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Perceptive movement of susceptible individuals with memory

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Abstract

The perception of susceptible individuals naturally lowers the transmission probability of an infectious disease but has been often ignored. In this paper, we formulate and analyze a diffusive SIS epidemic model with memory-based perceptive movement, where the perceptive movement describes a strategy for susceptible individuals to escape from infections. We prove the global existence and boundedness of a classical solution in an n-dimensional bounded smooth domain. We show the threshold-type dynamics in terms of the basic reproduction number R_0: when $R_0 < 1$, the unique disease-free equilibrium is globally asymptotically stable; when $R_0 > 1$, there is a unique constant endemic equilibrium, and the model is uniformly persistent. Numerical analysis exhibits that when $R_0 > 1$, solutions converge to the endemic equilibrium for slow memory-based movement and they converge to a stable periodic solution when memory-based movement is fast. Our results imply that the memory-based movement cannot determine the extinction or persistence of infectious disease, but it can change the persistence manner.

Keywords SIS epidemic model · Memory-based movement · Global existence · Extinction · Persistence

Mathematics Subject Classification 92D30 · 92D50 · 37N25 · 35B40

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1 Introduction

Mathematical modeling in epidemiology has made significant contributions to novel mathematical techniques and public health research (Hethcote 2000; Kermack and McKendrick 1927; Martcheva 2015; May and Anderson 1979). Theoretical studies can uncover the underlying mechanisms of infectious disease outbreaks and test the effectiveness of intervention strategies. For example, quantitative research on COVID-19 pandemics helped understand and control the spread of this infectious disease, see Feng et al. (2022), Hou et al. (2020), Huo et al. (2021), Li et al. (2020b) and many other related papers. Significant efforts have been done by Fred Brauer on modeling the disease spread and estimating the epidemic final size (Brauer 2019; Brauer and Castillo-Chavez 2012; Xiao et al. 2016). In this paper, we consider the SIS (susceptible-infected-susceptible) type model, where the population is divided into two groups: susceptible individuals and infective individuals. SIS-type models are appropriate for gonorrhea and other sexually transmitted or bacterial agent diseases, in which infected individuals do not develop immunity against re-infection so that they become susceptible almost immediately after recovery (Hethcote and Yorke 1984). The simplest deterministic SIS epidemic model with standard incidence is

$$\begin{cases} \frac{dS}{dt} = \gamma I - \frac{\beta SI}{S+I}, & t > 0, \\ \frac{dI}{dt} = -\gamma I + \frac{\beta SI}{S+I}, & t > 0, \end{cases} \tag{1.1}$$

where $S(t)$ and $I(t)$ are the population densities of susceptible and infective individuals at time t, respectively. Here $\beta > 0$ is the disease transmission rate and $\gamma > 0$ denotes the recovery rate. The initial values satisfy

$$S(0) + I(0) = N > 0.$$

Obviously, the total population is conserved, that is,

$$S(t) + I(t) = N, \ \text{for any } t > 0.$$

The basic reproduction number is a threshold parameter in disease transmission models and measures the number of new infective individuals produced by one infective individual during its lifetime. For model (1.1), the basic reproduction number is defined as

$$R_0 = \frac{\beta}{\gamma}. \tag{1.2}$$

Let $S(t) = N - I(t)$ and substitute it into the second equation of (1.1), then we have

$$\frac{dI}{dt} = I\left(\beta - \gamma - \frac{\beta}{N} I\right), \ t > 0.$$

Therefore, we see that if $R_0 \leq 1$, then the infectious disease dies out; if $R_0 > 1$, then the infectious disease can spread.

The spatial movement of populations plays a vital role in the spread of infectious diseases. As a mean-field approximation of collective individual movements, we consider the following SIS reaction-diffusion model:

$$\begin{cases} \frac{\partial S}{\partial t} = -div(J(x,t)) + \left(-\frac{\beta(x)S}{S+I} + \gamma(x)\right) I, & x \in \Omega,\ t > 0, \\ \frac{\partial I}{\partial t} = -div(K(x,t)) + \left(\frac{\beta(x)S}{S+I} - \gamma(x)\right) I, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

where $S = S(x,t)$ and $I = I(x,t)$ respectively represent the densities of susceptible and infective individuals at location x and time t. Here $J(x,t)$ and $K(x,t)$ are the population fluxes of susceptible and infective individuals, respectively. We assume that the rates of disease transmission and recovery at x, $\beta(x)$ and $\gamma(x)$ respectively, are positive Hölder-continuous functions on $\overline{\Omega}$. Naturally, susceptible individuals tend to stay away intentionally from infections. Following the same logic as in Shi et al. (2020), we utilize the modified Fick's law to specify the movement term of susceptible individuals as

$$J(x,t) = -d_S \nabla S(x,t) - \sigma d_S S(x,t) \nabla f(I),$$

where $d_S > 0$ is the Fickian diffusion coefficient, $\sigma d_S > 0$ is the memory-based diffusion coefficient, and f is a function representing the population density of infected individuals before the present time. On the other hand, we assume that the infected individuals do not care where susceptible individuals and other infections are. Therefore, the diffusion flux of the infected population is described as

$$K(x,t) = -d_I \nabla I(x,t),$$

where $d_I > 0$ is the random diffusion coefficient.

In Shi et al. (2020), it is assumed that f is the identity function, and only the memory at $\tau > 0$ time units ago is important, then

$$f(I) = I(x, t-\tau),$$

where τ is the average memory period. It would be more realistic to take

$$f(I) = (g ** I)(x,t) = \int_{-\infty}^{t} \int_{\Omega} G(x,y,t-s) g(t-s) I(y,s) dy ds, \tag{1.4}$$

where $G(x,y,t)$ is the spatial weighting function that measures the memory of susceptible individuals at location x for those infections at location y. The temporal kernel function $g(t-s)$ accounts for the weighting of infected population at time s on the memory of susceptible population at time $t > s$, see Britton (1990), Shi et al. (2021). Such a form of f describes that the memory over all previous time has a contribution to the movement at present, but the effect at each time unit may be not equally important; the memory of susceptible individuals also depends on the distance of past distribution of infected individuals from their current location. We further impose the following conditions on $G(x,y,t)$ and $g(t)$:

$$\int_{\Omega} G(x, y, t)dy = 1,\ x \in \Omega,\ t > 0,\ \text{and} \int_0^{\infty} g(t)dt = 1. \tag{1.5}$$

In this paper, we choose $G(x, y, t)$ as the solution of

$$\begin{cases} \frac{\partial G}{\partial t}(x, y, t) = d_I \Delta_x G(x, y, t), & x \in \Omega, t > 0, \\ \frac{\partial G}{\partial \nu}(x, y, t) = 0, & x \in \partial\Omega, t > 0, \\ G(x, y, 0) = \delta(x - y), \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded, connected open region with C^2 boundary $\partial\Omega$, ν is the outward unit normal vector at the boundary $\partial\Omega$, and $\delta(\cdot)$ is Dirac function. Thus,

$$G(x, y, t) = \sum_{k=0}^{\infty} e^{-d_I \lambda_k t} \phi_k(x)\phi_k(y), \tag{1.6}$$

where λ_k is the k-th eigenvalue of the following eigenvalue problem:

