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Controlling smoking: A smoking epidemic model with different smoking degrees in deterministic and stochastic environments

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ABSTRACT

Engaging in smoking not only leads to substantial health risks but also imposes considerable financial burdens. To deepen our understanding of the mechanisms behind smoking transmission and to address the tobacco epidemic, we examined a five-dimensional smoking epidemic model that accounts for different degrees of smoking under both deterministic and stochastic conditions. In the deterministic case, we determine the basic reproduction number, analyze the stability of equilibria with and without smoking, and investigate the existence of saddle–node bifurcation. Our analysis reveals that the basic reproduction number cannot completely determine the existence of smoking, and the model possesses bistability, indicating its dynamic is susceptible to interference from environmental noises. In the stochastic case, we establish sufficient conditions for the ergodic stationary distribution and the elimination of smokers by constructing appropriate Lyapunov functions. Numerical simulations suggest that the effects of inevitable random fluctuations in the natural environment on controlling the smoking epidemic may be beneficial, harmful, or negligible, which are closely related to the noise intensities, initial smoking population sizes, and the effective exposure rate of smoking transmission (β). Given the uncontrollable nature of environmental random effects, effective smoking control strategies can be achieved by: (1) accurate monitoring of initial smoking population sizes, and (2) implementing effective measures to reduce β. Therefore, it is both effective and feasible to implement a complete set of strong MPOWER measures to control smoking prevalence.

1. Introduction

Medical studies indicated that tobacco smoking inflicts severe harm to health. Tobacco contains nicotine, which is highly addictive and poses a substantial risk factor for a wide range of health issues [1–4], including cardiovascular and respiratory diseases, more than 20 types of cancer, and various other major complications [5]. According to the World Health Organization (WHO), tobacco is responsible for the death of more than half of its consumers, with over 8 million deaths annually [6]. Out of these, more than 7 million are direct smokers and around 1.3 million are non-smokers exposed to second-hand smoke. The tobacco epidemic, with around 1.3 billion users worldwide, over 80% of whom live in low- and middle-economic countries, stands as one of the most significant global public health threats [7]. Moreover, the use of tobacco exacerbates poverty by diverting household expenditures from essential needs, such as food and shelter, to purchasing tobacco [8,9]. Therefore, taking action to quit smoking is a matter of utmost urgency.

To address the tobacco epidemic, the World Health Assembly ratified the WHO Framework Convention on Tobacco Control (WHO FCTC) on 21 May 2003, which took effect on 27 February 2005 [10]. In 2008, WHO introduced the MPOWER initiative, a practical and cost-effective approach, to further expand the implementation of the main provisions of WHO FCTC [11]. Moreover, strengthening WHO FCTC implementation has also been explicitly incorporated into the United Nations' sustainable development goals [12]. Currently, the WHO FCTC has 182 contracting parties, covering over 90% of the global population [6]. Despite some progress in global tobacco control, smoking remains a leading risk factor for premature death and disability [13–15]. Therefore, gaining a deep understanding of the transmission mechanism of smoking is crucial in determining the optimal control strategies.

To comprehend the propagation of smoking behavior within a population, a common approach is to establish appropriate mathematical models. These models are often formulated using the concepts of epidemiology, treating smoking behavior as an infectious disease

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Table 1
Interpretation of the parameters of model (1.1).

Parameter	Interpretation	Unit
Λ	Recruitment rate of potential smokers $P(t)$	Day^{-1}
β	Effective contact rate between $P(t)$ and $S(t)$	Day^{-1}
μ	Natural mortality rate	Day^{-1}
α	Transformation rate from $O(t)$ to $S(t)$	Day^{-1}
λ	Relapse rate from $Q(t)$ to $S(t)$	Day^{-1}
γ	Smoking quitting rate	Day^{-1}
δ	Proportion of smokers who quit smoking permanently	Dimensionless

that can be transmitted through social interactions [16–27]. For example, Sharomi et al. [18] proposed a dynamic model for reducing smoking, including four sub-populations such as potential smokers (non-smokers), smokers, temporary and permanent quitters. Through a theoretical analysis, the authors established a threshold to determine the success of smoking cessation. Additionally, by considering the influence of smoking level or frequency on smoking-related diseases, the authors investigated an extended model and demonstrated that different levels of smoking have significant effects on controlling the number of smokers. Inspired by the impact of smoking levels on smoking cessation, Ullah et al. [21] included a fifth sub-population, occasional smokers, and proposed a five-dimensional deterministic smoking model

$$\begin{cases} \dfrac{\mathrm{d}P(t)}{\mathrm{d}t} = \Lambda - \beta P(t)S(t) - \mu P(t), \\ \dfrac{\mathrm{d}O(t)}{\mathrm{d}t} = \beta P(t)S(t) - \alpha O(t) - \mu O(t), \\ \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \alpha O(t) + \lambda Q(t)S(t) - \gamma S(t) - \mu S(t), \\ \dfrac{\mathrm{d}Q(t)}{\mathrm{d}t} = \gamma(1-\delta)S(t) - \lambda Q(t)S(t) - \mu Q(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma\delta S(t) - \mu R(t), \end{cases} \tag{1.1}$$

where $P(t)$, $O(t)$, $S(t)$, $Q(t)$ and $R(t)$ represent the population sizes of potential smokers (non-smokers), occasional smokers, smokers, temporary quitters, and permanent quitters at time t, with initial values $(P(0), O(0), S(0), Q(0), R(0))^T \in \mathbb{R}^5_+$. The biological interpretations of the model parameters are given in Table 1. Although the authors [21] performed a stability analysis of the smoking-free and positive equilibria of the model (1.1), they did not provide comprehensive descriptions of all dynamic behaviors of the model. Consequently, further exploration of the model's dynamics is needed.

In addition, biological populations are inherently influenced by stochastic effects in the real world [28]. As noted by May [29], many biological parameters in biomathematical models experience varying degrees of impact from stochastic fluctuations. Thus, the stochastic differential equation models can more accurately predict the evolution trend of populations, which has attracted widespread attention among scholars [24,30–40]. For instance, Sharma [24] considered a smoking epidemic model with demographic stochasticity under the external intervention of raising tobacco taxes, revealing that demographic stochasticity can be beneficial in controlling smoking prevalence. Madhusudanan et al. [27] introduced a smoking model with time delays and Gaussian white noise by considering the influence of psychological and social addictions. Their results indicate that tobacco is a sensitive social addiction. Moreover, the parameters of addiction models depend on the properties of each environmental mechanism and affect the different stages and categories of addiction. Therefore, it is more practical to consider the dynamic effects of random fluctuations in smoking models. To address this, we followed the classical approach of incorporating random effects [41–44] and proposed the following stochastic version based on model (1.1):

$$\begin{cases} \mathrm{d}P(t) = [\Lambda - \beta P(t)S(t) - \mu P(t)]\mathrm{d}t + \sigma_1 P(t)\mathrm{d}B_1(t), \\ \mathrm{d}O(t) = [\beta P(t)S(t) - \alpha O(t) - \mu O(t)]\mathrm{d}t + \sigma_2 O(t)\mathrm{d}B_2(t), \\ \mathrm{d}S(t) = [\alpha O(t) + \lambda Q(t)S(t) - \gamma S(t) - \mu S(t)]\mathrm{d}t + \sigma_3 S(t)\mathrm{d}B_3(t), \\ \mathrm{d}Q(t) = [\gamma(1-\delta)S(t) - \lambda Q(t)S(t) - \mu Q(t)]\mathrm{d}t + \sigma_4 Q(t)\mathrm{d}B_4(t), \\ \mathrm{d}R(t) = [\gamma\delta S(t) - \mu R(t)]\mathrm{d}t + \sigma_5 R(t)\mathrm{d}B_5(t), \end{cases} \tag{1.2}$$

with initial value $(P(0), O(0), S(0), Q(0), R(0))^T \in \mathbb{R}^5_+$. Here $\sigma_i^2 > 0$, $i = 1,2,3,4,5$ are the environmental noise intensities, $B_i(t)$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where Ω is a sample space, $\mathbb{P}$ is a probability measure, $\mathcal{F}$ is a σ-algebra on Ω, and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration possessing usual conditions, meaning it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

The aim of this paper is to propose effective, reasonable, and feasible measures for achieving successful smoking cessation within human society by investigating the dynamic behavior of the smoking population through theoretical and numerical analyses in both deterministic and stochastic environments. The rest of the paper is structured as follows. In Section 2, we investigate the dynamics of deterministic model (1.1), which includes the positivity and boundedness of the solution, the existence of local and global asymptotic stability of the smoking-free and smoking-present equilibria, the existence of saddle–node bifurcation, and the phenomenon of bistability. In Section 3, we explore the dynamics of the stochastic model (1.2), which includes analyzing the model's well-posedness, the existence and uniqueness of ergodic stationary distribution, conditions for smoker elimination, and a series of numerical simulations to investigate the influence of different degrees of environmental random effects on controlling the prevalence of smoking. Lastly, we present a brief discussion in Section 4.

2. The dynamics of deterministic model (1.1)

The dynamic behavior of deterministic model (1.1) is explored in this section which includes the positivity and boundedness of the solution, the stability analysis of smoking-free and smoking-present equilibria, and the existence of saddle–node bifurcation. Moreover, the existence of bistability is demonstrated between the smoking-free and the smoking-present equilibria through a numerical simulation. For a deeper understanding of the relevant theoretical concepts, please refer to [45–49] and the associated references therein.

2.1. The positivity and boundedness of solution

Theorem 2.1. *For all $t \geq 0$, any solution $(P(t), O(t), S(t), Q(t), R(t))^T$ of the model (1.1) with positive initial value is always positive and uniformly bounded.*

Proof. See Appendix A. □

2.2. The existence of equilibria

Theorem 2.2. *The model (1.1) has a smoking-free equilibrium* $E_0(\frac{\Lambda}{\mu}, 0, 0, 0, 0)$*. Furthermore, if* $\Re_0 = \frac{\alpha\beta\Lambda}{\mu(\alpha+\mu)(\gamma+\mu)}$*, then there exists*

(1) *a unique smoking-present equilibrium provided* $\Re_0 > 1$.

(2) *either two smoking-present or no positive equilibria provided* $\Re_0 < 1$.

Proof. See Appendix B. □

2.3. The stability of smoking-free equilibrium

Using the method of next generation matrix [50], we obtain the basic reproduction number of model (1.1), which is $\Re_0 = \frac{\alpha\beta\Lambda}{\mu(\alpha+\mu)(\gamma+\mu)}$ (see Appendix C). Thereby we give the conditions for the local and global stability of smoking-free equilibrium E_0 in the following theorems.

Theorem 2.3. *If* $\Re_0 < 1$*, then the smoking-free equilibrium* E_0 *of model (1.1) is locally asymptotically stable. However,* E_0 *is unstable if* $\Re_0 > 1$.

Proof. See Appendix D. □

Theorem 2.4. *Denoting* $\widetilde{\Re}_0 = \frac{\alpha\beta\Lambda}{\mu^2(\alpha+\mu)}$*, and if* $\widetilde{\Re}_0 < 1$*, then the smoking-free equilibrium* E_0 *of model (1.1) is globally asymptotically stable.*

Proof. See Appendix E. □

Remark 2.5. Since $\Re_0 = \frac{\alpha\beta\Lambda}{\mu(\alpha+\mu)(\gamma+\mu)} \leq \frac{\alpha\beta\Lambda}{\mu^2(\alpha+\mu)} = \widetilde{\Re}_0$, then from Theorems 2.2, 2.3 and 2.4 it follows that

1. E_0 is globally asymptotically stable for $\Re_0 \leq \widetilde{\Re}_0 < 1$, indicating there will be no smokers.
2. E_0 is unstable for $1 < \Re_0 \leq \widetilde{\Re}_0$, indicating smoking will extend.
3. E_0 is only locally and not globally asymptotically stable for $\Re_0 < 1 < \widetilde{\Re}_0$; furthermore,
   (i) the model has a multistability if there are two smoking-present equilibria.
   (ii) E_0 is the ultimate state of the model if there is no positive equilibria.

2.4. The local stability of smoking-present equilibrium

Theorem 2.6. *Assume that model (1.1) has a smoking-present equilibrium* $E^*(P^*, O^*, S^*, Q^*, R^*)$*, then* E^* *is locally asymptotically stable provided the following conditions hold:*

(1) $k_i > 0,\ i = 1, 2, 3, 4,$

(2) $k_1 k_2 - k_3 > 0,$

(3) $k_1(k_2 k_3 - k_1 k_4) - k_3^2 > 0,$

where $k_i,\ i = 1, 2, 3, 4$ *are defined in Appendix F.*

Proof. See Appendix F. □

2.5. The saddle–node bifurcation

Theorem 2.7. *If* β *is the bifurcation parameter with the critical value*

$$\widetilde{\beta} = \frac{\mu(\alpha+\mu)[\mu(\gamma+\mu-\lambda Q^*)+\lambda S^*(\mu+\gamma\delta)]}{\alpha\mu P^*(\lambda S^*+\mu) - S^*(\alpha+\mu)[\mu(\gamma+\mu-\lambda Q^*)+\lambda S^*(\mu+\gamma\delta)]} > 0, \tag{2.1}$$

where P^*, S^*, Q^* *are the elements in* E^**, then model (1.1) has*

(1) no transcritical and pitchfork bifurcation,

(2) a saddle–node bifurcation provided

$$\frac{\lambda(\alpha+\mu)[\gamma(1-\delta)-\lambda Q^*]}{\alpha(\lambda S^*+\mu)^2} - \frac{\widetilde{\beta}^2 P^*}{(\widetilde{\beta}S^*+\mu)^2} \neq 0. \tag{2.2}$$

Proof. See Appendix G. □

2.6. Numerical simulations of model (1.1)

We performed a series of numerical simulations to validate our theoretical results by keeping all the parameters fixed except α and β, where β was systematically varied across the experiments. The fixed initial condition and parameters are

$$\begin{aligned} &P(0)=4,\ O(0)=1,\ S(0)=3,\ Q(0)=1,\ R(0)=1,\\ &\Lambda=1,\ \mu=0.1,\ \lambda=0.8,\ \gamma=0.5,\ \delta=0.1. \end{aligned} \tag{2.3}$$

(1) Let $\alpha = 0.2$. Considering β as a bifurcation parameter that satisfies the critical value $\widetilde{\beta} > 0$ as defined by (2.1). We have the corresponding smoking-present equilibrium $\widetilde{E}^* \approx (7.492, 0.836, 0.791, 0.486, 0.395)$ and the critical value $\widetilde{\beta} \approx 0.042 > 0$, which validates the condition (2.2). According to Theorem 2.7, the model has a saddle–node bifurcation at $\widetilde{E}^*$ (Fig. 1(1)). By keeping all the parameters as (2.3) with $\alpha = 0.2$ and varying β, we observe that (i) for $\beta < \widetilde{\beta}$, a single smoking-free equilibrium point exists and is globally asymptotically stable; (ii) for $\widetilde{\beta} < \beta < 0.09$ (approximately), there is a smoking-free equilibrium point and two positive equilibrium points exist, one of the positive equilibrium points is unstable, while the other and the smoking-free equilibrium point are both locally asymptotically stable, representing bistability; (iii) for $\beta > 0.09$ (approximately), a smoking-free equilibrium point and a positive equilibrium point exist, with the positive equilibrium point is globally asymptotically stable and the smoking-free equilibrium point is unstable (Fig. 1(1)).

