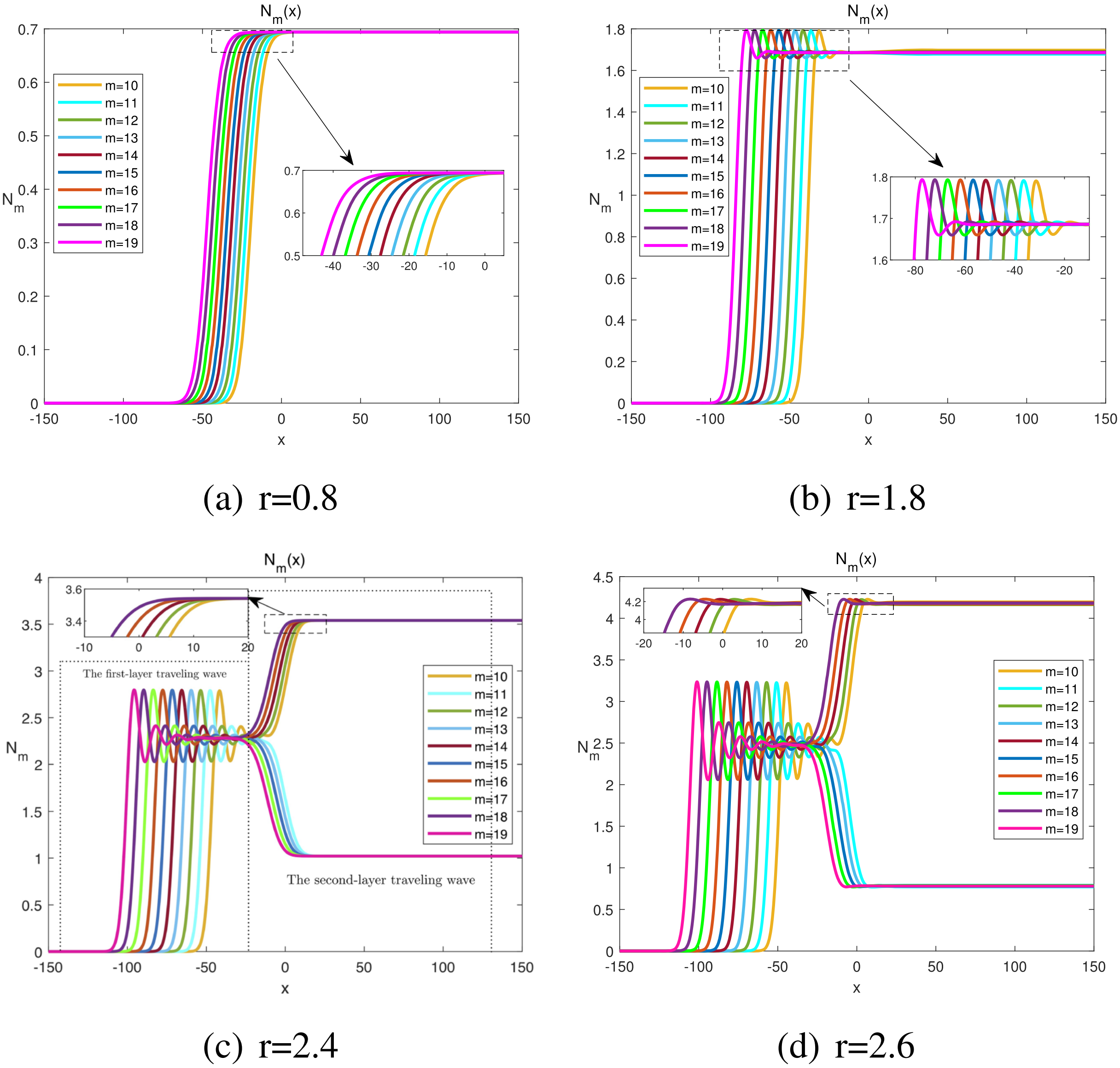


FIG. 1. Panels **a**, **b** show that Q has monotone and non-monotone traveling wave solutions, respectively. Panels **c**, **d** show that Q has non-monotone two-layer traveling wave solutions, but **c** depicts that the second-layer traveling wave for $Q \circ Q$ is monotonic

stages for species with overcompensation. The distribution patterns of species may differ greatly between adjacent reproduction cycles.