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.7}$$

and satisfies $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty$ as $k \to \infty$, $\phi_k(x)$ is the normalized eigenfunction corresponding to the eigenvalue λ_k such that (1.6) satisfies (1.5). Moreover, we from (1.4) see that

$$\frac{\partial f(I)}{\partial \nu} = \int_{-\infty}^{t} \int_{\Omega} \frac{\partial G}{\partial \nu}(x, y, t - s)g(t - s)I(y, s)dyds = 0,\ x \in \partial\Omega.$$

For the delay kernel $g(t)$, we choose the Gamma distribution function of order 0 and 1, that is,

$$g_0(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}},\quad g_1(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}}, \tag{1.8}$$

which are known as weak kernel and strong kernel, respectively. The weak kernel reflects that memory continues to decay as time goes by, while the strong kernel describes the processes of memory acquirement and decay as time goes by. For simplicity, we assume $\beta(x)$ and $\gamma(x)$ are positive constants meaning that the disease transmission and recovery are independent of space, and confine the movement to domain Ω. Then we have the following model:

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S + \sigma d_S \nabla \cdot [S \nabla f(I)] + \gamma I - \frac{\beta S I}{S + I}, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} = d_I \Delta I - \gamma I + \frac{\beta S I}{S + I}, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ S(x, 0) = S_0(x),\ I(x, t) = \eta(x, t), & x \in \Omega, t \in (-\infty, 0], \end{cases} \tag{1.9}$$

where f is in the form of (1.4). Throughout this paper, we assume that the initial values satisfy

(A1) $S_0(x),\ \eta(x,t) \geq (\not\equiv) 0$, for $x \in \Omega, t \in (-\infty, 0]$. $S_0(x) \in C(\overline{\Omega})$, $\eta(x,t) \in C((-\infty, 0]; W^{1,\infty}(\Omega))$, and

$$\int_\Omega [S_0(x) + \eta(x,0)]dx = N,$$

and

$$\int_{-\infty}^0 \int_\Omega G(x, y, -s) g_j(-s) \eta(y, s) dy ds \in C(\overline{\Omega}),\ \ j = 0, 1. \tag{1.10}$$

It is easy to see that the total number of individuals in Ω satisfies

$$\int_\Omega [S(x,t) + I(x,t)]dx = N,\ \text{ for any } t > 0. \tag{1.11}$$

In the case of $\sigma = 0$, there have been many studies on system (1.9) in spatially and/or temporally varying environments. When β and γ are spatially dependent, Allen et al. (2008) defined the basic reproduction number R_0, based on which they analyzed the stability of disease-free equilibrium and the existence of endemic equilibrium, and it turned out that a disease can persist by limiting the movement of susceptible individuals. The global stability of endemic equilibrium was not given in Allen et al. (2008), but it was established by Peng and Liu (2009) in two cases: $d_S = d_I$ and $\beta(x) = a\gamma(x)$ with $a \in (0, \infty)$. When β and γ are spatially heterogeneous and temporally periodic, Peng and Zhao (2012) investigated the extinction and persistence of disease in terms of R_0, and revealed that spatiotemporal heterogeneities can enhance the persistence of the disease. The interested readers can refer to Peng (2009), Cui et al. (2017), Chen et al. (2020), Deng and Wu (2016), Song and Xiao (2022), Gao and Dong (2020), Ge et al. (2015), Wang et al. (2022) for other efforts on (1.9) with $\sigma = 0$ and its variants.

In the case of $\sigma > 0$, Li et al. (2020a) showed that cross-diffusion had no contribution to the elimination of diseases. The cross-diffusion in Li et al. (2020a) can be understood as a strategy for susceptible individuals to avoid infection, which is similar to what we focus on here. However, we assume that such avoidance depends on the memory of susceptible individuals. It is a nonlocal, temporal distribution average of historic infective population, instead of exactly the current infective population considered in Li et al. (2020a). The spatial and temporal distributed delay has biological implications (see Britton 1990 for the derivation). Much attention was paid to the existence and properties of travelling wave solutions in reaction-diffusion models (see Ai 2007; Li et al. 2007; Fang et al. 2008 and the references therein). The effect of such delay on stability/instability of a positive steady state can be found in Gourley and So (2002), Chen and Yu (2016), Zuo and Shi (2021), Shi et al. (2021), Zuo and Song (2015), where Shi et al. (2021) firstly applied the form of (1.4) to a memory-based diffusive movement model.

Clearly, system (1.9) has two constant steady states: the disease-free equilibrium (DFE) given by $(\tilde{S}_0, 0) = \left(\frac{N}{|\Omega|}, 0\right)$, and the endemic equilibrium (EE) given by $(S_*, I_*) = \frac{N}{|\Omega|}\left(\frac{\gamma}{\beta}, 1 - \frac{\gamma}{\beta}\right)$ that is biologically meaningful if $\beta > \gamma$. In this paper, we aim to study the effect of memory-based perceptive movement of susceptible individuals on spatiotemporal dynamics. To be more specific, we define the basic reproduction number as (1.2) that is independent of the diffusion rates d_S, d_I and σd_S. We show that if $R_0 < 1$, then DFE is globally asymptotically stable; if $R_0 > 1$, then system (1.9) is uniformly persistent. It turns out that memory-based movement has no influence on the threshold dynamics, but numerical analysis exhibits that EE is asymptotically stable for small σ, and periodic solutions exist near EE for large σ. From the biological perspective, when the transmission rate is less than the recovery rate, the disease goes extinct; when the transmission rate is greater than the recovery rate, the disease persists. The avoidance movement of susceptible individuals cannot eradicate an infectious disease by itself. However, a high avoidance speed leads to the persistence in periodic oscillations, and a low avoidance speed can control the disease at a stable steady state.

The remaining paper is organized as follows. In Sect. 2, we show the global existence and boundedness of solutions of (1.9) by analyzing two systems without spatiotemporal delay, and then study the threshold dynamics in terms of R_0. In Sect. 3, we provide some numerical examples to illustrate and complement our theoretical results. In the last section, we summarize and discuss our paper.

2 Mathematical results

In this section, we study the global existence, boundedness and positivity of the classical solutions of system (1.9). We will also analyze the global stability of DFE and the uniform persistence of system (1.9).

According to Propositions 2.3 and 2.4 in Zuo and Shi (2021) (see also Lemmas 1 and 2 in Shi et al. 2021), we have the following relationships between system (1.9) with spatial kernel G given by (1.6) and temporal kernel g taken as (1.8) and systems without spatiotemporal delay. Denote

$$\begin{aligned} u_j(x,t) &= (g_j ** I)(x,t) \\ &= \int_{-\infty}^{t} \int_{\Omega} G(x,y,t-s) g_j(t-s) I(y,s) dy ds, \quad j = 0, 1. \end{aligned} \tag{2.1}$$

Lemma 2.1 *Suppose that $g(t) = g_0(t)$, where $g_0(t)$ is given in (1.8). Then the following statements hold:*

(i) *If $(S(x,t), I(x,t))$ is the solution of system (1.9), then $(S(x,t), I(x,t), u_0(x,t))$ is the solution of*

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S + \sigma d_S \nabla \cdot (S \nabla u) + \gamma I - \frac{\beta SI}{S+I}, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} = d_I \Delta I - \gamma I + \frac{\beta SI}{S+I}, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial t} = d_I \Delta u + \frac{1}{\tau}(I - u), & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ S(x,0) = S_0(x), \quad I(x,0) = \eta(x,0), & x \in \Omega, \\ u(x,0) = \int_{-\infty}^{0} \int_\Omega G(x,y,-s) g_0(-s) \eta(y,s) dy ds, & x \in \Omega, \end{cases} \tag{2.2}$$

where $u_0(x,t)$ *is given by* (2.1) *with* $j = 0$.