(2) Let $\alpha = 0.01$. By computations, it becomes evident that the condition (2.1) in Theorem 2.7 does not hold for the positive equilibrium. Consequently, there is no saddle–node bifurcation in model (1.1) (Fig. 1(2)).

In the subsequent simulations, we fix (2.3) with $\alpha = 0.2$, and further varying β to verify the correctness of the theoretical analysis for the model (1.1).

Example 2.8. Choosing $\beta = 0.014$, we have only one smoking-free equilibrium $E_0 = (10, 0, 0, 0, 0)$, with $\Re_0 \approx 0.156 < 1$ and $\widetilde{\Re}_0 \approx 0.933 < 1$. It follows from Theorem 2.4 that E_0 is globally asymptotically stable (Fig. 2(a)).

Example 2.9. Choosing $\beta = 0.04$, we have only a smoking-free equilibrium $E_0 = (10, 0, 0, 0, 0)$, with $\Re_0 \approx 0.444 < 1$ and $\widetilde{\Re}_0 \approx 2.667 > 1$. Following Remark 2.5, we conclude that E_0 is the ultimate state, which is consistent with Fig. 2(b).

Example 2.10. Choosing $\beta = 0.06$, we have one smoking-free equilibrium $E_0 = (10, 0, 0, 0, 0)$ and two smoking-present equilibria $E_1^* \approx (4.369, 1.877, 2.148, 0.532, 1.074)$ and $E_2^* \approx (9.280, 0.240, 0.129, 0.286, 0.065)$, with $\Re_0 \approx 0.667 < 1$ and $\widetilde{\Re}_0 = 4 > 1$. Thus, by Remark 2.5, the model has bistability, comprising a smoking-free equilibrium and a smoking-present equilibrium. Furthermore,

1. at E_1^*, we have, $k_1 \approx 2.522 > 0$, $k_2 \approx 1.346 > 0$, $k_3 \approx 0.182 > 0$, $k_4 \approx 0.009 > 0$, $k_1 k_2 - k_3 \approx 3.213 > 0$, and $k_1(k_2 k_3 - k_1 k_4) - k_3^2 \approx 0.525 > 0$, resulting E_1^* is locally stable by Theorem 2.6.
2. at E_2^*, we have, $k_4 \approx -0.001 < 0$, resulting E_2^* is unstable by Theorem 2.6.

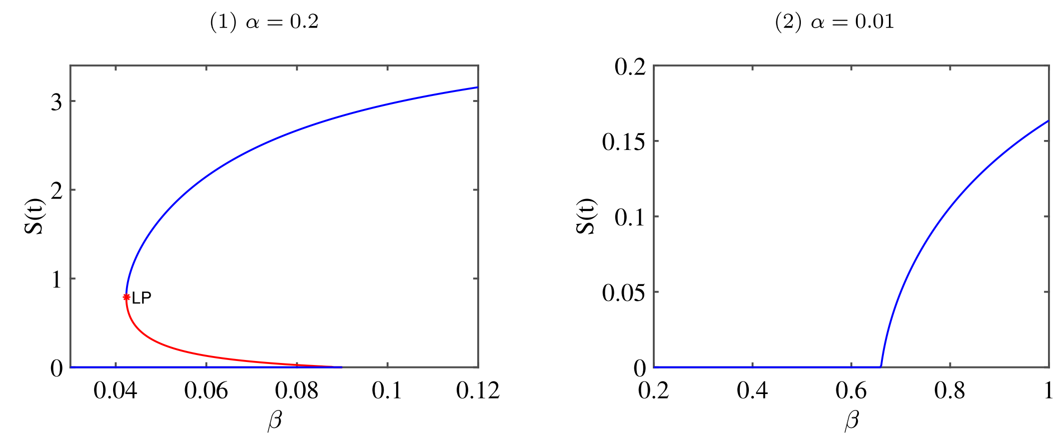

Fig. 1. (1) Bifurcation diagram where LP is the limit point, the red curve is unstable, and the blue curve is stable; (2) Curve of equilibrium points of $S(t)$.

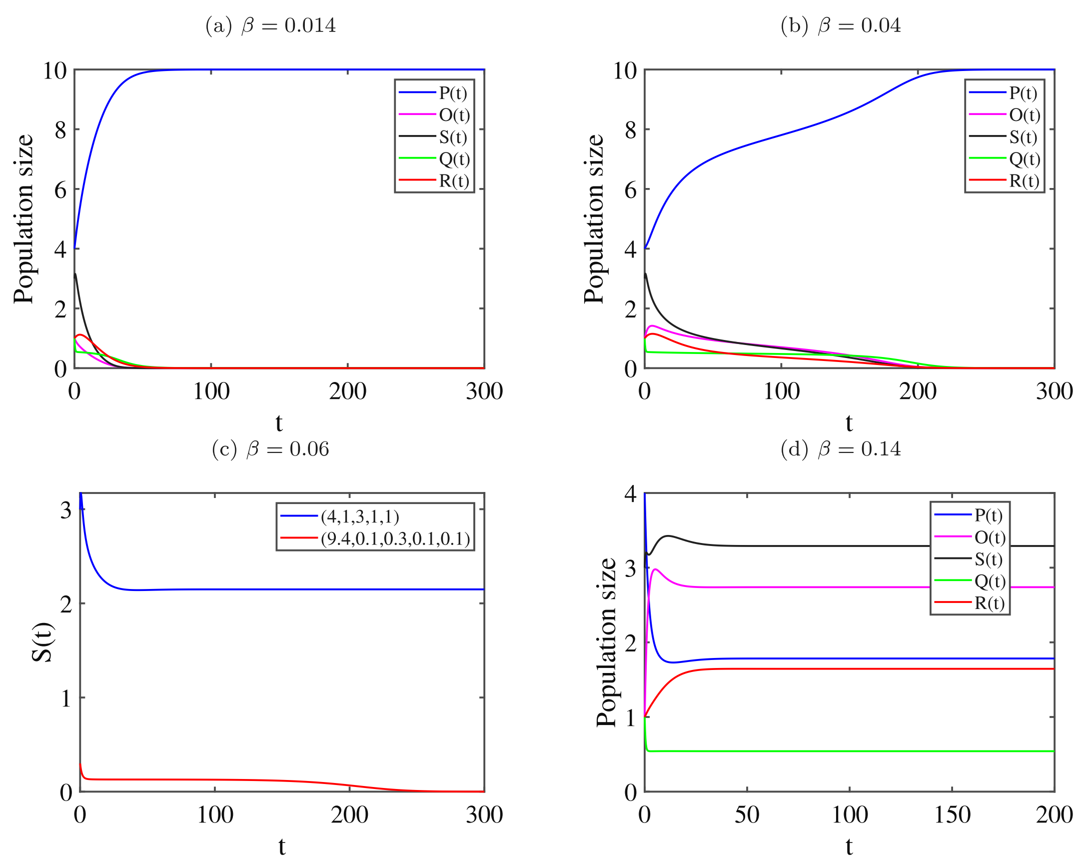

Fig. 2. (a), (b) Time series of the unique smoking-free equilibrium; (c) Bistable phenomenon: blue curve and red curve are time series of E_1^* and E_0, respectively; (d) Time series of the unique smoking-present equilibrium.

Therefore, the bistable phenomenon is composed of E_0 and E_1^*. To validate this phenomenon, we choose two different initial values $(4, 1, 3, 1, 1)$ and $(9.4, 0.1, 0.3, 0.1, 0.1)$, while keeping the parameters unchanged, and simulate the population of smokers $S(t)$ over time. The results of the simulation corroborate the existence of a smoking-present equilibrium E_1^* and a smoking-free equilibrium E_0 (Fig. 2(c)).

Example 2.11. Choosing $\beta = 0.14$, we have $E_0 = (10, 0, 0, 0, 0)$ and $E^* \approx (1.784, 2.739, 3.290, 0.542, 1.645)$, with $\mathfrak{R}_0 \approx 1.556 > 1$ and $\widetilde{\mathfrak{R}}_0 \approx 9.333 > 1$. According to Remark 2.5, E_0 is unstable. Furthermore, we determine that $k_1 \approx 3.759 > 0$, $k_2 \approx 3.025 > 0$, $k_3 \approx 0.700 > 0$, $k_4 \approx 0.056 > 0$, $k_1 k_2 - k_3 \approx 10.671 > 0$ and $k_1(k_2 k_3 - k_1 k_4) - k_3^2 \approx 6.687 > 0$. Following Theorem 2.6, E^* is stable. This implies that smoking will persist (Fig. 2(d)).

Remark 2.12. It is worth noting that there is an unresolved statement presented in Theorem 2.2(2), i.e., when $\mathfrak{R}_0 < 1$, it remains unclear under what conditions two smoking-present equilibria exist, and when there are no positive equilibria for model (1.1). However, it follows from Theorem 2.7 and Examples 2.8, 2.9, and 2.10 that under the premise of $\mathfrak{R}_0 < 1$, we need to further consider the critical value of saddle–node bifurcation $\widetilde{\beta}$ with the effective exposure rate of smoking transmission β as the bifurcation parameter. More specifically, when $\mathfrak{R}_0 < 1$,

- if $0 < \beta < \widetilde{\beta}$, then the model (1.1) does not have positive equilibria;
- if $\beta > \widetilde{\beta}$, then there are two smoking-present equilibria.

Remark 2.13. It is noteworthy that when $\mathfrak{R}_0 > 1$, the smoking-free equilibrium E_0 is unstable, and there exists a unique smoking-present equilibrium E^*. Furthermore, based on the numerical simulations, it is evident that the model does not exhibit a limit cycle. Therefore, when $\mathfrak{R}_0 > 1$, even though we did not present a specific proof for the global asymptotic stability of E^*, we still propose the following conjecture.

Conjecture 2.14. *For $\mathfrak{R}_0 > 1$, the unique smoking-present equilibrium E^* of model (1.1) is globally asymptotically stable.*

To sum up, the basic reproduction number $\mathfrak{R}_0$ cannot completely determine the existence of smoking behavior. This confirms a conclusion similar to that in [50]: even when the smoking-free equilibrium is locally asymptotically stable, smoking behavior may still be prevalent. Additionally, the presence of bistability indicates that the long-term dynamics of model (1.1) are influenced by different initial values, and the existence of saddle–node bifurcation implies that the dynamic behaviors of the model are quite sensitive to the effective exposure rate of smoking transmission β. Consequently, the dynamics of model (1.1) are significantly impacted by environmental white noise, and in the following section, we explore the dynamic effects of inevitable random fluctuations in the environment on the smoking epidemic model.

3. The dynamics of stochastic model (1.2)

The existence, uniqueness and boundedness of the stochastic positive solution of model (1.2) will be discussed first, as these aspects are crucial for the subsequent analysis of the main findings. We will use some correlation theories of stochastic differential equations (see [51–54] and references therein) to obtain these results.

3.1. Existence and uniqueness of global positive solution of model (1.2)

Theorem 3.1. *For any given initial value $(P(0), O(0), S(0), Q(0), R(0))^T \in \mathbb{R}^5_+$, the model (1.2) has a unique global solution $(P(t), O(t), S(t), Q(t), R(t))^T \in \mathbb{R}^5_+$ for all $t \geq 0$ almost surely (a.s.).*

Proof. See Appendix H. □

3.2. Ultimate boundedness of stochastic positive solution for model (1.2)

Theorem 3.2. *The solution $(P(t), O(t), S(t), Q(t), R(t))^T$ of the model (1.2) established by Theorem 3.1 satisfies*

$$\limsup_{t\to\infty}[P(t) + O(t) + S(t) + Q(t) + R(t)] < \infty \ a.s. \tag{3.1}$$

Proof. See Appendix I. □

In the next sections, we theoretically establish the specific conditions for the persistence or eradication of smoking populations, which will help to further explore the efficient and feasible control strategies for giving up smoking.

3.3. Persistence of smokers

To establish the criterion for the persistence of smokers in the following theorem, we denote,

$$\varpi = \frac{\alpha\beta\Lambda}{(\mu + \frac{\sigma_1^2}{2})(\alpha + \mu + \frac{\sigma_2^2}{2})(\gamma + \mu + \frac{\sigma_3^2}{2})}.$$

Theorem 3.3. *If $\varpi > 1$, then model (1.2) has a unique ergodic stationary distribution.*

Proof. Following the definition of ergodic stationary distribution [55, Theorems 4.1 and 4.2], we divided the proof into two parts. In the first part we constructed a $\mathbf{C}^2$-function $V(P, O, S, Q, R) : \mathbb{R}^5_+ \to \mathbb{R}_+$ and a bounded open set U_ϵ such that $\mathcal{L}V < 0$ for all $(P, O, S, Q, R)^T \in \mathbb{R}^5_+ \backslash U_\epsilon$. In the second part, we verified the uniform elliptic criterion.