According to Theorem 4.3, system (4.1) has a pair of stable standing waves in the translation sense, which means that the system state converges to a standing wave solution $W(x)$ or its translation $W(x+s)$, where s depends on the initial function. Choose $L = 150$, $g(N) = Ne^{2.4-N}$. To simulate this conclusion, we select the following four front-like functions as the initial function $N_0(x)$ to satisfy $\inf_{x\in\mathbb{R}} N_0(x) > 0$ and $\limsup_{x\to-\infty} N_0(x) < \beta < \liminf_{x\to-\infty} N_0(x)$:

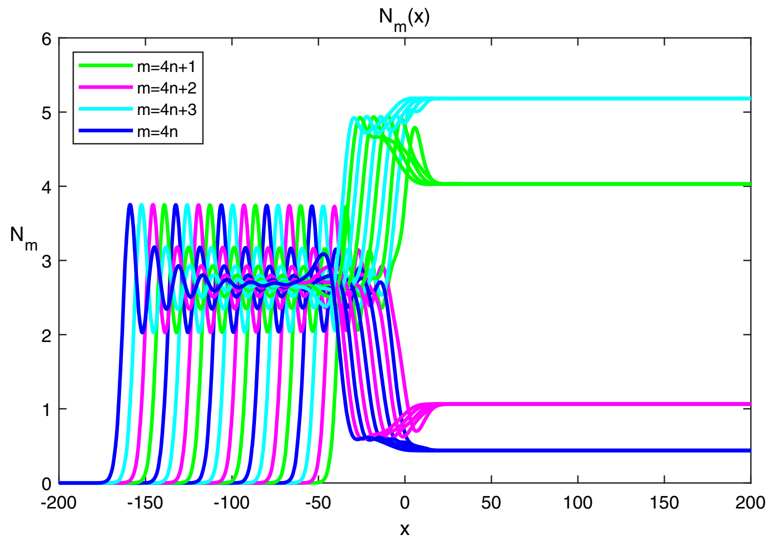


FIG. 2. System (1.5) has a traveling four-point cycle

$$f_1(x) = \begin{cases} 1.9, & -150 \le x \le 1; \\ 0.12x + 1.78, & 1 < x < 11; \\ 3.1, & 11 \le x \le 150. \end{cases} \tag{5.1}$$

$$f_2(x) = \begin{cases} 0.8, & -150 \le x \le 1; \\ 0.24x + 0.56, & 1 < x < 11; \\ 3.2, & 11 \le x \le 150. \end{cases} \tag{5.2}$$

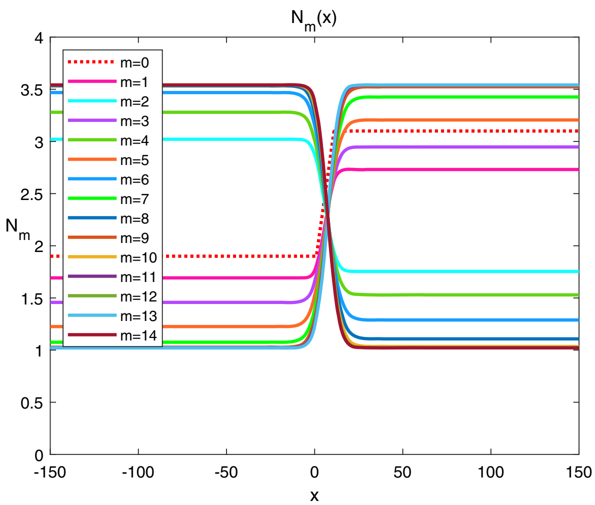
$$f_3(x) = \begin{cases} 0.5, & -150 \le x \le 0; \\ 0.15x + 0.5, & 0 < x < 20; \\ 3.5, & 20 \le x \le 150. \end{cases} \tag{5.3}$$

$$f_4(x) = \begin{cases} 0.6, & -150 \le x \le 0; \\ 0.06x(sin(0.2\pi x) + 2) + 0.6, & 0 < x < 20; \\ 3, & 20 \le x \le 150. \end{cases} \tag{5.4}$$

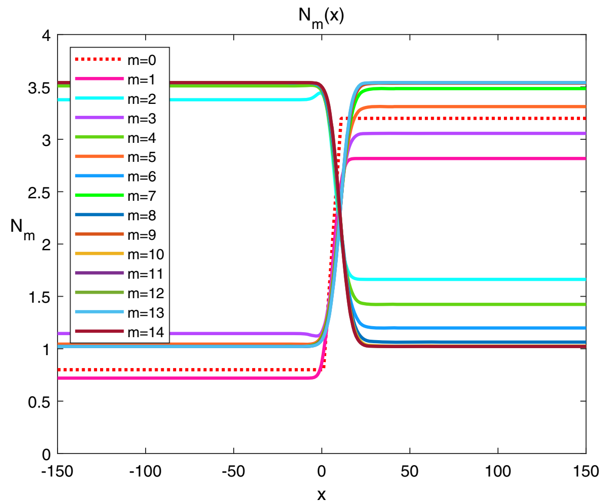
Figure 3 shows that the system starting from different initial states approaches the unique pair of standing waves. In addition, compared with Fig. 1c, solutions converge to different waveforms when $\inf_{x\in\mathbb{R}} N_0(x) > 0$.