(ii) *If* $(S(x,t), I(x,t), u(x,t))$ *is the solution of*

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S + \sigma d_S \nabla \cdot (S \nabla u) + \gamma I - \frac{\beta SI}{S+I}, & x \in \Omega, t \in \mathbb{R}, \\ \frac{\partial I}{\partial t} = d_I \Delta I - \gamma I + \frac{\beta SI}{S+I}, & x \in \Omega, t \in \mathbb{R}, \\ \frac{\partial u}{\partial t} = d_I \Delta u + \frac{1}{\tau}(I - u), & x \in \Omega, t \in \mathbb{R}, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t \in \mathbb{R}, \\ S(x,0) = S_0(x), \quad I(x,s) = \eta(x,s), & x \in \Omega, s \in (-\infty, 0], \\ u(x,t_0) = \int_{-\infty}^{t_0} \int_\Omega G(x,y,t_0 - s) g_0(t_0 - s) \eta(y,s) dy ds, & x \in \Omega, t_0 \le 0, \end{cases} \tag{2.3}$$

then $(S(x,t), I(x,t))$ *satisfies* (1.9). *In addition, if* $(S(x,t), I(x,t), u(x,t))$ *is a steady state of system* (2.3)*, then* $(S(x,t), I(x,t))$ *is a steady state of system* (1.9)*; if* $(S(x,t), I(x,t), u(x,t))$ *is a periodic solution of system* (2.3) *with period* T*, then* $(S(x,t), I(x,t))$ *is a periodic solution of system* (1.9) *with period* T.

Lemma 2.2 *Suppose that* $g(t) = g_1(t)$*, where* $g_1(t)$ *is given in* (1.8)*. Then the following statements hold:*

(i) *If* $(S(x,t), I(x,t))$ *is the solution of system* (1.9)*, then* $(S(x,t), I(x,t), u_1(x,t), u_0(x,t))$ *is the solution of*

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S + \sigma d_S \nabla \cdot (S \nabla u) + \gamma I - \frac{\beta SI}{S+I}, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} = d_I \Delta I - \gamma I + \frac{\beta SI}{S+I}, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial t} = d_I \Delta u + \frac{1}{\tau}(v - u), & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_I \Delta v + \frac{1}{\tau}(I - v), & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ S(x,0) = S_0(x), \ I(x,0) = \eta(x,0), & x \in \Omega, \\ u(x,0) = \int_{-\infty}^{0} \int_\Omega G(x,y,-s) g_1(-s) \eta(y,s) dy ds, & x \in \Omega, \\ v(x,0) = \int_{-\infty}^{0} \int_\Omega G(x,y,-s) g_0(-s) \eta(y,s) dy ds, & x \in \Omega, \end{cases} \tag{2.4}$$

where $u_1(x,t)$ *and* $u_0(x,t)$ *are given by* (2.1).

(ii) *If* $(S(x,t), I(x,t), u(x,t), v(x,t))$ *is the solution of*

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S + \sigma d_S \nabla \cdot (S \nabla u) + \gamma I - \frac{\beta SI}{S+I}, & x \in \Omega, t \in \mathbb{R}, \\ \frac{\partial I}{\partial t} = d_I \Delta I - \gamma I + \frac{\beta SI}{S+I}, & x \in \Omega, t \in \mathbb{R}, \\ \frac{\partial u}{\partial t} = d_I \Delta u + \frac{1}{\tau}(v - u), & x \in \Omega, t \in \mathbb{R}, \\ \frac{\partial v}{\partial t} = d_I \Delta v + \frac{1}{\tau}(I - v), & x \in \Omega, t \in \mathbb{R}, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in \mathbb{R}, \\ S(x,0) = S_0(x), \ I(x,s) = \eta(x,s), & x \in \Omega, s \in (-\infty, 0], \\ u(x,t_0) = \int_{-\infty}^{t_0} \int_\Omega G(x,y,t_0 - s) g_1(t_0 - s) \eta(y,s) dy ds, & x \in \Omega, t_0 \le 0, \\ v(x,t_0) = \int_{-\infty}^{t_0} \int_\Omega G(x,y,t_0 - s) g_0(t_0 - s) \eta(y,s) dy ds, & x \in \Omega, t_0 \le 0, \end{cases} \tag{2.5}$$

then $(S(x,t), I(x,t))$ *satisfies* (1.9). *In addition, if* $(S(x,t), I(x,t), u(x,t), v(x,t))$ *is a steady state of system* (2.5)*, then* $(S(x,t), I(x,t))$ *is a steady state of system* (1.9)*; if* $(S(x,t), I(x,t), u(x,t), v(x,t))$ *is a periodic solution of system* (2.5) *with period* T*, then* $(S(x,t), I(x,t))$ *is a periodic solution of system* (1.9) *with period* T.

2.1 Wellposedness

Denote $P_0(x) = (S_0(x), \eta(x,0), u(x,0))$, $Z_0(x) = (S_0(x), \eta(x,0), u(x,0), v(x,0))$, and

$$\begin{aligned} X_w &= \left\{ P_0(x) \in [W^{1,\infty}(\Omega)]^3 : \int_\Omega [S_0(x) + \eta(x,0)] dx = N \right\}, \\ X_s &= \left\{ Z_0(x) \in [W^{1,\infty}(\Omega)]^4 : \int_\Omega [S_0(x) + \eta(x,0)] dx = N \right\}. \end{aligned} \tag{2.6}$$

When the initial values of system (2.2) are in X_w, and the initial values of system (2.4) are in X_s, similar to (Horstmann and Winkler 2005, Theorem 3.1) (see also Li et al. 2020a, Lemma 2.2), we can obtain the local existence of solutions for systems (2.2) and (2.4), respectively. Then by the strong maximum principle, the solutions are positive on their existence intervals if $P_0(x) \ge (\not\equiv) 0$ and $Z_0(x) \ge (\not\equiv) 0$ for $x \in \Omega$. The result is summarized as below.

Lemma 2.3 *For any* $P_0(x) \in X_w$ *with* $P_0(x) \ge (\not\equiv) 0$ *for* $x \in \Omega$*, there exists* $T_m \in (0, \infty]$ *such that system* (2.2) *has a unique positive classical solution* $(S(x,t), I(x,t), u(x,t))$*, which satisfies*

$$\begin{aligned} &S(x,t), I(x,t) \in C(\overline{\Omega} \times [0, T_m)) \cap C^{2,1}(\overline{\Omega} \times (0, T_m)), \\ &u(x,t) \in C(\overline{\Omega} \times [0, T_m)) \cap C^{2,1}(\overline{\Omega} \times (0, T_m)) \cap L^\infty_{loc}([0, T_m); W^{1,p}(\Omega)), \end{aligned}$$

for any $p > 1$. *In addition, either* $T_m = \infty$*, or for any* $p > 1$,

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} + \|u(\cdot,t)\|_{W^{1,p}(\Omega)} \to \infty, \ \text{as } t \nearrow T_m.$$

The same conclusion on local-in-time wellposedness holds for system (2.4) *with initial value* $Z_0(x) \in X_s$ *and* $Z_0(x) \geq (\not\equiv) 0$ *for* $x \in \Omega$.