Part I. Applying Itô's formula to model (1.2) yields

$$\begin{aligned}
\mathcal{L}(-\ln P) &= -\frac{1}{P}(\Lambda - \beta PS - \mu P) + \frac{\sigma_1^2}{2} \\
&= -\frac{\Lambda}{P} + \beta S + \mu + \frac{\sigma_1^2}{2} \\
&= -\frac{\Lambda}{\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)} \cdot \frac{\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)}{P} + \mu + \frac{\sigma_1^2}{2} + \beta S \\
&\leq -\left(\mu + \frac{\sigma_1^2}{2}\right)\left[\ln \frac{\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)}{P} + 1\right] + \mu + \frac{\sigma_1^2}{2} + \beta S \\
&= -\left(\mu + \frac{\sigma_1^2}{2}\right)\ln \frac{\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)}{P} + \beta S,
\end{aligned} \tag{3.2}$$

where the above inequality holds, because $x \geq \ln x + 1$ for $x > 0$. Similarly, we have

$$\begin{aligned}
\mathcal{L}(-\ln O) &= -\frac{1}{O}(\beta PS - \alpha O - \mu O) + \frac{\sigma_2^2}{2} \\
&= -\beta\frac{PS}{O} + \alpha + \mu + \frac{\sigma_2^2}{2} \\
&= -\left(\alpha + \mu + \frac{\sigma_2^2}{2}\right) \cdot \frac{P}{\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)} \cdot \frac{\left(\frac{\beta}{\alpha+\mu+\frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)}{O} \cdot S + \alpha + \mu + \frac{\sigma_2^2}{2} \\
&\leq -\left(\alpha + \mu + \frac{\sigma_2^2}{2}\right)\left\{\ln\left[\frac{P}{\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)} \cdot \frac{\left(\frac{\beta}{\alpha+\mu+\frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)}{O} \cdot S\right] + 1\right\} + \alpha \\
&\quad + \mu + \frac{\sigma_2^2}{2} \\
&= -\left(\alpha + \mu + \frac{\sigma_2^2}{2}\right)\ln \frac{P}{\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)} - \left(\alpha + \mu + \frac{\sigma_2^2}{2}\right)\ln \frac{\left(\frac{\beta}{\alpha+\mu+\frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)}{O} \\
&\quad - \left(\alpha + \mu + \frac{\sigma_2^2}{2}\right)\ln S,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\mathcal{L}(-\ln S) &= -\frac{1}{S}(\alpha O + \lambda QS - \gamma S - \mu S) + \frac{\sigma_3^2}{2} \\
&= -\alpha\frac{O}{S} - \lambda Q + \gamma + \mu + \frac{\sigma_3^2}{2} \\
&\leq -\alpha\left(\frac{\beta}{\alpha + \mu + \frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu + \frac{\sigma_1^2}{2}}\right) \cdot \frac{O}{\left(\frac{\beta}{\alpha+\mu+\frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)} \cdot \frac{1}{S} + \gamma + \mu + \frac{\sigma_3^2}{2} \\
&\leq -\alpha\left(\frac{\beta}{\alpha + \mu + \frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu + \frac{\sigma_1^2}{2}}\right)\left[\ln \frac{O}{\left(\frac{\beta}{\alpha+\mu+\frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)} + \ln\frac{1}{S} + 1\right] \\
&\quad + \gamma + \mu + \frac{\sigma_3^2}{2}.
\end{aligned} \tag{3.4}$$

In addition,

$$\mathcal{L}(-\ln Q) = -\frac{1}{Q}[\gamma(1-\delta)S - \lambda QS - \mu Q] + \frac{\sigma_4^2}{2} = -\frac{\gamma(1-\delta)S}{Q} + \lambda S + \mu + \frac{\sigma_4^2}{2}, \tag{3.5}$$

$$\mathcal{L}(-\ln R) = -\frac{1}{R}(\gamma\delta S - \mu R) + \frac{\sigma_5^2}{2} = -\frac{\gamma\delta S}{R} + \mu + \frac{\sigma_5^2}{2}. \tag{3.6}$$

Thus, we take

$$V_1(P,O,S) = \frac{\alpha\beta\Lambda}{(\mu+\frac{\sigma_1^2}{2})(\alpha+\mu+\frac{\sigma_2^2}{2})}\left[-\frac{1}{\mu+\frac{\sigma_1^2}{2}}\ln P - \frac{1}{\alpha+\mu+\frac{\sigma_2^2}{2}}\ln O\right] - \ln S,$$

it then follows from (3.2), (3.3) and (3.4) that

$$\begin{aligned}\mathcal{L}V_1 \le & \frac{\alpha\beta\Lambda}{(\mu+\frac{\sigma_1^2}{2})(\alpha+\mu+\frac{\sigma_2^2}{2})}\left[-\ln\frac{\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)}{P} + \frac{\beta S}{\mu+\frac{\sigma_1^2}{2}} - \ln\frac{P}{\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)} - \ln\frac{\left(\frac{\beta}{\alpha+\mu+\frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)}{O} - \ln S\right] \\ & - \alpha\left(\frac{\beta}{\alpha+\mu+\frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)\left[\ln\frac{O}{\left(\frac{\beta}{\alpha+\mu+\frac{\sigma_2^2}{2}}\right)\left(\frac{\Lambda}{\mu+\frac{\sigma_1^2}{2}}\right)} + \ln\frac{1}{S} + 1\right] \\ & + \gamma + \mu + \frac{\sigma_3^2}{2} \\ = & -\frac{\alpha\beta\Lambda}{(\mu+\frac{\sigma_1^2}{2})(\alpha+\mu+\frac{\sigma_2^2}{2})} + \gamma + \mu + \frac{\sigma_3^2}{2} + \frac{\beta S}{\mu+\frac{\sigma_1^2}{2}} \\ = & -(\gamma+\mu+\frac{\sigma_3^2}{2})(\varpi - 1) + \frac{\beta S}{\mu+\frac{\sigma_1^2}{2}}.\end{aligned} \tag{3.7}$$

Moreover, we consider

$$V_2(P,O,S,Q,R) = \frac{1}{\eta+1}(P+O+S+Q+R)^{\eta+1},$$

where $\eta \in \left(0, \frac{2\mu}{\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2,\sigma_5^2\}}\right)$ is a sufficiently small constant. Using Itô's formula to $V_2(P,O,S,Q,R)$, we have

$$\begin{aligned}\mathcal{L}V_2 = & (P+O+S+Q+R)^\eta \cdot [\Lambda - \mu(P+O+S+Q+R)] \\ & + \frac{\eta}{2}(P+O+S+Q+R)^{\eta-1}\cdot(\sigma_1^2P^2+\sigma_2^2O^2+\sigma_3^2S^2+\sigma_4^2Q^2+\sigma_5^2R^2) \\ \le & \Lambda(P+O+S+Q+R)^\eta - \mu(P+O+S+Q+R)^{\eta+1} \\ & + \frac{\eta}{2}\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2,\sigma_5^2\}(P+O+S+Q+R)^{\eta+1} \\ \le & -\frac{\ell}{2}(P+O+S+Q+R)^{\eta+1} + H_1 \\ \le & -\frac{\ell}{2}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) + H_1,\end{aligned} \tag{3.8}$$

where $\ell := \mu - \frac{\eta}{2}\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2,\sigma_5^2\}$ and

$$H_1 = \sup_{(P,O,S,Q,R)^T\in\mathbb{R}_+^5}\left\{-\frac{\ell}{2}(P+O+S+Q+R)^{\eta+1} + \Lambda(P+O+S+Q+R)^\eta\right\} < \infty.$$

Thus we construct a $\mathbf{C}^2$-function $\overline{V}(P,O,S,Q,R) : \mathbb{R}_+^5 \to \mathbb{R}$ by

$$\begin{aligned}&\overline{V}(P,O,S,Q,R) \\ &= MV_1(P,O,S) - \ln P - \ln O - \ln Q - \ln R + V_2(P,O,S,Q,R) \\ &= M\left[\frac{\alpha\beta\Lambda}{(\mu+\frac{\sigma_1^2}{2})(\alpha+\mu+\frac{\sigma_2^2}{2})}\left(-\frac{1}{\mu+\frac{\sigma_1^2}{2}}\ln P - \frac{1}{\alpha+\mu+\frac{\sigma_2^2}{2}}\ln O\right) - \ln S\right] \\ &\quad - \ln P - \ln O - \ln Q - \ln R + \frac{1}{\eta+1}(P+O+S+Q+R)^{\eta+1},\end{aligned}$$

where M is a sufficiently large positive constant such that

$$-M(\gamma+\mu+\frac{\sigma_3^2}{2})(\varpi-1) + H_2 \le -2, \tag{3.9}$$

where

$$H_2 = \sup_{(P,O,S,Q,R)^T\in\mathbb{R}_+^5}\left\{-\frac{\ell}{2}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) + H_1 + (\lambda+\beta)S + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2}\right\} < \infty.$$

It is straightforward that $\overline{V}(P,O,S,Q,R)$ is continuous and approaches to infinity as $(P,O,S,Q,R)^T$ tends to the boundary of $\mathbb{R}_+^5$. Thus $\overline{V}(P,O,S,Q,R)$ has a minimum lower bound, and further we assume that $(\overline{P},\overline{O},\overline{S},\overline{Q},\overline{R})^T$ denotes a point in the interior of $\mathbb{R}_+^5$ at which $\overline{V}(P,O,S,Q,R)$ reaches this lowest bound. Then we define a non-negative $\mathbf{C}^2$-function $V(P,O,S,Q,R) : \mathbb{R}_+^5 \to \mathbb{R}_+$ by

$$V(P,O,S,Q,R) = \overline{V}(P,O,S,Q,R) - \overline{V}(\overline{P},\overline{O},\overline{S},\overline{Q},\overline{R}). \tag{3.10}$$

From (3.2)–(3.8), and applying the Itô's formula to (3.10), we have

$$\begin{aligned}\mathcal{L}V \le & -M(\gamma+\mu+\frac{\sigma_3^2}{2})(\varpi-1) + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\Lambda}{P} \\ & - \beta\frac{PS}{O} - \frac{\gamma(1-\delta)S}{Q} - \frac{\gamma\delta S}{R} + (\beta+\lambda)S \\ & - \frac{\ell}{2}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\ & + H_1 + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2}.\end{aligned} \tag{3.11}$$

We construct the following bounded open set

$$\begin{aligned}U_\varepsilon = \Big\{ & (P,O,S,Q,R)^T \in \mathbb{R}_+^5 : P \in (\varepsilon, \frac{1}{\varepsilon}), O \in (\varepsilon^3, \frac{1}{\varepsilon^3}), \\ & S \in (\varepsilon, \frac{1}{\varepsilon}), Q \in (\varepsilon^2, \frac{1}{\varepsilon^2}), R \in (\varepsilon^2, \frac{1}{\varepsilon^2}) \Big\},\end{aligned} \tag{3.12}$$

where $\varepsilon \in (0,1)$ is a sufficiently small constant. In the complementary set, $U_\varepsilon^C = \mathbb{R}_+^5 \backslash U_\varepsilon$, we take the sufficiently small value of ε such that the following conditions hold

$$-\frac{\Lambda}{\varepsilon} + H_3 \le -1, \tag{3.13}$$

$$-\frac{\beta}{\varepsilon} + H_3 \le -1, \tag{3.14}$$

$$\varepsilon \le \frac{\mu+\sigma_1^2/2}{M\beta}, \tag{3.15}$$

$$-\frac{\gamma(1-\delta)}{\varepsilon} + H_3 \le -1, \tag{3.16}$$

$$-\frac{\ell}{4\varepsilon^{\eta+1}} + H_3 \le -1, \tag{3.17}$$

$$-\frac{\gamma\delta}{\varepsilon} + H_3 \le -1, \tag{3.18}$$

$$-\frac{\ell}{4\varepsilon^{3(\eta+1)}} + H_3 \le -1, \tag{3.19}$$

$$-\frac{\ell}{4\varepsilon^{2(\eta+1)}} + H_3 \le -1, \tag{3.20}$$

where

$$H_3 = \sup_{(P,O,S,Q,R)^T\in\mathbb{R}_+^5}\left\{-\frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) + \frac{M\beta S}{\mu+\sigma_1^2/2} + H_1 + (\lambda+\beta)S + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2}\right\} < \infty.$$

For convenience, we divide U_ε^C into the following ten domains

$$U_{\varepsilon,1}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 : P \le \varepsilon\right\};$$

$$U_{\varepsilon,2}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 \,:\, P \geq \varepsilon, S \geq \varepsilon, O \leq \varepsilon^3\right\};$$
$$U_{\varepsilon,3}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 \,:\, S \leq \varepsilon\right\};$$
$$U_{\varepsilon,4}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 \,:\, S \geq \varepsilon, Q \leq \varepsilon^2\right\};$$
$$U_{\varepsilon,5}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 \,:\, P \geq 1/\varepsilon\right\};$$
$$U_{\varepsilon,6}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 \,:\, S \geq \varepsilon, R \leq \varepsilon^2\right\};$$
$$U_{\varepsilon,7}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 \,:\, O \geq 1/\varepsilon^3\right\};$$
$$U_{\varepsilon,8}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 \,:\, S \geq 1/\varepsilon\right\};$$
$$U_{\varepsilon,9}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 \,:\, Q \geq 1/\varepsilon^2\right\};$$
$$U_{\varepsilon,10}^C = \left\{(P,O,S,Q,R)^T \in \mathbb{R}_+^5 \,:\, R \geq 1/\varepsilon^2\right\}.$$

It is obvious that, $U_\varepsilon^C = \bigcup_{i=1}^{10} U_{\varepsilon,i}^C$. The following cases are used to verify that $\mathcal{L}V \leq -1$ always holds on each partitioned domain $U_{\varepsilon,i}^C$, $i = 1,2,\ldots,10$.