6. Concluding remarks

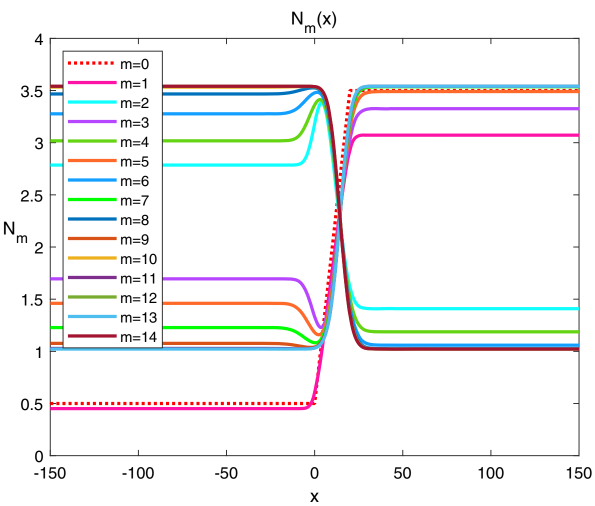
This paper studied the propagation dynamics of a hybrid system consisting of a partial differential equation and a non-monotone discrete-time map. Few efforts have been made on the non-monotone traveling wave solution of an impulsive PDE system. We proved the upward convergence of the oscillating traveling wave when the birth function is a unimodal function. In the study of the propagation dynamics of second iterators, the existence of monotonic and non-monotonic traveling wave solutions of second iterators was obtained. Furthermore, we proved the existence, uniqueness, and stability of the standing wave solution for second iterative operators. Numerical simulations are performed to complement the theoretical results.



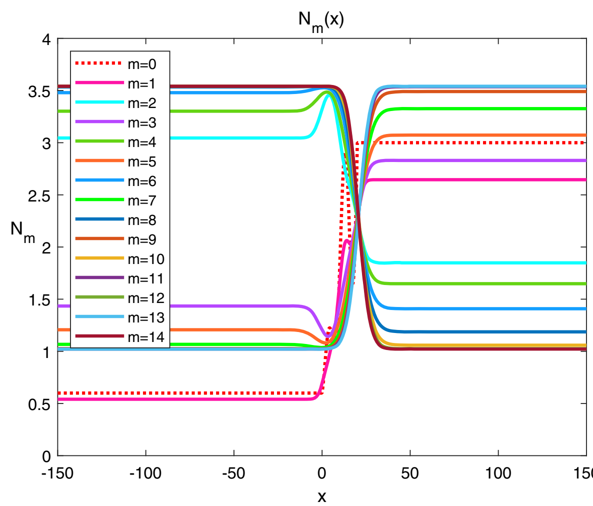
(a) $N_0(x) = f_1(x)$



(b) $N_0(x) = f_2(x)$



(c) $N_0(x) = f_3(x)$



(d) $N_0(x) = f_4(x)$

Fig. 3. The system begins to evolve from different initial values and finally converges to a unique pair of standing wave solutions

A stable periodic state is beneficial to the sustainable development of the population. Bifurcation and chaos can lead to large changes in population density and threaten the stability of ecosystems. The conclusions in this work provide the direction and theoretical basis for the formulation of a control strategy. In the simplest case, we can design a new function $G(N) = g(N) - KN$ instead of $g(N)$, where $-KN$ is a classic linear controller. Impulse control can induce switching between different coexistence states. By means of such impulse control, the bifurcation phenomenon can be suppressed and chaos can be limited, which has important practical significance for improving the stability of ecosystems.

The supplementary explanation is that we cannot give a condition that explicitly includes f and g without knowing the exact expression for f. We only give the properties of the composition operator H and its derivatives, but these conditions are essential, which is in line with other conditions already in [6,34], and we found more critical parameters that can cause the waveform to mutate. For a class of multipulse reaction–diffusion equation model, Liang et al. [15] found that the density distributions of the population are quite different for odd and even generations, the density distributions of the population are quite different for odd and even generations, and the population density distributions of odd and even generations have symmetry and opposite densities. Our results show that this phenomenon arises from overcompensation dynamics. Note that the second-iterate operator may have four steady states, two of which are stable steady states. In this case, the notion of a single front is not sufficient to understand the dynamics of solutions, and we instead observe an appearance similar to so-called propagating terraces. However, unlike the propagating terrace described in [4,5,10], the platform presented in our article is connected by unstable steady states. Even though bistable traveling waves are stable, the expected propagating terrace with a bistable upper layer and monostable lower layer does not occur. This phenomenon of multilayer traveling waves deserves further exploration. How to rigorously obtain the spreading speed for each layer is a challenging problem.

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