To study the global existence and uniform-in-time boundedness for systems (2.2) and (2.4), we introduce Lemma 2.4 and Lemma 2.6 as follows. The proof of Lemma 2.4 can be found in (Winkler 2010, Lemma 1.3) and (Peng and Zhao 2015, Lemma 2.1). Lemma 2.6 directly follows (Peng and Zhao 2012, Lemma 3.1), see also Dung (1997, Theorem 1 and Corollary 1).

Lemma 2.4 *Let* $(e^{td\Delta})_{t\geq 0}$ *be the Neumann heat semigroup on* $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, *and denote* λ_1 *as the first positive eigenvalue of* $-\Delta$ *on* Ω *under Neumann boundary conditions. Then there exist positive constants* c_i, $i = 1, 2, 3, 4$, *which depend on* Ω, *such that the following results hold:*

(i) *If* $1 \leq q \leq p \leq \infty$, *then*

$$\left\| e^{td\Delta} f \right\|_{L^p(\Omega)} \leq c_1 e^{-\lambda_1 dt} \left(1 + t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right) \|f\|_{L^q(\Omega)}, \text{ for all } t > 0,$$

holds for all $f \in L^q(\Omega)$ *satisfying* $\int_\Omega f\,dx = 0$.

(ii) *If* $1 \leq q \leq p \leq \infty$, *then*

$$\left\| \nabla e^{td\Delta} f \right\|_{L^p(\Omega)} \leq c_2 e^{-\lambda_1 dt} \left(1 + t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right) \|f\|_{L^q(\Omega)}, \text{ for all } t > 0,$$

holds for all $f \in L^q(\Omega)$.

(iii) *If* $2 \leq q \leq p < \infty$, *then*

$$\left\| \nabla e^{td\Delta} f \right\|_{L^p(\Omega)} \leq c_3 e^{-\lambda_1 dt} \left(1 + t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right) \|\nabla f\|_{L^q(\Omega)}, \text{ for all } t > 0,$$

holds for all $f \in W^{1,q}(\Omega)$.

(iv) *If* $1 < q \leq p \leq \infty$, *then*

$$\left\| e^{td\Delta} \nabla \cdot f \right\|_{L^p(\Omega)} \leq c_4 e^{-\lambda_1 dt} \left(1 + t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right) \|f\|_{L^q(\Omega)}, \text{ for all } t > 0,$$

holds for all $f \in (L^q(\Omega))^n$.

Remark 2.5 Based on Lemma 2.4(i), we claim that if $1 \leq q \leq p \leq \infty$, then there exists a positive constant $\tilde{c}_1$ such that

$$\left\| e^{td\Delta} f \right\|_{L^p(\Omega)} \leq \tilde{c}_1 \left(1 + t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right) \|f\|_{L^q(\Omega)}, \text{ for all } t > 0, \tag{2.7}$$

holds for all $f \in L^q(\Omega)$.

Proof Let $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f dx$, then $e^{td\Delta}\bar{f} = \bar{f}$. It follows from Lemma 2.4(i) and Minkowski inequality that

$$\begin{aligned}\left\|e^{td\Delta} f\right\|_{L^p(\Omega)} &= \left\|e^{td\Delta}(f-\bar{f})+\bar{f}\right\|_{L^p(\Omega)}\\ &\leq \left\|e^{td\Delta}(f-\bar{f})\right\|_{L^p(\Omega)} + \|\bar{f}\|_{L^p(\Omega)}\\ &\leq c_1 e^{-\lambda_1 dt}\left(1+t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)\|f-\bar{f}\|_{L^q(\Omega)} + \|\bar{f}\|_{L^p(\Omega)}\\ &\leq c_1 e^{-\lambda_1 dt}\left(1+t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)(\|f\|_{L^q(\Omega)}+\|\bar{f}\|_{L^q(\Omega)}) + \|\bar{f}\|_{L^p(\Omega)}.\end{aligned} \tag{2.8}$$

By Hölder inequality, we have

$$\begin{aligned}\|\bar{f}\|_{L^q(\Omega)} &= |\bar{f}||\Omega|^{\frac{1}{q}} \leq |\Omega|^{\frac{1}{q}-1}\int_\Omega |f| dx\\ &\leq |\Omega|^{\frac{1}{q}-1}\left(\int_\Omega |f|^q dx\right)^{\frac{1}{q}}\left(\int_\Omega 1 dx\right)^{1-\frac{1}{q}} = \|f\|_{L^q(\Omega)},\end{aligned} \tag{2.9}$$

Similarly to (2.9), we can obtain

$$\|\bar{f}\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(\Omega)}. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8), the claim is proved. □

Lemma 2.6 *Consider the parabolic system*

$$\begin{cases}\frac{\partial u_i}{\partial t} = d_i\Delta u_i + f_i(x,t,u), & x\in\Omega,\ t>0,\ i=1,\ldots,m,\\ \frac{\partial u_i}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_i(x,0)=u_i^0(x), & x\in\Omega,\end{cases}$$

where $u=(u_1,\ldots,u_m)$, $u_i^0(x)\in C(\bar{\Omega})$ and $d_i>0$ are constants, $i=1,\ldots,m$. Assume that for each $i=1,\ldots,m$, $f_i(x,t,u)$ satisfy the polynomial growth condition:

$$|f_i(x,t,u)| \leq \kappa_1\sum_{k=1}^m |u_k|^\delta + \kappa_2$$

for some nonnegative constants κ_1 and κ_2, and positive constant δ. Let p_0 be a positive constant such that $p_0 > \frac{n}{2}\max\{0,(\delta-1)\}$, and denote the maximal existence time of the solution $u(x,t)$ corresponding to the initial data u^0 by $r(u^0)$. Suppose that there exists a positive constant $C_{p_0}(u^0)$ such that

$$\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \leq C_{p_0}(u^0),\ \textit{for all } t\in(0,r(u^0)).$$

Then the solution $u(x,t)$ exists for all time and there is a positive constant C_∞ such that

$$\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \le C_\infty(u^0), \text{ for all } t \in (0,\infty).$$

Moreover, if there exists finite numbers ρ and K_ρ independent of the initial data u^0 such that

$$\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \le K_{p_0}(\rho), \text{ for all } t \in [\rho,\infty),$$

then there exists a positive constant $K_\infty(\rho)$ independent of the initial data u^0 such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le K_\infty(\rho), \text{ for all } t \in [\rho,\infty).$$

The following theorem shows the global existence of classical solutions for systems (2.2) and (2.4).

Theorem 2.7 *For any initial value $P_0(x) \in X_w$ with $P_0(x) \ge (\not\equiv)0$, $x \in \Omega$, system (2.2) has a unique global classical solution $(S(x,t), I(x,t), u(x,t))$ which is positive and uniformly bounded in $\overline{\Omega} \times (0,\infty)$ in the sense that there exists a constant $K > 0$, which depends on $P_0(x)$ and parameters in the system, such that*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} + \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le K, \ t > 0. \tag{2.11}$$

Moreover, there exists a constant $\tilde{K} > 0$ independent of $P_0(x)$ but dependent on parameters in the system such that for some large $T > 0$,

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \tilde{K}, \ t > T. \tag{2.12}$$

The same global existence and uniform-in-time boundedness holds for the solution of system (2.4) with the initial value $Z_0(x) \in X_s$ and $Z_0(x) \ge (\not\equiv)0$ for $x \in \Omega$.

Proof Based on Lemma 2.3, it suffices to show $T_m = \infty$ and the global boundedness of the solution $(S(x,t), I(x,t), u(x,t))$ or $(S(x,t), I(x,t), u(x,t), v(x,t))$.