Case 1. When $(P,O,S,Q,R)^T \in U_{\varepsilon,1}^C$, then following (3.11) and (3.13), we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{\Lambda}{P} + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\
& + H_1 + (\beta+\lambda)S \\
& + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\
\leq & -\frac{\Lambda}{\varepsilon} + H_3 \\
\leq & -1.
\end{aligned}$$

Case 2. When $(P,O,S,Q,R)^T \in U_{\varepsilon,2}^C$, then following (3.11) and (3.14), we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{\beta PS}{O} + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\
& + H_1 + (\beta+\lambda)S \\
& + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\
\leq & -\beta\cdot\frac{\varepsilon\cdot\varepsilon}{\varepsilon^3} + H_3 = -\frac{\beta}{\varepsilon} + H_3 \\
\leq & -1.
\end{aligned}$$

Case 3. When $(P,O,S,Q,R)^T \in U_{\varepsilon,3}^C$, then following (3.9), (3.11) and (3.15), we have

$$\begin{aligned}
\mathcal{L}V \leq & -M(\gamma+\mu+\frac{\sigma_3^2}{2})(\varpi-1) + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} \\
& - \frac{\ell}{2}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) + H_1 \\
& + (\beta+\lambda)S + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\
\leq & -M(\gamma+\mu+\frac{\sigma_3^2}{2})(\varpi-1) + \frac{M\beta\varepsilon}{\mu+\frac{\sigma_1^2}{2}} + H_2 \\
\leq & -1.
\end{aligned}$$

Case 4. When $(P,O,S,Q,R)^T \in U_{\varepsilon,4}^C$, then following (3.11) and (3.16), we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{\gamma(1-\delta)S}{Q} + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\
& + H_1 + (\beta+\lambda)S \\
& + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\
\leq & -\frac{\gamma(1-\delta)\varepsilon}{\varepsilon^2} + H_3 = -\frac{\gamma(1-\delta)}{\varepsilon} + H_3 \\
\leq & -1.
\end{aligned}$$

Case 5. When $(P,O,S,Q,R)^T \in U_{\varepsilon,5}^C$, then following (3.11) and (3.17), we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{\ell}{4}P^{\eta+1} + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\
& + H_1 + (\beta+\lambda)S \\
& + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\
\leq & -\frac{\ell}{4\varepsilon^{\eta+1}} + H_3 \\
\leq & -1.
\end{aligned}$$

Case 6. When $(P,O,S,Q,R)^T \in U_{\varepsilon,6}^C$, then following (3.11) and (3.18), we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{\gamma\delta S}{R} + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\
& + H_1 + (\beta+\lambda)S \\
& + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\
\leq & -\frac{\gamma\delta\varepsilon}{\varepsilon^2} + H_3 = -\frac{\gamma\delta}{\varepsilon} + H_3 \\
\leq & -1.
\end{aligned}$$

Case 7. When $(P,O,S,Q,R)^T \in U_{\varepsilon,7}^C$, then following (3.11) and (3.19), we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{\ell}{4}O^{\eta+1} + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\
& + H_1 + (\beta+\lambda)S \\
& + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\
\leq & -\frac{\ell}{4\varepsilon^{3(\eta+1)}} + H_3 \\
\leq & -1.
\end{aligned}$$

Case 8. When $(P,O,S,Q,R)^T \in U_{\varepsilon,8}^C$, then following (3.11) and (3.17) we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{\ell}{4}S^{\eta+1} + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\
& + H_1 + (\beta+\lambda)S \\
& + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\
\leq & -\frac{\ell}{4\varepsilon^{\eta+1}} + H_3 \\
\leq & -1.
\end{aligned}$$

Case 9. When $(P,O,S,Q,R)^T \in U_{\varepsilon,9}^C$, then following (3.11) and (3.20), we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{\ell}{4}Q^{\eta+1} + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\
& + H_1 + (\beta+\lambda)S \\
& + \alpha + 4\mu + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + \frac{\sigma_5^2}{2} \\
\leq & -\frac{\ell}{4\varepsilon^{2(\eta+1)}} + H_3 \\
\leq & -1.
\end{aligned}$$

Case 10. When $(P,O,S,Q,R)^T \in U_{\varepsilon,10}^C$, then following (3.11) and (3.20), we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{\ell}{4}R^{\eta+1} + \frac{M\beta S}{\mu+\frac{\sigma_1^2}{2}} - \frac{\ell}{4}(P^{\eta+1}+O^{\eta+1}+S^{\eta+1}+Q^{\eta+1}+R^{\eta+1}) \\
& + H_1 + (\beta+\lambda)S
\end{aligned}$$

$$
\begin{aligned}
&+\alpha+4\mu+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_4^2}{2}+\frac{\sigma_5^2}{2}\\
\le& -\frac{\ell}{4\varepsilon^{2(\eta+1)}}+H_3\\
\le& -1.
\end{aligned}
$$

Therefore, we conclude $\mathcal{L}V(P,O,S,Q,R)\le -1$, for any $(P,O,S,Q,R)^T\in\mathbb{R}_+^5\backslash U_\varepsilon$, which establish the first part of having an ergodic stationary distribution.

Part II. The diffusion matrix of model (1.2) is

$$
G=\begin{pmatrix}
\sigma_1^2P^2 & 0 & 0 & 0 & 0\\
0 & \sigma_2^2O^2 & 0 & 0 & 0\\
0 & 0 & \sigma_3^2S^2 & 0 & 0\\
0 & 0 & 0 & \sigma_4^2Q^2 & 0\\
0 & 0 & 0 & 0 & \sigma_5^2R^2
\end{pmatrix}.
$$

Let $\varrho=\min_{(P,O,S,Q,R)^T\in U_\varepsilon}\{\sigma_1^2P^2,\sigma_2^2O^2,\sigma_3^2S^2,\sigma_4^2Q^2,\sigma_5^2R^2\}$, then

$$
\sum_{i,j=1}^{5}g_{ij}(P,O,S,Q,R)\xi_i\xi_j=\sigma_1^2P^2\xi_1^2+\sigma_2^2O^2\xi_2^2+\sigma_3^2S^2\xi_3^2+\sigma_4^2Q^2\xi_4^2+\sigma_5^2R^2\xi_5^2\ge\varrho\|\xi\|^2,
$$

for all $(P,O,S,Q,R)^T\in U_\varepsilon$ and $\xi=(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)^T\in\mathbb{R}^5$, where U_ε is defined by (3.12). Thus the second part of ergodic stationary distribution is satisfied.

By combining both of the parts, we conclude that the model (1.2) has a unique ergodic stationary distribution, which completes the proof of Theorem 3.3. □

Remark 3.4. Theorem 3.3 gives the sufficient criterion for the existence of ergodic stationary distribution, which indicates the weak stability and persistence of stochastic model (1.2). In addition, when the effect of environmental random fluctuations on smoking models is not considered, namely $\sigma_i=0$, $i=1,2,\dots,5$, we deduce $\varpi=\frac{\alpha\beta\Lambda}{\mu(\alpha+\mu)(\gamma+\mu)}=\Re_0$, indicating that the positive solution of the deterministic model (1.1) is globally asymptotically stable for $\varpi=\Re_0>1$, which is consistent with Conjecture 2.14.

3.4. Elimination of smokers

To derive the conditions for the elimination of the smoking population, we denote,

$$
\phi=\mu(\Re-1)\mathbb{I}_{\{\Re\le1\}}+(\alpha+\mu)(\Re-1)\mathbb{I}_{\{\Re>1\}}
+\sigma_1\sqrt{\frac{\alpha\beta\Lambda}{(\alpha+\mu)(2\mu-\sigma_1^2)}}-\left[2(\sigma_2^{-2}+\sigma_3^{-2}+\sigma_4^{-2}+\sigma_5^{-2})\right]^{-1},
$$

where $\Re=\sqrt{\widetilde{\Re}_0}=\sqrt{\frac{\alpha\beta\Lambda}{\mu^2(\alpha+\mu)}}$, and $\mathbb{I}_{\{\cdot\}}$ is the indicator function of $\{\cdot\}$. We use this notation to establish the following theorem.

Theorem 3.5. *Let $(P(t),O(t),S(t),Q(t),R(t))^T$ be the solution of model (1.2) with initial value $(P(0),O(0),S(0),Q(0),R(0))^T\in\mathbb{R}_+^5$. If $\mu>\frac{\sigma_1^2}{2}$, then for almost all $\omega\in\Omega$, we have*

$$
\limsup_{t\to\infty}\frac{1}{t}\ln\left[\mu\sqrt{\alpha}O+\sqrt{(\alpha+\mu)\beta\Lambda}(S+Q+R)\right]\le\phi\ \ a.s.
$$

If $\phi<0$, then there are no smoking populations. Moreover, the distribution of $P(t)$ converges weakly a.s. to an invariant measure with density

$$
\pi(x)=\left[\sigma_1^{-2}\left(\frac{\sigma_1^2}{2\Lambda}\right)^{\frac{2\mu}{\sigma_1^2}+1}\Gamma\left(\frac{2\mu}{\sigma_1^2}+1\right)\right]^{-1}x^{-2-\frac{2\mu}{\sigma_1^2}}e^{-\frac{2\Lambda}{\sigma_1^2x}},\quad x>0. \tag{3.21}
$$

Proof. It follows from Theorem 3.1 that for any given initial value $(P(0),O(0),S(0),Q(0),R(0))^T\in\mathbb{R}_+^5$, the positive solution $(P(t),O(t),S(t),Q(t),R(t))^T$ of model (1.2) exists and is unique. From the first equation of model (1.2), we have

$$
\mathrm{d}P(t)\le[\Lambda-\mu P(t)]\mathrm{d}t+\sigma_1P(t)\mathrm{d}B_1(t).
$$

If $X(t)$ is the solution of a one-dimensional stochastic differential equation

$$
\begin{cases}
\mathrm{d}X(t)=[\Lambda-\mu X(t)]\mathrm{d}t+\sigma_1X(t)\mathrm{d}B_1(t),\\
X(0)=P(0),
\end{cases}
$$

then by following the comparison theorem for one-dimensional Itô's processes [56, Theorem 6.1.1], we have $P(t)\le X(t)$ a.s. In addition, when $\mu>\frac{\sigma_1^2}{2}$, the stochastic process $X(t)$ converges weakly a.s. to an ergodic stationary distribution with the invariant density $\pi(x)$ given by (3.21) [57, Theorems 3.1 and 4.1]. Consequently, from the ergodic theorem, we have

$$
\lim_{t\to\infty}\frac{1}{t}\int_0^t X(\theta)\mathrm{d}\theta=\int_{\mathbb{R}_+}x\pi(x)\mathrm{d}x=\frac{\Lambda}{\mu}\ \ a.s.
$$

We define a $\mathbf{C}^2$-function $V(O,S,Q,R):\mathbb{R}_+^4\to\mathbb{R}_+$ by

$$
V(O,S,Q,R)=k_1O+k_2(S+Q+R),
$$

where $k_1=\mu\sqrt{\alpha}$ and $k_2=\sqrt{\beta\Lambda(\alpha+\mu)}$. Using the Itô's formula to $\ln V$, we obtain

$$
\mathrm{d}(\ln V)=\mathcal{L}(\ln V)\mathrm{d}t+\frac{k_1\sigma_2O}{V}\mathrm{d}B_2(t)+\frac{k_2\sigma_3S}{V}\mathrm{d}B_3(t)+\frac{k_2\sigma_4Q}{V}\mathrm{d}B_4(t)+\frac{k_2\sigma_5R}{V}\mathrm{d}B_5(t), \tag{3.22}
$$

where

$$
\begin{aligned}
\mathcal{L}(\ln V)=&\frac{k_1}{V}[\beta PS-(\alpha+\mu)O]+\frac{k_2}{V}(\alpha O-\mu S-\mu Q-\mu R)\\
&-\frac{k_1^2\sigma_2^2O^2}{2V^2}-\frac{k_2^2\sigma_3^2S^2}{2V^2}-\frac{k_2^2\sigma_4^2Q^2}{2V^2}-\frac{k_2^2\sigma_5^2R^2}{2V^2},
\end{aligned}
$$

with

$$
\begin{aligned}
V^2=&\left[k_1\sigma_2O\frac{1}{\sigma_2}+k_2\left(\sigma_3S\frac{1}{\sigma_3}+\sigma_4Q\frac{1}{\sigma_4}+\sigma_5R\frac{1}{\sigma_5}\right)\right]^2\\
\le&\left[k_1^2\sigma_2^2O^2+k_2^2\left(\sigma_3^2S^2+\sigma_4^2Q^2+\sigma_5^2R^2\right)\right]\left(\frac{1}{\sigma_2^2}+\frac{1}{\sigma_3^2}+\frac{1}{\sigma_4^2}+\frac{1}{\sigma_5^2}\right),
\end{aligned} \tag{3.23}
$$

and

$$
\begin{aligned}
&\frac{1}{V}\left\{k_1[\beta PS-(\alpha+\mu)O]+k_2(\alpha O-\mu S-\mu Q-\mu R)\right\}\\
=&\frac{k_1\beta S}{V}\left(P-\frac{\Lambda}{\mu}\right)+\frac{1}{V}\left\{k_1\left[\frac{\beta\Lambda S}{\mu}-(\alpha+\mu)O\right]+k_2(\alpha O-\mu S-\mu Q-\mu R)\right\}\\
\le&\frac{k_1\beta}{k_2}\left|X-\frac{\Lambda}{\mu}\right|+\frac{1}{V}\left\{k_1\left[\frac{\beta\Lambda S}{\mu}-(\alpha+\mu)O\right]+k_2(\alpha O-\mu S)\right\}\\
\le&\beta\mu\sqrt{\frac{\alpha}{\beta\Lambda(\alpha+\mu)}}\left|X-\frac{\Lambda}{\mu}\right|+\frac{1}{V}\left\{\mu\sqrt{\beta\Lambda(\alpha+\mu)}(\Re-1)S+(\alpha+\mu)(\Re-1)O\right\}\\
\le&\mu\sqrt{\frac{\alpha\beta}{\Lambda(\alpha+\mu)}}\left|X-\frac{\Lambda}{\mu}\right|+\mu(\Re-1)\mathbb{I}_{\{\Re\le1\}}+(\alpha+\mu)(\Re-1)\mathbb{I}_{\{\Re>1\}}.
\end{aligned} \tag{3.24}
$$

Combining (3.23) and (3.24) we obtain

$$
\begin{aligned}
\mathcal{L}(\ln V)\le&\mu\sqrt{\frac{\alpha\beta}{\Lambda(\alpha+\mu)}}\left|X-\frac{\Lambda}{\mu}\right|+\mu(\Re-1)\mathbb{I}_{\{\Re\le1\}}+(\alpha+\mu)(\Re-1)\mathbb{I}_{\{\Re>1\}}\\
&-\left[2(\sigma_2^{-2}+\sigma_3^{-2}+\sigma_4^{-2}+\sigma_5^{-2})\right]^{-1}.
\end{aligned}
$$

It then follows from (3.22) that

$$
\begin{aligned}
\mathrm{d}(\ln V)\le&\left\{\mu\sqrt{\frac{\alpha\beta}{\Lambda(\alpha+\mu)}}\left|X-\frac{\Lambda}{\mu}\right|+\mu(\Re-1)\mathbb{I}_{\{\Re\le1\}}+(\alpha+\mu)(\Re-1)\mathbb{I}_{\{\Re>1\}}\right.\\
&\left.-\left[2(\sigma_2^{-2}+\sigma_3^{-2}+\sigma_4^{-2}+\sigma_5^{-2})\right]^{-1}\right\}\mathrm{d}t\\
&+\frac{k_1\sigma_2O}{V}\mathrm{d}B_2(t)+\frac{k_2\sigma_3S}{V}\mathrm{d}B_3(t)+\frac{k_2\sigma_4Q}{V}\mathrm{d}B_4(t)+\frac{k_2\sigma_5R}{V}\mathrm{d}B_5(t).
\end{aligned} \tag{3.25}
$$

Integrating from 0 to t and dividing by t on the both sides of (3.25) yields

$$\begin{aligned}
\frac{\ln V(t)}{t} \leq& \frac{\ln V(0)}{t} + \mu(\mathfrak{R}-1)\mathbb{I}_{\{\mathfrak{R}\leq 1\}} + (\alpha+\mu)(\mathfrak{R}-1)\mathbb{I}_{\{\mathfrak{R}>1\}} \\
&- \left[2(\sigma_2^{-2}+\sigma_3^{-2}+\sigma_4^{-2}+\sigma_5^{-2})\right]^{-1} \\
&+ \mu\sqrt{\frac{\alpha\beta}{\Lambda(\alpha+\mu)}}\frac{1}{t}\int_0^t \left|X(\theta)-\frac{\Lambda}{\mu}\right|\mathrm{d}\theta \\
&+ \frac{1}{t}\int_0^t \frac{k_1\sigma_2 O(\theta)}{V(\theta)}\mathrm{d}B_2(\theta) + \frac{1}{t}\int_0^t \frac{k_2\sigma_3 S(\theta)}{V(\theta)}\mathrm{d}B_3(\theta) \\
&+ \frac{1}{t}\int_0^t \frac{k_2\sigma_4 Q(\theta)}{V(\theta)}\mathrm{d}B_4(\theta) + \frac{1}{t}\int_0^t + \frac{k_2\sigma_5 R(\theta)}{V(\theta)}\mathrm{d}B_5(\theta) \\
=& \frac{\ln V(0)}{t} + \mu(\mathfrak{R}-1)\mathbb{I}_{\{\mathfrak{R}\leq 1\}} + (\alpha+\mu)(\mathfrak{R}-1)\mathbb{I}_{\{\mathfrak{R}>1\}} \\
&- \left[2(\sigma_2^{-2}+\sigma_3^{-2}+\sigma_4^{-2}+\sigma_5^{-2})\right]^{-1} \\
&+ \mu\sqrt{\frac{\alpha\beta}{\Lambda(\alpha+\mu)}}\frac{1}{t}\int_0^t \left|X(\theta)-\frac{\Lambda}{\mu}\right|\mathrm{d}\theta \\
&+ \frac{\mathcal{M}_2(t)}{t} + \frac{\mathcal{M}_3(t)}{t} + \frac{\mathcal{M}_4(t)}{t} + \frac{\mathcal{M}_5(t)}{t},
\end{aligned} \tag{3.26}$$

where

$$\mathcal{M}_2(t) := \int_0^t \frac{k_1\sigma_2 O(\theta)}{V(\theta)}\mathrm{d}B_2(\theta),\ \mathcal{M}_3(t) := \int_0^t \frac{k_2\sigma_3 S(\theta)}{V(\theta)}\mathrm{d}B_3(\theta),$$
$$\mathcal{M}_4(t) := \int_0^t \frac{k_2\sigma_4 Q(\theta)}{V(\theta)}\mathrm{d}B_4(\theta),\ \mathcal{M}_5(t) := \int_0^t \frac{k_2\sigma_5 R(\theta)}{V(\theta)}\mathrm{d}B_5(\theta),$$

are locally continuous martingales, and their corresponding quadratic variations are

$$\langle\mathcal{M}_2,\mathcal{M}_2\rangle_t = \sigma_2^2\int_0^t \left(\frac{k_1 O(\theta)}{V(\theta)}\right)^2 \mathrm{d}\theta \leq \sigma_2^2 t,$$
$$\langle\mathcal{M}_3,\mathcal{M}_3\rangle_t = \sigma_3^2\int_0^t \left(\frac{k_2 S(\theta)}{V(\theta)}\right)^2 \mathrm{d}\theta \leq \sigma_3^2 t,$$
$$\langle\mathcal{M}_4,\mathcal{M}_4\rangle_t = \sigma_4^2\int_0^t \left(\frac{k_1 Q(\theta)}{V(\theta)}\right)^2 \mathrm{d}\theta \leq \sigma_4^2 t,$$
$$\langle\mathcal{M}_5,\mathcal{M}_5\rangle_t = \sigma_5^2\int_0^t \left(\frac{k_2 R(\theta)}{V(\theta)}\right)^2 \mathrm{d}\theta \leq \sigma_5^2 t.$$

It is straightforward that $\limsup_{t\to\infty}\frac{\langle\mathcal{M}_i,\mathcal{M}_i\rangle_t}{t} \leq \sigma_i^2 < \infty$, $i = 2,3,4,5$. Subsequently, it follows the strong law of large numbers for local martingales [52], i.e.,

$$\lim_{t\to\infty}\frac{\mathcal{M}_i(t)}{t} = 0\ a.s.,\ i = 2,3,4,5. \tag{3.27}$$

In addition, from the ergodic theorem, we obtain

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_0^t \left|X(\theta)-\frac{\Lambda}{\mu}\right|\mathrm{d}\theta &= \int_{\mathbb{R}_+}\left|x-\frac{\Lambda}{\mu}\right|\pi(x)\mathrm{d}x \leq \left[\int_0^\infty \left(x-\frac{\Lambda}{\mu}\right)^2\pi(x)\mathrm{d}x\right]^{\frac{1}{2}} \\
&\leq \left[\int_0^\infty x^2\pi(x)\mathrm{d}x - \frac{2\Lambda}{\mu}\int_0^\infty x\pi(x)\mathrm{d}x + \left(\frac{\Lambda}{\mu}\right)^2\right]^{\frac{1}{2}} \\
&= \left[\frac{2\Lambda^2}{\mu(2\mu-\sigma_1^2)} - \frac{2\Lambda^2}{\mu^2} + \left(\frac{\Lambda}{\mu}\right)^2\right]^{\frac{1}{2}} \\
&= \frac{\sigma_1\Lambda}{\mu\sqrt{2\mu-\sigma_1^2}}.
\end{aligned} \tag{3.28}$$

Taking the limit superior on both sides of (3.26), and considering (3.27) and (3.28), we obtain

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\ln V(t)}{t} \leq& \mu(\mathfrak{R}-1)\mathbb{I}_{\{\mathfrak{R}\leq 1\}} + (\alpha+\mu)(\mathfrak{R}-1)\mathbb{I}_{\{\mathfrak{R}>1\}} \\
&- \left[2(\sigma_2^{-2}+\sigma_3^{-2}+\sigma_4^{-2}+\sigma_5^{-2})\right]^{-1} \\
&+ \sigma_1\sqrt{\frac{\alpha\beta\Lambda}{(\alpha+\mu)(2\mu-\sigma_1)}} \\
:=& \phi\ a.s.
\end{aligned}$$

Therefore, when $\phi < 0$, we have

$$\lim_{t\to\infty}\left[\mu\sqrt{\alpha}O(t) + \sqrt{(\alpha+\mu)\beta\Lambda}\big(S(t)+Q(t)+R(t)\big)\right] = 0\ a.s.,$$

leading us to have

$$\lim_{t\to\infty}O(t) = 0,\ \lim_{t\to\infty}S(t) = 0,\ \lim_{t\to\infty}Q(t) = 0,\ \lim_{t\to\infty}R(t) = 0\ a.s. \tag{3.29}$$

Thus for any small $0 < \varepsilon \ll 1$, it follows from (3.29) that there exist a time $\mathcal{T}_\varepsilon > 0$ and a set $\Omega_\varepsilon \subset \Omega$, satisfying $\mathbb{P}(\Omega_\varepsilon) > 1-\varepsilon$ and $\beta PS \leq \varepsilon P$ for all $t \geq \mathcal{T}_\varepsilon$ and $\omega \in \Omega_\varepsilon$. Consequently, from the first equation of model (1.2), we obtain

$$[\Lambda - \varepsilon P(t) - \mu P(t)]\mathrm{d}t + \sigma_1 P(t)\mathrm{d}B_1(t) \leq \mathrm{d}P(t) \leq [\Lambda - \mu P(t)]\mathrm{d}t + \sigma_1 P(t)\mathrm{d}B_1(t).$$

From the stochastic comparison theorem [56, Theorem 6.1.1, PP437-438] and for sufficiently small ε, we conclude that $P(t)$ has the same ergodic stationary distribution and invariant density as $X(t)$. This completes the proof of Theorem 3.5. □

3.5. Numerical simulations of model (1.2)

In order to numerically verify the theoretical results of model (1.2), we used Milstein's higher-order method [58], and investigated the influence of unavoidable environmental random effects on smoking epidemic dynamics. The discretized equations of the stochastic model (1.2) are

$$\begin{cases}
P_{k+1} = P_k + (\Lambda - \beta P_k S_k - \mu P_k)\Delta t + \frac{\sigma_1^2}{2}P_k(\mathcal{W}_{1,k}^2 - 1)\Delta t + \sigma_1 P_k\sqrt{\Delta t}\mathcal{W}_{1,k}, \\
O_{k+1} = O_k + (\beta P_k S_k - \alpha O_k - \mu O_k)\Delta t + \frac{\sigma_2^2}{2}O_k(\mathcal{W}_{2,k}^2 - 1)\Delta t + \sigma_2 O_k\sqrt{\Delta t}\mathcal{W}_{2,k}, \\
S_{k+1} = S_k + (\alpha O_k + \lambda Q_k S_k - \gamma S_k - \mu S_k)\Delta t + \frac{\sigma_3^2}{2}S_k(\mathcal{W}_{3,k}^2 - 1)\Delta t + \sigma_3 S_k\sqrt{\Delta t}\mathcal{W}_{3,k}, \\
Q_{k+1} = Q_k + [\gamma(1-\delta)S_k - \lambda Q_k S_k - \mu Q_k]\Delta t + \frac{\sigma_4^2}{2}Q_k(\mathcal{W}_{4,k}^2 - 1)\Delta t + \sigma_4 Q_k\sqrt{\Delta t}\mathcal{W}_{4,k}, \\
R_{k+1} = R_k + (\gamma\delta S_k - \mu R_k)\Delta t + \frac{\sigma_5^2}{2}R_k(\mathcal{W}_{5,k}^2 - 1)\Delta t + \sigma_5 R_k\sqrt{\Delta t}\mathcal{W}_{5,k},
\end{cases}$$

where $P_{k+1}, O_{k+1}, S_{k+1}, Q_{k+1}$, and R_{k+1} are respectively the population sizes of potential smokers, occasional smokers, smokers, temporary quitters, and permanent quitters obtained by $k+1$ iteration, $\Delta t > 0$ is the time increment, and $\mathcal{W}_{j,k}$, $j = 1,2,3,4,5$ are mutually independent Gaussian random variables with standard normal distribution $\mathcal{N}(0,1)$.

We explored the impacts of environmental random fluctuations on the dynamics of smoking model under two different scenarios of deterministic model (1.1): (1) a single smoking-present equilibrium, (2) a bistable state comprising a smoking-free equilibrium and a smoking-present equilibrium. The details are as follows.

Example 3.6. Using the system parameters from (2.3) with $\alpha = 0.2$ and setting $\beta = 0.14$, as established in Example 2.11, we found that model (1.1) exhibits only a stable smoking-present equilibrium. We now investigate the impacts of different intensities of environmental white noise on the dynamics of model (1.2)

(1) Setting $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 0.05$, we obtain $\varpi \approx 1.527 > 1$, indicating the existence of a unique ergodic stationary distribution for stochastic model (1.2), as supported by Theorem 3.3 and Fig. 3. This indicates that the effect of small environmental noise intensities on smoking control is negligible.