Rewrite the I-equation in system (2.2) as

$$\begin{cases} \frac{\partial I}{\partial t} = d_I \Delta I + B(x,t) I, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ I(x,0) = \eta(x,0), & x \in \Omega, \end{cases} \tag{2.13}$$

where

$$B(x,t) = \frac{\beta S}{S+I} - \gamma.$$

It can be seen that $B(x,t) \le \beta+\gamma$ for all $(x,t) \in \Omega\times(0,T_m)$, and is locally Lipschitz in (x,t). On the other hand, (1.11) indicates that $\|I(\cdot,t)\|_{L^1(\Omega)} \le N$ for $t \in (0,T_m)$. It then follows from (Alikakos 1979, Theorem 3.1) that there exists a constant $M_1 > 0$ such that

$$\|I(\cdot,t)\|_{L^\infty(\Omega)} \le M_1, \quad t \in (0,T_m), \tag{2.14}$$

where M_1 depends on N and $\|\eta(\cdot,0)\|_{L^\infty(\Omega)}$.

Via the variation-of-constants formula, we have

$$u(\cdot,t) = e^{(d_I\Delta-1/\tau)t}u(\cdot,0) + \frac{1}{\tau}\int_0^t e^{(d_I\Delta-1/\tau)(t-s)}I(\cdot,s)ds,\ t \in (0,T_m). \tag{2.15}$$

Take supremum on both sides of (2.15), then we have for any $t \in (0,T_m)$,

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\Omega)} \le &\left\|e^{(d_I\Delta-1/\tau)t}u(\cdot,0)\right\|_{L^\infty(\Omega)}\\ &+ \frac{1}{\tau}\int_0^t \left\|e^{(d_I\Delta-1/\tau)(t-s)}I(\cdot,s)\right\|_{L^\infty(\Omega)} ds.\end{aligned} \tag{2.16}$$

By Remark 2.5, we have for any $t \in (0,T_m)$,

$$\left\|e^{(d_I\Delta-1/\tau)t}u(\cdot,0)\right\|_{L^\infty(\Omega)} = e^{-t/\tau}\left\|e^{td_I\Delta}u(\cdot,0)\right\|_{L^\infty(\Omega)} \le 2\tilde{c}_1e^{-t/\tau}\|u(\cdot,0)\|_{L^\infty(\Omega)}. \tag{2.17}$$

Moreover, for any $t \in (0,T_m)$,

$$\frac{1}{\tau}\int_0^t \left\|e^{(d_I\Delta-1/\tau)(t-s)}I(\cdot,s)\right\|_{L^\infty(\Omega)} ds \le \frac{2M_1\tilde{c}_1}{\tau}\int_0^t e^{-(t-s)/\tau}ds < 2M_1\tilde{c}_1.$$

Therefore, there exists a constant $M_2 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le M_2, \quad t \in (0,T_m). \tag{2.18}$$

Again using (2.15), we have

$$\begin{aligned}\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} \le\ & e^{-t/\tau}\left\|\nabla e^{td_I\Delta}u(\cdot,0)\right\|_{L^\infty(\Omega)}\\ &+ \frac{1}{\tau}\int_0^t e^{-(t-s)/\tau}\left\|\nabla e^{(t-s)d_I\Delta}I(\cdot,s)\right\|_{L^\infty(\Omega)} ds.\end{aligned} \tag{2.19}$$

In light of Lemma 2.4(iii), for any $p \in [2,\infty)$ and $t \in (0,T_m)$,

$$\begin{aligned}\|\nabla e^{td_I\Delta}u(\cdot,0)\|_{L^p(\Omega)} &\le 2c_3e^{-\lambda_1d_It}\|\nabla u(\cdot,0)\|_{L^p(\Omega)}\\ &\le 2c_3e^{-\lambda_1d_It}\max\left\{1,|\Omega|^{\frac{1}{2}}\right\}\|\nabla u(\cdot,0)\|_{L^\infty(\Omega)}.\end{aligned}$$

Consequently, there exists a constant $M_3 > 0$ such that

$$e^{-t/\tau}\|\nabla e^{td_I\Delta}u(\cdot,0)\|_{L^\infty(\Omega)} \le M_3\|u(\cdot,0)\|_{W^{1,\infty}(\Omega)}, \quad t\in(0,T_m). \tag{2.20}$$

In addition, from Lemma 2.4(ii), we see that for $t\in(s,T_m)$

$$\left\|\nabla e^{(t-s)d_I\Delta}I(\cdot,s)\right\|_{L^\infty(\Omega)} \le c_2e^{-\lambda_1 d_I(t-s)}\left(1+(t-s)^{-\frac{1}{2}}\right)\|I(\cdot,s)\|_{L^\infty(\Omega)}. \tag{2.21}$$

Based on (2.19), (2.20) and (2.21) and (2.14), we have

$$\begin{aligned}\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} &\le M_4\int_0^t e^{-(\lambda_1 d_I+1/\tau)(t-s)}\left(1+(t-s)^{-\frac{1}{2}}\right)ds\\ &\quad + M_3\|u(\cdot,0)\|_{W^{1,\infty}(\Omega)} \le M_5, \text{ for } t\in(0,T_m).\end{aligned} \tag{2.22}$$

For any $T\in(0,T_m)$ and $p>n$, the S-equation satisfies

$$\frac{\partial S}{\partial t} - d_S\Delta S + S = \sigma d_S\nabla\cdot(S\nabla u) + S + \gamma I - \frac{\beta SI}{S+I}.$$

It follows from the variation-of-constants formula that

$$\begin{aligned}S(\cdot,t) &= e^{t(d_S\Delta-1)}S_0(\cdot) + \sigma d_S\int_0^t e^{(t-s)(d_S\Delta-1)}\nabla\cdot(S(\cdot,s)\nabla u(\cdot,s))ds\\ &\quad + \int_0^t e^{(t-s)(d_S\Delta-1)}\left(S+\gamma I-\frac{\beta SI}{S+I}\right)(\cdot,s)ds, \quad t\in(0,T).\end{aligned}$$

Taking supremum on both sides of the above equation yields

$$\begin{aligned}\|S(\cdot,t)\|_{L^\infty(\Omega)} &\le \|e^{t(d_S\Delta-1)}S_0(\cdot)\|_{L^\infty(\Omega)}\\ &\quad + \sigma d_S\int_0^t \left\|e^{(t-s)(d_S\Delta-1)}\nabla\cdot(S(\cdot,s)\nabla u(\cdot,s))\right\|_{L^\infty(\Omega)}ds\\ &\quad + \int_0^t \left\|e^{(t-s)(d_S\Delta-1)}\left(S+\gamma I-\frac{\beta SI}{S+I}\right)(\cdot,s)\right\|_{L^\infty(\Omega)}ds\\ &:= \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3, \quad t\in(0,T).\end{aligned} \tag{2.23}$$

Similar to (2.17), we have

$$\mathrm{I}_1 = e^{-t}\|e^{td_S\Delta}S_0(\cdot)\|_{L^\infty(\Omega)} \le 2\tilde{c}_1e^{-t}\|S_0(\cdot)\|_{L^\infty(\Omega)}, \text{ for all } t\in(0,T). \tag{2.24}$$

Define

$$\check{S} = \check{S}(T) := \sup_{t\in(0,T)}\|S(\cdot,t)\|_{L^\infty(\Omega)}.$$