(2) Setting $\sigma_1 = 0.05$ and $\sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 2.5$, we compute $\mu = 0.1 > 0.00125 = \frac{\sigma_1^2}{2}$ and $\phi \approx -0.130 < 0$, satisfying the sufficient conditions of Theorem 3.5. Consequently, smoking-associated populations will extinct, leaving only non-smokers with an ergodic stationary distribution, which is consistent with Fig. 4. This indicates that large stochastic fluctuations in the environment are beneficial for controlling smoking behavior.

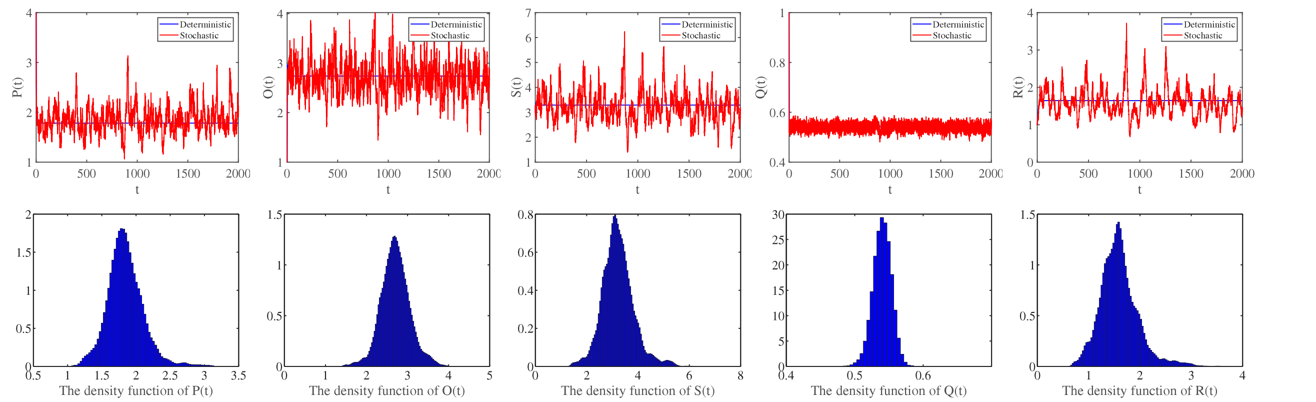

Fig. 3. Time series diagrams of five populations (top row) and density functions of the corresponding species (bottom row) for model (1.2) with $\sigma_i = 0.05$ for $i = 1, 2, 3, 4, 5$.

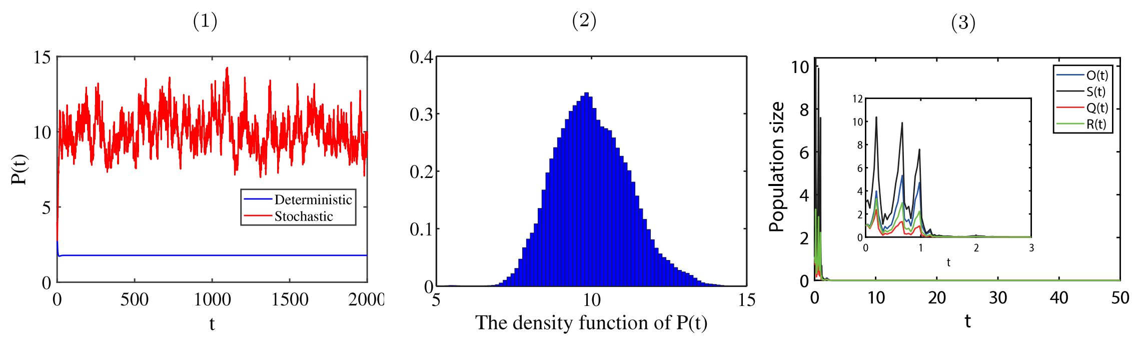

Fig. 4. (1) Time series of non-smokers $P(t)$; (2) Density function of $P(t)$ corresponding to (1); (3) Eliminated time series of four populations related to smoking in model (1.2) with $\sigma_1 = 0.05$ and $\sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 2.5$.

Example 3.7. We fixed the parameters as (2.3) and $\alpha = 0.2$, with $\beta = 0.06$. In Example 2.10, we established that the model (1.1) exhibits bistability, with a smoking-present equilibrium $E_1^* \approx (4.369, 1.877, 2.148, 0.532, 1.074)$ and a smoking-free equilibrium $E_0 = (10, 0, 0, 0, 0)$ for different initial values: $(4, 1, 3, 1, 1)$ and $(9.4, 0.1, 0.3, 0.1, 0.1)$. Moreover, according to Theorem 3.5, large noise can eliminate smoking behavior. Now, we investigate the impact of small fluctuations in the natural environment on controlling smoking behavior through numerical simulations.

(i) When the initial value is $(9.4, 0.1, 0.3, 0.1, 0.1)$, smoking is certain to vanish provided random fluctuations in the environment are not taken into account (Fig. 2(b)). However, when the inevitable random fluctuations in the environment are assumed as $\sigma_i = 0.05$, $i = 1, 2, 3, 4, 5$, we simulated 5000 sample trajectories over the same time period, resulting in Fig. 5(a), which includes 2734 smoking-free and 2266 smoking-present sample paths. This indicates that even small random fluctuations in the environment can lead to the prevalence of smoking behavior.

(ii) When the initial value is set to $(4, 1, 3, 1, 1)$, smoking continues to prevail in deterministic model (1.1) (Fig. 2(b)). Similarly, with $\sigma_i = 0.05$, Fig. 5(b) shows the existence of 21 smoking-free and 4979 smoking-present sample orbits. This suggests that small noises are also beneficial to smoking control to some extent.

In conclusion, large random fluctuations in the environment are always conducive to the control of smoking, while the influences of small noises on smoking control can be favorable, or harmful, or negligible. The conclusions are closely related to the intensities of environmental noises, the initial sizes of smoking populations, and the effective exposure rate of smoking transmission β.

4. Discussion

To explore the population dynamics of smoking behavior comprehensively and realistically, this paper examined a 5-dimensional smoking model in both deterministic and stochastic environments, including potential smokers $P(t)$, occasional smokers $O(t)$, smokers $S(t)$, temporary quitters $Q(t)$, and permanent quitters $R(t)$.

In deterministic model (1.1), we obtained the basic reproduction number $\Re_0 = \frac{\alpha\beta\Lambda}{\mu(\alpha+\mu)(\gamma+\mu)}$ using the next-generation matrix method [50]. Then, we established conditions for the local and global asymptotic stability of the smoking-free equilibrium and the local asymptotic stability of the smoking-present equilibrium. Additionally, we investigated the existence of saddle–node bifurcation. Our findings align with those in [50], showing that $\Re_0$ alone cannot completely determine the existence of smoking behavior. In other words, even when the smoking-free equilibrium is locally asymptotically stable, smoking behavior can persist. This implies that the model has a bistable phenomenon composed of a smoking-free equilibrium and a smoking-present equilibrium. Consequently, the initial sizes of smoking populations can significantly influence the presence or elimination of smoking behavior.

Compared to [18], we found that considering different levels of smoking can completely change the dynamics of smoking models, leading to the following insights:

1. The basic reproduction number $\Re_0$, obtained using the next-generation matrix method, cannot be completely used as a threshold condition for determining the existence of smoking behavior.
2. The existence of saddle–node bifurcation and the emergence of bistability reflect more realistically that the different initial sizes of smoking populations can also determine whether smoking prevails to a certain extent.

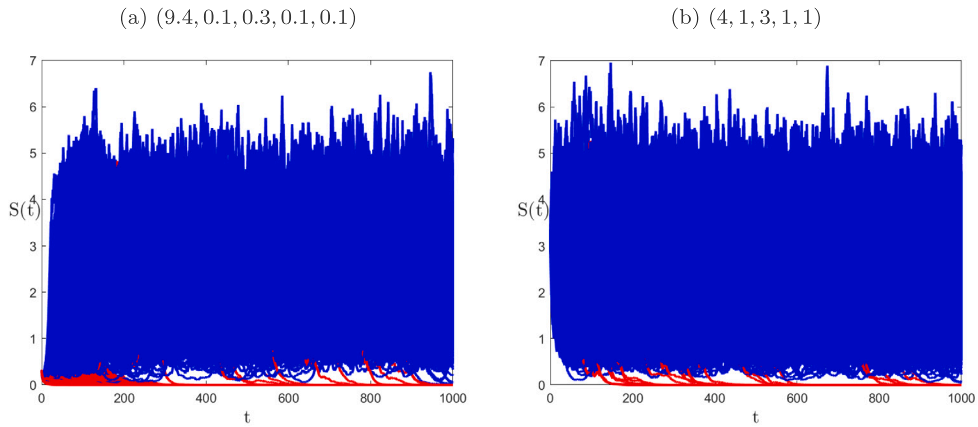

Fig. 5. (a) and (b) represent 5000 sample paths of smokers $S(t)$ for stochastic model (1.2) with same noise intensities $\sigma_i = 0.05$ but different initial values.

These significant differences indicate that considering different levels of smoking can make smoking models more realistic and comprehensive in describing the prevalence of smoking in the real world. Furthermore, through rigorous mathematical derivations, we improved the analysis of the dynamics of model (1.1) originally presented in [21].

In stochastic model (1.2), we first analyzed the well-posedness, including the existence, uniqueness, and ultimate boundedness of a globally positive solution. Then, we established the sufficient condition for the existence and uniqueness of the ergodic stationary distribution. Especially, when random effects in the environment are not considered, i.e., the intensities of noises are zero, we validated Conjecture 2.14 for the deterministic model (1.1). Furthermore, we provided sufficient criteria for the eradication of smoking behavior. Finally, through numerical simulations, we concluded that the effects of inevitable random fluctuations in the natural environment on the control of the smoking epidemic can be favorable, or harmful, or negligible. The ultimate effects of environmental noises on smoking control are closely related to noise intensities, initial smoking population sizes, and the effective exposure rate of smoking transmission β. More specifically,

- When β is larger, such that $\Re_0 > 1$, significant stochastic environmental fluctuations are conducive to the control of smoking, however, the effect of small noise intensities is negligible.
- When β is reduced, such that $\Re_0 < 1$, the deterministic model (1.1) has a bistability composed of a smoking-free equilibrium and a smoking-present equilibrium. Large random fluctuations in the environment can still effectively control the smoking epidemic, while the control effects of small noise intensities can be favorable or harmful. Specifically,
  1. (1) All smokers are removed in the deterministic model (1.1) for a low initial size of smoking populations. Furthermore, for mild environmental noises, smokers will continue to epidemic with a certain positive probability. This indicates that random effects are detrimental to smoking control.
  2. (2) Conversely, when the initial size of smoking populations is large, smoking is prevalent in deterministic model (1.1). Further taking small noise intensities into account, we found that smokers will be eliminated with a positive probability. This demonstrates that random effects are beneficial for smoking control.

As environmental changes are inherent in natural ecosystems, it becomes challenging to accurately regulate the intensities of inevitable environmental noises. To successfully eradicate smoking behavior, two key approaches are crucial. First, conducting early and accurate monitoring of the initial sizes of smoking populations is essential. Second, implementing effective, reasonable, and feasible measures to reduce the effective exposure rate of smoking transmission β, is vital. Notably, when environmental random effects are not considered, with a decrease of β, the originally prevalent smoking populations gradually enter a bistable transition period (i.e., the coexistence of smoking-present and smoking-free equilibria). Only during the transition period, the different initial sizes of smoking populations can effectively regulate the presence or absence of smoking. Therefore, taking effective measures to reduce the effective exposure rate of smoking transmission β is the key to controlling the smoking epidemic.

To explore effective smoking control measures, we compared the strategies outlined in the WHO FCTC [10] with the MPOWER approach introduced by WHO [11]. Both approaches align in their smoking control measures [6], leading us to focus our discussion on the MPOWER approach.

(1) **M**onitor tobacco use and prevention policies [11]. The measure aims to capture periodic data on key indicators of tobacco use among adolescents and adults nationwide.
(2) **P**rotect people from tobacco use [11]. The measure aims to establish and implement completely smoking-free environments in all indoor public places, such as medical and educational institutions.
(3) **O**ffer help to quit tobacco use [11]. The measure aims to strengthen health systems to provide comprehensive support for smoking cessation.
(4) **W**arn about the dangers of tobacco [11]. The measure aims to promote the dangers of tobacco in the media.
(5) **E**nforce bans on tobacco advertising, promotion and sponsorship [11]. The measure aims to establish and implement effective legislation to comprehensively prohibit any form of direct or indirect tobacco publicizing.
(6) **R**aise taxes on tobacco [11]. The measure aims to increase the tax rate and strengthen tax administration on tobacco products.

In [59], the authors emphasized the effectiveness of the MPOWER series of policies in reducing smoking prevalence. Tax and price increases are considered as the most influential tobacco control policies among various alternative MPOWER measures [60–63]. Additionally, WHO MPOWER measures have been shown to save lives and reduce healthcare costs [6]. Therefore, there is an urgent need to accelerate the implementation of a strong set of MPOWER measures for smoking control. However, intriguing and unexplored questions remain regarding how to quantify the impacts of different effective policies within MPOWER measures in controlling β, and how to effectively combine these measures to achieve the greatest reduction in the effective exposure rate of smoking transmission β. These will be the focus of our future research.

CRediT authorship contribution statement

Shengqiang Zhang: Conceptualization, Formal analysis, Investigation, Methodology, Software, Visualization, Writing – original draft, Writing – review & editing. **Yanling Meng:** Conceptualization, Funding acquisition, Investigation, Project administration. **Amit Kumar Chakraborty:** Investigation, Methodology, Validation, Visualization, Writing – review & editing. **Hao Wang:** Conceptualization, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No datasets were generated or analyzed in this study.

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Appendix A. Proof of Theorem 2.1

Proof. To verify the positivity of the solution of model (1.1) with a positive initial value, we rewrite the model as

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = F(Y(t)), \tag{A.1}$$

where $Y = (P, O, S, Q, R)^T \in \mathbb{R}^5$, and

$$F(Y) = \begin{pmatrix} F_1(Y) \\ F_2(Y) \\ F_3(Y) \\ F_4(Y) \\ F_5(Y) \end{pmatrix} = \begin{pmatrix} \Lambda - \beta P(t)S(t) - \mu P(t) \\ \beta P(t)S(t) - \alpha O(t) - \mu O(t) \\ \alpha O(t) + \lambda Q(t)S(t) - \gamma S(t) - \mu S(t) \\ \gamma(1-\delta)S(t) - \lambda Q(t)S(t) - \mu Q(t) \\ \gamma\delta S(t) - \mu R(t) \end{pmatrix}.$$

Since $F(Y) : \mathbb{R}^5 \to \mathbb{R}$ satisfies the local Lipschitz condition, it follows from the fundamental theorem of ordinary differential equations that the solution of model (A.1) with positive initial value exists and is unique. In addition, it is worth noting that $F_i(Y)|_{y_i=0} \geq 0$, where $y_1 = P(t)$, $y_2 = O(t)$, $y_3 = S(t)$, $y_4 = Q(t)$, $y_5 = R(t)$. Using Nagumo theorem [64], we obtain for all $t \geq 0$, the solution of model (1.1) with positive initial value remains positive.

To investigate the boundedness of the solutions of model (1.1), we consider $(P(t), O(t), S(t), Q(t), R(t))^T$ is a solution of model (1.1) with positive initial value $(P(0), O(0), S(0), Q(0), R(0))^T$. By adding equations of model (1.1), we obtain the total population size, $N(t) = P(t) + O(t) + S(t) + Q(t) + R(t)$, satisfies $\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \Lambda - \mu N(t)$, and then $\lim_{t\to\infty} N(t) = \frac{\Lambda}{\mu}$ *a.s.* Thus the positively invariant region of model (1.1) is given by

$$\Gamma = \left\{ (P, O, S, Q, R)^T : 0 \leq P, O, S, Q, R \text{ and } P + O + S + Q + R = \frac{\Lambda}{\mu} \right\}.$$

This completes the proof of Theorem 2.1. □

Appendix B. Proof of Theorem 2.2

Proof. For any nonnegative equilibrium $E(P, O, S, Q, R)$ of model (1.1), the system of equations satisfy

$$\begin{cases} \Lambda - \beta PS - \mu P = 0, \\ \beta PS - \alpha O - \mu O = 0, \\ \alpha O + \lambda QS - \gamma S - \mu S = 0, \\ \gamma(1-\delta)S - \lambda QS - \mu Q = 0, \\ \gamma\delta S - \mu R = 0, \end{cases} \Rightarrow \begin{cases} P = \frac{\Lambda}{\beta S + \mu}, \\ O = \frac{\Lambda \beta S}{(\alpha+\mu)(\beta S + \mu)}, \\ Q = \frac{\gamma(1-\delta)S}{\lambda S + \mu}, \\ R = \frac{\gamma\delta S}{\mu}, \end{cases}$$

and

$$\frac{\alpha\beta\Lambda S}{(\alpha+\mu)(\beta S+\mu)} + \frac{\gamma(1-\delta)\lambda S^2}{\lambda S + \mu} = (\gamma+\mu)S. \tag{B.1}$$

When the population size of smokers is zero, i.e., $S = 0$, we obtain the smoke-free equilibrium $E_0(\frac{\Lambda}{\mu}, 0, 0, 0, 0)$. However, when $S > 0$, from (B.1) we can derive that

$$\begin{aligned} &\beta\lambda(\alpha+\mu)(\mu+\gamma\delta)S^2 + (\alpha\lambda\mu^2 + \alpha\lambda\mu\gamma\delta + \lambda\mu^3 \\ &\quad + \lambda\mu^2\gamma\delta + \alpha\beta\mu\gamma + \beta\mu^2\gamma + \alpha\beta\mu^2 + \beta\mu^3 - \alpha\beta\lambda\Lambda)S \\ &\quad + \mu[\mu(\alpha+\mu)(\mu+\gamma) - \alpha\beta\Lambda] = 0. \end{aligned} \tag{B.2}$$

It follows from Vieta theorem [65] that two roots (S_1 and S_2) of the quadratic Eq. (B.2) satisfy $S_1 \cdot S_2 = \frac{\mu[\mu(\alpha+\mu)(\mu+\gamma)-\alpha\beta\Lambda]}{\beta\lambda(\alpha+\mu)(\mu+\gamma\delta)}$. Therefore, we have two situations:

(1) when $\mu(\alpha+\mu)(\mu+\gamma) < \alpha\beta\Lambda$, that is $\mathfrak{R}_0 := \frac{\alpha\beta\Lambda}{\mu(\alpha+\mu)(\gamma+\mu)} > 1$, then $S_1 \cdot S_2 < 0$, which implies that (B.2) has a unique positive root $E_1^*(P_1^*, O_1^*, S_1^*, Q_1^*, R_1^*)$;

(2) when $\mathfrak{R}_0 < 1$, then $S_1 \cdot S_2 > 0$, which implies that (B.2) either has two positive roots $E_{21}^*(P_{21}^*, O_{21}^*, S_{21}^*, Q_{21}^*, R_{21}^*)$ and $E_{22}^*(P_{22}^*, O_{22}^*, S_{22}^*, Q_{22}^*, R_{22}^*)$ or none.

This completes the proof of Theorem 2.2. □

Appendix C. The basic reproduction number $\mathfrak{R}_0$ of model (1.1)

Since the subpopulations with smoking behavior in model (1.1) are $O(t)$, $S(t)$, and $Q(t)$, and the transitions from $O(t)$ to $S(t)$, from $S(t)$ to $Q(t)$, and from $Q(t)$ back to $S(t)$ are not considered as new smokers, then following the next generation matrix method [50], we have

$$\mathfrak{F} = \begin{pmatrix} 0 \\ \beta PS \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathcal{V} = \begin{pmatrix} \beta PS + \mu P - \Lambda \\ \alpha O + \mu O \\ \gamma S + \mu S - \alpha O - \lambda QS \\ \lambda QS + \mu Q - \gamma(1-\delta)S \\ \mu R - \gamma\delta S \end{pmatrix}.$$

When there are no smoking populations, i.e., $O(t) = S(t) = Q(t) = 0$, the non smoking equilibrium is $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)^T$, and at E_0 we have

$$\mathbb{F} = \begin{pmatrix} 0 & \frac{\beta\Lambda}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathbb{V} = \begin{pmatrix} \alpha+\mu & 0 & 0 \\ -\alpha & \gamma+\mu & 0 \\ 0 & -\gamma(1-\delta) & \mu \end{pmatrix}.$$

Using $\mathbb{F}$ and $\mathbb{V}$, it is straightforward to compute

$$\mathbb{V}^{-1} = \begin{pmatrix} \frac{1}{\alpha+\mu} & 0 & 0 \\ \frac{\alpha}{(\alpha+\mu)(\gamma+\mu)} & \frac{1}{\gamma+\mu} & 0 \\ \frac{\alpha\gamma(1-\delta)}{\mu(\alpha+\mu)(\gamma+\mu)} & \frac{\gamma(1-\delta)}{\mu(\gamma+\mu)} & \frac{1}{\mu} \end{pmatrix} \text{ and}$$

$$\mathbb{F}\mathbb{V}^{-1} = \begin{pmatrix} \frac{\alpha\beta\Lambda}{\mu(\alpha+\mu)(\gamma+\mu)} & \frac{\beta\Lambda}{\mu(\gamma+\mu)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the basic reproduction number of model (1.1) is

$$\mathfrak{R}_0 = \rho(\mathbb{F}\mathbb{V}^{-1}) = \frac{\alpha\beta\Lambda}{\mu(\alpha+\mu)(\gamma+\mu)},$$

where $\rho(\cdot)$ is the spectral radius of a matrix $(\cdot)$.

Appendix D. Proof of Theorem 2.3

Proof. For model (1.1), the Jacobian matrix evaluated at the smoke-free equilibrium E_0 is

$$J|_{E_0} = \begin{pmatrix} -\mu & 0 & -\frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & -\alpha-\mu & \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & \alpha & -\gamma-\mu & 0 & 0 \\ 0 & 0 & \gamma(1-\delta) & -\mu & 0 \\ 0 & 0 & \gamma\delta & 0 & -\mu \end{pmatrix}.$$

The characteristic equation of $J|_{E_0}$ is

$$(\psi+\mu)^3\left[\psi^2 + (\alpha+2\mu+\gamma)\psi + (\alpha+\mu)(\mu+\gamma) - \frac{\alpha\beta\Lambda}{\mu}\right] = 0.$$

One of the eigenvalue of $J|_{E_0}$ is $-\mu$ (triple root). To establish the local asymptotic stability of smoke-free equilibrium E_0, we need to determine the characteristic roots of the equation

$$\psi^2 + (\alpha+2\mu+\gamma)\psi + (\alpha+\mu)(\mu+\gamma) - \frac{\alpha\beta\Lambda}{\mu} = 0. \tag{D.1}$$

The discriminant of (D.1) is

$$\Delta = (\alpha+2\mu+\gamma)^2 - 4\left[(\alpha+\mu)(\mu+\gamma) - \frac{\alpha\beta\Lambda}{\mu}\right] = (\alpha-\gamma)^2 + 4(\alpha+\mu)(\mu+\gamma)\mathfrak{R}_0 > 0.$$

Which implies that Eq. (D.1) has two real roots ψ_1 and ψ_2, where $\mathfrak{R}_0 = \frac{\alpha\beta\Lambda}{\mu(\alpha+\mu)(\gamma+\mu)}$. Further, it follows from the Vieta theorem [65] that $\psi_1+\psi_2 = -(\alpha+2\mu+\gamma) < 0$ and $\psi_1\cdot\psi_2 = (\alpha+\mu)(\mu+\gamma)(1-\mathfrak{R}_0)$. Therefore, there are two scenarios:

(1) when $\mathfrak{R}_0 < 1$, $\psi_1\cdot\psi_2 > 0$, where $\psi_1 < 0$ and $\psi_2 < 0$, then the smoke-free equilibrium E_0 is locally asymptotically stable;
(2) when $\mathfrak{R}_0 > 1$, $\psi_1\cdot\psi_2 < 0$, which implies that there exists a positive root and a negative root, then E_0 losses stability.

This completes the proof of Theorem 2.3. □

Appendix E. Proof of Theorem 2.4

Proof. Based on the positively invariant region Γ in Appendix A, we can reduce model (1.1) to lower dimensions, resulting in the following systems

$$\begin{cases} \frac{dO(t)}{dt} = \beta\left(\frac{\Lambda}{\mu} - O(t) - S(t) - Q(t) - R(t)\right)S(t) - \alpha O(t) - \mu O(t), \\ \frac{dS(t)}{dt} = \alpha O(t) + \lambda Q(t)S(t) - \gamma S(t) - \mu S(t), \\ \frac{dQ(t)}{dt} = \gamma(1-\delta)S(t) - \lambda Q(t)S(t) - \mu Q(t), \\ \frac{dR(t)}{dt} = \gamma\delta S(t) - \mu R(t), \end{cases} \tag{E.1}$$

with positive initial value $(O(0), S(0), Q(0), R(0))^T \in \mathbb{R}^4_+$. The system (E.1) always has a zero solution, and to validate the global asymptotic stability of the zero solution, we construct a $\mathbf{C}^2$-function as

$$V(O,S,Q,R) = \ell_1 O(t) + S(t) + Q(t) + R(t)$$

The total derivative of V is

$$\begin{aligned} \frac{dV}{dt} =& \ell_1\left[\beta\left(\frac{\Lambda}{\mu} - O - S - Q - R\right)S - \alpha O - \mu O\right] + \alpha O - \mu S - \mu Q - \mu R \\ \le& \left(\frac{\ell_1\beta\Lambda}{\mu} - \mu\right)S + [\alpha - \ell_1(\alpha+\mu)]O - \mu Q - \mu R. \end{aligned}$$

Let

$$\begin{cases} \frac{\ell_1\beta\Lambda}{\mu} - \mu < 0, \\ \alpha - \ell_1(\alpha+\mu) < 0, \end{cases} \Rightarrow \frac{\alpha}{\alpha+\mu} < \ell_1 < \frac{\mu^2}{\beta\Lambda},$$

that is

$$\widetilde{\mathfrak{R}}_0 = \frac{\alpha\beta\Lambda}{\mu^2(\alpha+\mu)} < 1.$$

Thus, when $\widetilde{\mathfrak{R}}_0 < 1$, choosing $\ell_1 = \frac{1}{2}(\frac{\alpha}{\alpha+\mu} + \frac{\mu^2}{\beta\Lambda})$, we have $\frac{dV}{dt} \le 0$, and when $O = S = Q = R = 0$, we have $\frac{dV}{dt} = 0$. It further follows from the Routh–Hurwitz criteria that when $\mathfrak{R}_0 < 1$, the zero solution of system (E.1) is globally asymptotically stable. This indicates that the smoking-free equilibrium E_0 of model (1.1) is globally asymptotically stable. Hence, the proof of Theorem 2.4 is completed. □

Appendix F. Proof of Theorem 2.6

Proof. The Jacobian matrix of model (1.1) evaluated at the positive equilibrium $E^*(P^*, O^*, S^*, Q^*, R^*)$ is

$$J|_{E^*} = \begin{pmatrix} -\beta S^*-\mu & 0 & -\beta P^* & 0 & 0 \\ \beta S^* & -\alpha-\mu & \beta P^* & 0 & 0 \\ 0 & \alpha & \lambda Q^*-\gamma-\mu & \lambda S^* & 0 \\ 0 & 0 & \gamma(1-\delta)-\lambda Q^* & -\lambda S^*-\mu & 0 \\ 0 & 0 & \gamma\delta & 0 & -\mu \end{pmatrix}.$$