Along with Lemma 2.4(iv) and (2.22), we see that

$$
\begin{aligned}
\mathrm{I}_2 &\leq \sigma d_S \int_0^t e^{-(t-s)} \left\| e^{(t-s)d_S\Delta} \nabla \cdot (S(\cdot,s)\nabla u(\cdot,s)) \right\|_{L^\infty(\Omega)} ds \\
&\leq \sigma d_S c_4 \int_0^t e^{-(1+\lambda_1 d_S)(t-s)} \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) \|S(\cdot,s)\nabla u(\cdot,s)\|_{L^p(\Omega)} ds \\
&\leq M_6 \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) \|S(\cdot,s)\|_{L^p(\Omega)} ds \\
&\leq M_6 \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) \|S(\cdot,s)\|_{L^\infty(\Omega)}^{1-\frac{1}{p}} \|S(\cdot,s)\|_{L^1(\Omega)}^{\frac{1}{p}} ds \\
&\leq M_6 \check{S}^{1-\frac{1}{p}} N^{\frac{1}{p}} \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) ds \\
&\leq M_6 \tilde{M}_6 \check{S}^{1-\frac{1}{p}} N^{\frac{1}{p}}, \text{ for all } t \in (0,T),
\end{aligned} \tag{2.25}
$$

where the condition $p > n$ guarantees that

$$
\begin{aligned}
\tilde{M}_6 &= \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) ds \\
&\leq 1 + \int_0^\infty e^{-t} t^{-\frac{1}{2}-\frac{n}{2p}} dt = 1 + \Gamma\left(\frac{1}{2} - \frac{n}{2p}\right) < \infty.
\end{aligned}
$$

By virtue of Remark 2.5 and (1.11), it holds that

$$
\begin{aligned}
\mathrm{I}_3 &\leq \int_0^t e^{-(t-s)} \left\| e^{(t-s)d_S\Delta} S(\cdot,s) \right\|_{L^\infty(\Omega)} ds \\
&\quad + \int_0^t e^{-(t-s)} \left\| e^{(t-s)d_S\Delta} \left(\gamma I - \frac{\beta S I}{S+I}\right)(\cdot,s) \right\|_{L^\infty(\Omega)} ds \\
&\leq \tilde{c}_1 \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{n}{2p}}\right) \|S(\cdot,s)\|_{L^p(\Omega)} ds \\
&\quad + 2\tilde{c}_1(\beta+\gamma) \int_0^t e^{-(t-s)} \|I(\cdot,s)\|_{L^\infty(\Omega)} ds \\
&\leq \tilde{c}_1 \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{n}{2p}}\right) \|S(\cdot,s)\|_{L^\infty(\Omega)}^{1-\frac{1}{p}} \|S(\cdot,s)\|_{L^1(\Omega)}^{\frac{1}{p}} ds + M_7 \\
&\leq M_8 \tilde{M}_8 \check{S}^{1-\frac{1}{p}} N^{\frac{1}{p}} + M_7,
\end{aligned} \tag{2.26}
$$

where

$$
\tilde{M}_8 = \int_0^\infty e^{-t} \left(1 + t^{-\frac{n}{2p}}\right) dt < \infty.
$$

Substituting (2.24), (2.25) and (2.26) into (2.23), we obtain

$$\begin{aligned}\check{S} &\leq (M_6\tilde{M}_6 + M_8\tilde{M}_8)N^{\frac{1}{p}}\check{S}^{1-\frac{1}{p}} + \|S_0(\cdot)\|_{L^\infty(\Omega)} + M_7 \\ &:= M_9\check{S}^{1-\frac{1}{p}} + M_{10}.\end{aligned}$$

Thus, we have

$$\check{S} \leq \max\left\{\left(\frac{M_{10}}{M_9}\right)^{\frac{p}{p-1}}, (2M_9)^p\right\}.$$

Because $T \in (0, T_m)$ is arbitrary, there exists a constant $M_{11} > 0$ such that

$$\|S(\cdot, t)\|_{L^\infty(\Omega)} \leq M_{11}, \quad t \in (0, T_m). \tag{2.27}$$

This together with (2.14) and (2.18) indicates that $T_m = \infty$.

We next show for large t, $\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} + \|u(\cdot, t)\|_{L^\infty(\Omega)}$ has an upper bound that is independent of initial value $P_0(x)$. From Lemma 2.6 with $m = 1$, $p_0 = 1$ and $\delta = 1$ and (1.11), there exists $T_1 > 0$ and a constant $M_{12} > 0$ independent of $P_0(x)$ such that

$$\|I(\cdot, t)\|_{L^\infty(\Omega)} \leq M_{12}, \; t > T_1.$$

Solving u on $[T_1, \infty)$ by variation-of-constants formula, we have

$$u(\cdot, t) = e^{(d_I\Delta - 1/\tau)(t-T_1)}u(\cdot, T_1) + \frac{1}{\tau}\int_{T_1}^t e^{(d_I\Delta-1/\tau)(t-s)}I(\cdot, s)ds, \; t > T_1. \tag{2.28}$$

Applying the similar argument for (2.18), we see that there exists $T_2 > T_1$ and a constant $M_{13} > 0$ independent of $P_0(x)$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq M_{13}, \; \text{for } t > T_2.$$

Moreover, similar to (2.22), we can find a $T_3 \geq T_2$ and a constant $M_{14} > 0$ independent of $P_0(x)$, such that

$$\|\nabla u(\cdot, t)\|_{L^\infty(\Omega)} \leq M_{14}, \; \text{for } t > T_3. \tag{2.29}$$

The component S of the solution on $[T_3, \infty)$ is

$$\begin{aligned}S(\cdot, t) = e^{(t-T_3)(d_S\Delta-1)}S(\cdot, T_3) + \sigma d_S\int_{T_3}^t e^{(t-s)(d_S\Delta-1)}\nabla\cdot(S(\cdot, s)\nabla u(\cdot, s))ds \\ + \int_{T_3}^t e^{(t-s)(d_S\Delta-1)}\left(S + \gamma I - \frac{\beta SI}{S+I}\right)(\cdot, s)ds, \; t > T_3.\end{aligned}$$

Based on (2.29) and repeating the procedure for the proof of (2.27), we can prove that there is an upper bound of $\|S(\cdot,t)\|_{L^\infty(\Omega)}$ independent of $P_0(x)$ for $t > T_3$.

From the derivation of (2.18), we can obtain the L^∞-boundedness of v in system (2.4). Then by the variation-of-constants formula and Lemma 2.4, the conclusion on system (2.4) can be proved as for system (2.2). □

Based on Lemmas 2.1 and 2.2 and Theorem 2.7, we have the following result for system (1.9).

Corollary 2.8 *Suppose that initial value satisfies (A1) and N is fixed. Then system (1.9) has a positive unique global classical solution $(S(x,t), I(x,t))$ satisfying that there exists a constant $K_1 > 0$, which depends on the initial value and parameters in the system, such that*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \le K_1,\ \ t > 0.$$

Moreover, there exists a constant $\tilde{K}_1 > 0$ independent of the initial value but dependent on parameters in the system satisfying

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \le \tilde{K}_1,\ \ t > T,$$

for some large $T > 0$.

2.2 Threshold dynamics

In this section, we investigate the effect of memory-based movement on the global dynamics of system (1.9) with weak kernel and strong kernel, respectively.

In the case of $R_0 < 1$, system (1.9) exactly has a constant steady state DFE $(\tilde{S}_0, 0)$, and we will show that DFE is globally asymptotically stable among solutions of system (1.9) with initial conditions (A1) motivated by Allen et al. (2008, Lemma 2.4).

Lemma 2.9 *If $R_0 < 1$, then DFE is linearly stable; if $R_0 > 1$, then DFE is unstable.*

Proof Letting $\eta(x,t) = S(x,t) - \tilde{S}_0$ and $\xi(x,t) = I(x,t)$ for system (1.9), and linearizing the system at equilibrium (0, 0), we have

$$\begin{cases} \frac{\partial \eta}{\partial t} = d_S \Delta \eta + (\gamma - \beta)\xi, & x \in \Omega, t > 0, \\ \frac{\partial \xi}{\partial t} = d_I \Delta \xi - (\gamma - \beta)\xi, & x \in \Omega, t > 0, \\ \frac{\partial \eta}{\partial \nu} = \frac{\partial \xi}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ \int_\Omega [\eta(x,t) + \xi(x,t)] dx = 0,\ \text{for any } t > 0. \end{cases}$$

Substituting $(\eta, \xi) = \big(e^{-\mu t}\phi(x), e^{-\mu t}\psi(x)\big)$ into the above system yields the following eigenvalue problem

$$\begin{cases} d_S \Delta \phi + (\gamma - \beta)\psi + \mu\phi = 0, & x \in \Omega, \\ d_I \Delta \psi - (\gamma - \beta)\psi + \mu\psi = 0, & x \in \Omega, \\ \frac{\partial \phi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{2.30}$$

and

$$\int_\Omega [\phi(x) + \psi(x)]dx = 0. \tag{2.31}$$

We claim that if (μ, ϕ, ψ) is a solution of (2.30) and (2.31) with $(\phi, \psi) \not\equiv (0, 0)$, then $\psi \not\equiv 0$. If it is not true, then substituting $\psi \equiv 0$ for $x \in \Omega$ into equation (2.30) gives

$$d_S \Delta\phi + \mu\phi = 0 \text{ for } x \in \Omega,\ \frac{\partial \phi}{\partial \nu} = 0 \text{ for } x \in \partial\Omega.$$

It is well-known that the above equation has a principle eigenvalue $\mu^* = 0$ and the corresponding eigenfunction ϕ^* is a constant. In view of (2.31), we have $\int_\Omega \phi^* dx = \phi^*|\Omega| = 0$, which implies that $\phi^* \equiv 0$. It is a contraction, so the claim is proved.

Note that the second equation of (2.30) decouples from the first one. Thus, to obtain the linear stability of DFE, it suffices to study the distribution of eigenvalues of the following problem

$$\begin{cases} d_I \Delta\psi - (\gamma - \beta)\psi + \mu\psi = 0, & x \in \Omega, \\ \frac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega \end{cases} \tag{2.32}$$

with restriction (2.31). Notice that the eigenvalues of (2.32) satisfy $\gamma - \beta = \mu_0 < \mu_1 \le \mu_2 \le \cdots \to \infty$, then $R_0 < 1$ leads to all eigenvalues of (2.32) are greater than zero, which shows that DFE is linearly stable. When $R_0 > 1$, $\mu_0 < 0$ and the associated eigenfunction can be denoted as ψ_0. Similar to Allen et al. (2008), we see that

$$\begin{cases} d_S \Delta\phi + \mu_0\phi = -(\gamma - \beta)\psi_0, & x \in \Omega, \\ \frac{\partial \phi}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

has a unique solution ϕ_0, and $\int_\Omega [\phi_0(x) + \psi_0(x)]dx = 0$. Thus, (2.30) has a solution (μ_0, ϕ_0, ψ_0) with $\mu_0 < 0$. This implies that if $R_0 > 1$, then DFE is unstable. $\square$

Lemma 2.10 *Suppose that $R_0 < 1$, the initial value satisfies (A1) and N is fixed. Then the unique solution $(S(x, t), I(x, t))$ of system (1.9) with weak kernel or strong kernel satisfies that*

$$(S(x, t), I(x, t)) \to (\tilde{S}_0, 0) \text{ in } C(\overline{\Omega}) \text{ as } t \to \infty.$$

Proof We first prove the result for system (1.9) with weak kernel. Define

$$V(x, t) = S(x, t) - \tilde{S}_0 + I(x, t),$$

then we have

$$\begin{cases} \frac{\partial V}{\partial t} = d_S \Delta V + \sigma d_S \nabla \cdot (S \nabla u) + (d_I - d_S)\Delta I, & x \in \Omega, t > 0, \\ \frac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ V(x,0) = S_0(x) - \tilde{S}_0 + \eta(x,0), & x \in \Omega, \end{cases} \tag{2.33}$$

where $u(x,t) = (g_0 ** I)(x,t) = \int_{-\infty}^{t} \int_\Omega G(x,y,t-s) g_0(t-s) I(y,s) dy ds$.

Step 1: We claim that $I(x,t) \to 0$ and $u(x,t) \to 0$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$. In view of the I-equation, we have

$$I_t = d_I \Delta I + \beta \frac{SI}{S+I} - \gamma I \le d_I \Delta I - (\gamma - \beta) I, \ x \in \Omega, \ t > 0. \tag{2.34}$$

Let μ_0 be the principal eigenvalue of (2.32), and denote $\psi_0 > 0$ as the eigenfunction associated with μ_0. Take a large constant $M > 0$ such that $\eta(x,0) \le M\psi_0$ for $x \in \overline{\Omega}$, then by the comparison principle, we have

$$I(x,t) \le M e^{-\mu_0 t} \psi_0, \quad x \in \overline{\Omega}, \ t > 0. \tag{2.35}$$

This, together with the positivity of $I(x,t)$, implies that $\lim_{t\to\infty} I(x,t) = 0$ uniformly for $x \in \overline{\Omega}$. Therefore, for any small $\varepsilon > 0$, there exists $T_* > 0$ such that

$$0 \le I(x,t) \le \varepsilon, \ \text{for } x \in \overline{\Omega}, t > T_*.$$

Let $z^*(t)$ be the solution of

$$\begin{cases} \frac{dz}{dt} = \frac{1}{\tau}(\varepsilon - z), \ t > T_*, \\ z(T_*) = \|u(\cdot, T_*)\|_{L^\infty(\Omega)}. \end{cases}$$

It is easy to see that

$$\frac{\partial z^*(t)}{\partial t} - d_I \Delta z^*(t) - \frac{1}{\tau}(\varepsilon - z^*(t)) = 0 \ge 0.$$

This and the u-equation show that $u(x,t) \le z^*(t)$ for $x \in \overline{\Omega}$ and $t > T_*$. Due to the arbitrary of ε and the positivity of $u(x,t)$, we have $\lim_{t\to\infty} u(x,t) = 0$ uniformly for $x \in \overline{\Omega}$.