The characteristic equation of $J|_{E^*}$ is

$$(\psi+\mu)(\psi^4 + k_1\psi^3 + k_2\psi^2 + k_3\psi + k_4) = 0,$$

where

$$\begin{aligned} k_1 =& \gamma + 4\mu + \lambda S^* - \lambda Q^* + \alpha + \beta S^* > 0, \\ k_2 =& \mu(\gamma+\mu-\lambda Q^*) + \lambda S^*(\mu+\gamma\delta) + (\alpha+\beta S^*+2\mu)(\gamma+2\mu+\lambda S^*-\lambda Q^*) \\ &+ (\beta S^*+\mu)(\alpha+\mu) - \alpha\beta P^*, \\ k_3 =& (\alpha+\beta S^*+2\mu)[\mu(\gamma+\mu-\lambda Q^*) + \lambda S^*(\mu+\gamma\delta)] - \alpha\beta P^*(\lambda S^*+2\mu) \\ &+ (\beta S^*+\mu)(\alpha+\mu)(\gamma+2\mu+\lambda S^*-\lambda Q^*), \\ k_4 =& (\beta S^*+\mu)(\alpha+\mu)[\mu(\gamma+\mu-\lambda Q^*) + \lambda S^*(\mu+\gamma\delta)] - \alpha\beta P^*\mu(\lambda S^*+\mu). \end{aligned}$$

Thus, it follows from the Routh–Hurwitz criteria [47] that E^* is locally asymptotically stable provided the conditions of Theorem 2.6 are satisfied. This completes the proof. □

Appendix G. Proof of Theorem 2.7

Proof. It follows from Appendix F that when $\widetilde{\beta} = \frac{\mu(\alpha+\mu)[\mu(\gamma+\mu-\lambda Q^*)+\lambda S^*(\mu+\gamma\delta)]}{\alpha\mu P^*(\lambda S^*+\mu)-S^*(\alpha+\mu)[\mu(\gamma+\mu-\lambda Q^*)+\lambda S^*(\mu+\gamma\delta)]} > 0$, we have $k_4 = 0$, which implies that the matrix $J|_{E^*}$ has a zero eigenvalue. Thus we have the following modified matrix:

$$J(E^*,\widetilde{\beta}) = \begin{pmatrix} -\widetilde{\beta}S^*-\mu & 0 & -\widetilde{\beta}P^* & 0 & 0 \\ \widetilde{\beta}S^* & -\alpha-\mu & \widetilde{\beta}P^* & 0 & 0 \\ 0 & \alpha & \lambda Q^*-\gamma-\mu & \lambda S^* & 0 \\ 0 & 0 & \gamma(1-\delta)-\lambda Q^* & -\lambda S^*-\mu & 0 \\ 0 & 0 & \gamma\delta & 0 & -\mu \end{pmatrix}.$$

The eigenvectors of matrices $J(E^*,\widetilde{\beta})$ and $J^T(E^*,\widetilde{\beta})$ with respect to zero eigenroot is

$$\mathbf{U} = \left(-\frac{\widetilde{\beta}P^*u}{\widetilde{\beta}S^*+\mu},\ \frac{\mu\widetilde{\beta}P^*u}{(\alpha+\mu)(\widetilde{\beta}S^*+\mu)},\ u,\ \frac{\gamma(1-\delta)-\lambda Q^*}{\lambda S^*+\mu}u,\ \frac{\gamma\delta}{\mu}u\right)^T,$$

$$\mathbf{W} = \left(\frac{\widetilde{\beta}S^*}{\widetilde{\beta}S^*+\mu}w,\ w,\ \frac{\alpha+\mu}{\alpha}w,\ \frac{(\alpha+\mu)\lambda S^*}{\alpha(\lambda S^*+\mu)}w,\ 0\right)^T,$$

where u and w are two non-zero numbers. Then we have the following results:

1. $F_\beta(Y,\beta) = (-P(t)S(t),\ P(t)S(t),\ 0,\ 0,\ 0)^T$, $F_\beta(E^*,\widetilde{\beta}) = (-P^*S^*,\ P^*S^*,\ 0,\ 0,\ 0)^T$ and $\mathbf{W}^T\cdot F_\beta(E^*,\widetilde{\beta}) = \frac{\mu P^*S^*w}{\widetilde{\beta}S^*+\mu} \neq 0$. By Sotomayor's theorem [48], we obtain that both the transcritical bifurcation and pitchfork bifurcation are nonexistent.

2. When (2.2) holds, then $\mathbf{W}^T \cdot \left(D^2F_\beta(E^*,\widetilde{\beta})(U,U)\right) = 2\mu u^2 w\left(\frac{\lambda(\alpha+\mu)[\gamma(1-\delta)-\lambda Q^*]}{\alpha(\lambda S^*+\mu)^2} - \frac{\widetilde{\beta}^2P^*}{(\widetilde{\beta}S^*+\mu)^2}\right) \neq 0$. Using Sotomayor's theorem [48], we can verify that model (1.1) undergoes a saddle–node bifurcation when the parameter β crosses threshold value $\beta = \widetilde{\beta}$.

This completes the proof of Theorem 2.7. □

Appendix H. Proof of Theorem 3.1

Proof. Since the coefficients of model (1.2) satisfy the local Lipschitz conditions, then for any given positive initial value $(P(0), O(0), S(0), Q(0), R(0))^T$, the model admits a unique local solution $P(t), O(t), S(t), Q(t), R(t)^T \in \mathbb{R}^5_+$ on $0 \le t < \tau_e$ a.s., where τ_e denotes the explosion time. In order to establish the global property of the solution, we only need to verify $\tau_e = \infty$ a.s. To do so, let $m_0 \ge 1$ be sufficiently large such that $P(0), O(0), S(0), Q(0)$ and $R(0)$ lie within $[\frac{1}{m_0}, m_0]$. For any integer $m \ge m_0$, we define the following stopping time

$$\tau_m = \inf\left\{t\in[0,\tau_e) : \min\{P(t),O(t),S(t),Q(t),R(t)\} \le \frac{1}{m} \right.$$
$$\left. \text{or } \max\{P(t),O(t),S(t),Q(t),R(t)\} \ge m\right\}.$$

We consider $\inf\emptyset = \infty$ (generally, $\emptyset$ is the empty set). It follows from the definition that τ_m is increasing as $m \to \infty$. Let $\tau_\infty = \lim_{m\to\infty}\tau_m$, obviously $\tau_\infty \le \tau_e$ a.s. When $\tau_\infty = \infty$ a.s. holds, then $\tau_e = \infty$ a.s. and $(P(t), O(t), S(t), Q(t), R(t))^T \in \mathbb{R}^5_+$ a.s. for all $t \ge 0$. To proceed with the proof by contradiction, we assume that $\tau_\infty = \infty$ a.s. is false, then there exist two values $\mathcal{T} > 0$ and $\varepsilon \in (0, 1)$ satisfying

$$\mathbb{P}\{\tau_\infty \le \mathcal{T}\} > \varepsilon.$$

Then there is an integer $m_1 \ge m_0$ such that

$$\mathbb{P}\{\tau_m \le \mathcal{T}\} \ge \varepsilon \text{ for any } m \ge m_1.$$

Further, we define a non-negative $\mathbf{C}^2$-function $V(P,O,S,Q,R)$ as

$$V(P,O,S,Q,R) = \left[P - \frac{\mu}{2\beta} - \frac{\mu}{2\beta}\ln\frac{P}{(\frac{\mu}{2\beta})}\right] + (O - 1 - \ln O) + (S - 1 - \ln S) + \left[Q - \frac{\mu}{2\lambda} - \frac{\mu}{2\lambda}\ln\frac{Q}{(\frac{\mu}{2\lambda})}\right] + (R - 1 - \ln R).$$

Using the Itô's formula [52] to $V(P,O,S,Q,R)$, we have

$$\mathrm{d}V(P,O,S,Q,R) = \mathcal{L}V(P,O,S,Q,R)\mathrm{d}t + \sigma_1\left(P - \frac{\mu}{2\beta}\right)\mathrm{d}B_1(t) + \sigma_2(O-1)\mathrm{d}B_2(t) + \sigma_3(S-1)\mathrm{d}B_3(t) + \sigma_4\left(Q - \frac{\mu}{2\lambda}\right)\mathrm{d}B_4(t) + \sigma_5(R-1)\mathrm{d}B_5(t),$$

where $\mathcal{L}V(P,O,S,Q,R) : \mathbb{R}^5_+ \to \mathbb{R}$ is described as

$$\begin{aligned}
\mathcal{L}V =& \left(1 - \frac{\mu}{2\beta P}\right)(\Lambda - \beta PS - \mu P) + \frac{\mu\sigma_1^2}{4\beta} \\
&+ \left(1 - \frac{1}{O}\right)(\beta PS - \alpha O - \mu O) + \frac{\sigma_2^2}{2} \\
&+ \left(1 - \frac{1}{S}\right)(\alpha O + \lambda QS - \gamma S - \mu S) + \frac{\sigma_3^2}{2} \\
&+ \left(1 - \frac{\mu}{2\lambda Q}\right)[\gamma(1-\delta)S - \lambda QS - \mu Q] + \frac{\mu\sigma_4^2}{4\lambda} \\
&+ \left(1 - \frac{1}{R}\right)(\gamma\delta S - \mu R) + \frac{\sigma_5^2}{2} \\
=& \Lambda - \beta PS - \mu P - \frac{\Lambda\mu}{2\beta P} + \frac{\mu S}{2} + \frac{\mu^2}{2\beta} + \frac{\mu\sigma_1^2}{4\beta} \\
&+ \beta PS - \alpha O - \mu O - \frac{\beta PS}{O} + \alpha + \mu + \frac{\sigma_2^2}{2} \\
&+ \alpha O + \lambda QS - \gamma S - \mu S - \frac{\alpha O}{S} - \lambda Q + \gamma \\
&+ \mu + \frac{\sigma_3^2}{2} + \gamma(1-\delta)S - \lambda QS - \mu Q \\
&- \frac{\mu\gamma(1-\delta)S}{2\lambda Q} + \frac{\mu S}{2} + \frac{\mu^2}{2\lambda} + \frac{\mu\sigma_4^2}{4\lambda} + \gamma\delta S - \mu R - \frac{\gamma\delta S}{R} + \mu + \frac{\sigma_5^2}{2} \\
=& \Lambda + \frac{\mu^2}{2\beta} + \frac{\mu\sigma_1^2}{4\beta} + \alpha + \mu + \frac{\sigma_2^2}{2} + \gamma + \mu + \frac{\sigma_3^2}{2} + \frac{\mu^2}{2\lambda} + \frac{\mu\sigma_4^2}{4\lambda} + \mu + \frac{\sigma_5^2}{2} \\
&+ (-\beta PS + \beta PS) + \left(\frac{\mu S}{2} - \mu S + \frac{\mu S}{2}\right) + (-\alpha O + \alpha O) \\
&+ (\lambda QS - \lambda QS) + \left(-\gamma S + \gamma(1-\delta)S + \gamma\delta S\right) \\
&+ \left(-\mu P - \frac{\Lambda\mu}{2\beta P} - \mu O - \frac{\beta PS}{O} - \frac{\alpha O}{S} - \lambda Q - \mu Q - \frac{\mu\gamma(1-\delta)S}{2\lambda Q} - \mu R - \frac{\gamma\delta S}{R}\right) \\
\le& \Lambda + \frac{\mu^2}{2\beta} + \frac{\mu\sigma_1^2}{4\beta} + \alpha + \mu + \frac{\sigma_2^2}{2} + \gamma + \mu + \frac{\sigma_3^2}{2} + \frac{\mu^2}{2\lambda} + \frac{\mu\sigma_4^2}{4\lambda} + \mu + \frac{\sigma_5^2}{2} \\
:=& K.
\end{aligned}$$

Here, K is a positive constant independent of the variables P, O, S, Q, and R. The remaining steps of the proof, following a similar approach to that of [66, Theorem 3.1], are omitted here, thereby completing the proof of Theorem 3.1. □

Appendix I. Proof of Theorem 3.2

Proof. Let $N(t) = P(t) + O(t) + S(t) + Q(t) + R(t)$, using the equations of model (1.2), we obtain

$$\begin{cases}\mathrm{d}N(t) = [\Lambda - \mu N(t)]\mathrm{d}t + \sigma_1 P(t)\mathrm{d}B_1(t) + \sigma_2 O(t)\mathrm{d}B_2(t) \\ \quad +\sigma_3 S(t)\mathrm{d}B_3(t) + \sigma_4 Q(t)\mathrm{d}B_4(t) + \sigma_5 S(t)\mathrm{d}B_5(t), \\ N(0) = P(0) + O(0) + S(0) + Q(0) + R(0).\end{cases}$$

The solution of $N(t)$ is given by

$$N(t) = N(0)e^{-\mu t} + \frac{\Lambda}{\mu} - \frac{\Lambda}{\mu}e^{-\mu t} + \mathcal{M}(t), \tag{I.1}$$

where $\mathcal{M}(t) = \int_0^t e^{-\mu(t-\theta)}[\sigma_1 P(\theta)\mathrm{d}B_1(\theta) + \sigma_2 O(\theta)\mathrm{d}B_2(\theta) + \sigma_3 S(\theta)\mathrm{d}B_3(\theta) + \sigma_4 Q(\theta)\mathrm{d}B_4(\theta) + \sigma_5 S(\theta)\mathrm{d}B_5(\theta)]$ is a locally continuous martingale with $\mathcal{M}(0) = 0$. In addition, Eq. (I.1) can be rewritten as

$$N(t) = N(0) + A(t) - B(t) + \mathcal{M}(t),$$

where $A(t) = \frac{\Lambda}{\mu}\left(1 - e^{-\mu t}\right)$ and $B(t) = N(0)\left(1 - e^{-\mu t}\right)$. Obviously, when $t \ge 0$, $A(t)$ and $B(t)$ are two continuous bounded increasing processes with $A(0) = B(0) = 0$. By [52, Theorem 1.3.9], we deduce that $\lim_{t\to\infty} N(t)$ exists and is finite a.s., thereby confirming the validity of (3.1). Thus, the proof is completed. □

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