Step 2: We claim that

$$\int_0^\infty \int_\Omega |\nabla V|^2 dx dt < \infty. \tag{2.36}$$

Multiplying both sides of the I-equation by I, and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega I^2 dx = -d_I \int_\Omega |\nabla I|^2 + \int_\Omega I^2 \left(\frac{\beta S}{S+I} - \gamma \right) dx,$$

which and (2.35) yield that

$$
\begin{aligned}
d_I \int_0^T \int_\Omega |\nabla I|^2 dxdt &= \frac{1}{2}\int_\Omega \eta^2(x,0)dx - \frac{1}{2}\int_\Omega I^2(x,T)dx \\
&\quad + \int_0^T \int_\Omega I^2 \left(\frac{\beta S}{S+I} - \gamma\right) dxdt \\
&\le \frac{1}{2}\int_\Omega \eta^2(x,0)dx + (\beta+\gamma)\int_0^T \int_\Omega \left(Me^{-\mu_0 t}\psi_0\right)^2 dxdt \\
&\le C_1, \text{ for any } T > 0.
\end{aligned}
\tag{2.37}
$$

By a similar process, we can get the estimate on $\nabla u(x,t)$,

$$
\begin{aligned}
d_I \int_0^T \int_\Omega |\nabla u|^2 dxdt &= \frac{1}{2}\int_\Omega u^2(x,0)dx - \frac{1}{2}\int_\Omega u^2(x,T)dx \\
&\quad + \frac{1}{\tau}\int_0^T \int_\Omega u(I-u)dxdt \\
&\le \frac{1}{2}\int_\Omega u^2(x,0)dx + \frac{1}{\tau}\int_0^T \int_\Omega Me^{-\mu_0 t}\psi_0 u dxdt \\
&\le \frac{1}{2}\int_\Omega u^2(x,0)dx + \frac{M}{\tau\mu_0}(1-e^{-\mu_0 T})\|u\|_{L^\infty(\Omega\times(0,\infty))} \\
&\le C_2, \text{ for any } T > 0.
\end{aligned}
\tag{2.38}
$$

Moreover, it follows from (2.33) and Cauchy-Schwarz inequality that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega V^2 dx &= -d_S \int_\Omega |\nabla V|^2 dx - \sigma d_S \int_\Omega S\nabla u \cdot \nabla V dx \\
&\quad - (d_I - d_S)\int_\Omega \nabla V \cdot \nabla I dx \\
&\le -d_S \int_\Omega |\nabla V|^2 dx + \frac{d_S}{4}\int_\Omega |\nabla V|^2 dx + \sigma^2 d_S \int_\Omega S^2 |\nabla u|^2 dx \\
&\quad + \frac{d_S}{4}\int_\Omega |\nabla V|^2 dx + \frac{(d_I - d_S)^2}{d_S}\int_\Omega |\nabla I|^2 dx \\
&\le -\frac{d_S}{2}\int_\Omega |\nabla V|^2 dx + \sigma^2 d_S \|S(\cdot,t)\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u|^2 dx \\
&\quad + \frac{(d_I - d_S)^2}{d_S}\int_\Omega |\nabla I|^2 dx.
\end{aligned}
\tag{2.39}
$$

Integrate (2.39) in time variable on $(0,\infty)$, and apply Theorem 2.7, (2.37) and (2.38), then the claim is proved.

Step 3: We claim that $\int_0^\infty \|V_t(\cdot,t)\|^2_{(H^1(\Omega))^*}dt < \infty$, where $(H^1(\Omega))^*$ is the dual space of $H^1(\Omega)$, and the norm is

$$\|f\|_{(H^1(\Omega))^*} = \sup\{\langle f, y\rangle : y \in H^1(\Omega), \|y\|_{H^1(\Omega)} \le 1\}.$$

For any test function $\varphi \in H^1(\Omega)$, by Hölder inequality, we have

$$\begin{aligned}
\int_\Omega V_t\varphi &= -d_S\int_\Omega \nabla\varphi\cdot\nabla V dx - \sigma d_S\int_\Omega \nabla\varphi\cdot S\nabla u dx - (d_I - d_S)\int_\Omega \nabla\varphi\cdot\nabla I dx \\
&\le d_S\|\nabla V(\cdot,t)\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + \sigma d_S\|S(\cdot,t)\|_{L^\infty(\Omega)}\|\nabla u(\cdot,t)\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \\
&\quad + (d_I + d_S)\|\nabla I(\cdot,t)\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \\
&\le \Big[d_S\|\nabla V(\cdot,t)\|_{L^2(\Omega)} + \sigma d_S\|S(\cdot,t)\|_{L^\infty(\Omega)}\|\nabla u(\cdot,t)\|_{L^2(\Omega)} \\
&\quad + (d_I + d_S)\|\nabla I(\cdot,t)\|_{L^2(\Omega)}\Big]\|\varphi\|_{H^1(\Omega)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|V_t(\cdot,t)\|_{(H^1(\Omega))^*} \le{}& d_S\|\nabla V(\cdot,t)\|_{L^2(\Omega)} + \sigma d_S\|S(\cdot,t)\|_{L^\infty(\Omega)}\|\nabla u(\cdot,t)\|_{L^2(\Omega)} \\
&+ (d_I + d_S)\|\nabla I(\cdot,t)\|_{L^2(\Omega)}.
\end{aligned}$$

Based on Theorem 2.7, (2.36), (2.37), and (2.38). The claim is proved.

Step 4: We claim that $S(x,t) \to \tilde{S}_0$ uniformly for $x \in \overline{\Omega}$. Based on **Step 1**, we only need to prove

$$V(\cdot,t) \to 0 \text{ in } C(\overline{\Omega}),\ \text{ as } t \to \infty. \tag{2.40}$$

Based on the claims in **Step 2** and **Step 3** and following the procedure of Li et al. (2020a, Theorem 3.5), we can obtain (2.40).

Now, we consider system (1.9) with strong kernel. The inequality (2.35) still holds in this case, so $I(x,t) \to 0$ uniformly for $x \in \overline{\Omega},\ t \to \infty$.

Same as the proof of the decay of $u(x,t)$ in weak kernel, we have $v(x,t) \to 0$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$. Then using the exponentially decaying property of v, we obtain that $u(x,t) \to 0$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$. Moreover, we have

$$\begin{aligned}
d_I\int_0^T\int_\Omega |\nabla u|^2 dxdt &= \frac{1}{2}\int_\Omega u^2(x,0)dx - \frac{1}{2}\int_\Omega u^2(x,T)dx \\
&\quad + \frac{1}{\tau}\int_0^T\int_\Omega u(v-u)dxdt \\
&\le \frac{1}{2}\int_\Omega u^2(x,0)dx + \frac{1}{\tau}\int_0^T\int_\Omega vudxdt \\
&\le \frac{1}{2}\int_\Omega u^2(x,0)dx + \frac{1}{\tau}\|v\|_{L^\infty(\Omega\times(0,\infty))}\|u\|_{L^\infty(\Omega\times(0,\infty))} \\
&\le \tilde{C}_1,\ \text{ for all } T > 0.
\end{aligned} \tag{2.41}$$

With these key points obtained, repeat the procedure in weak kernel, then it follows that

$$\|V(\cdot,t)\|_{C(\bar{\Omega})} \to 0, \text{ as } t \to \infty.$$

The proof is completed. □

Form Lemmas 2.9 and 2.10, we summarize the global stability of DFE as follows.

Theorem 2.11 *For system* (1.9) *with weak kernel or strong kernel, suppose that initial values satisfy (A1) and N is fixed. Then DFE is globally asymptotically stable if $R_0 < 1$, and it is unstable if $R_0 > 1$.*

We next focus on the dynamical behaviours of the system (1.9) when $R_0 > 1$. We will analyze the persistence of system (1.9) with weak kernel and strong kernel by systems (2.2) and (2.4), respectively.

We first deal with the case of weak kernel. For $\phi = (\phi_1, \phi_2, \phi_3) \in X_w$, where X_w is given by (2.6), let the norm in X_w be

$$\|\phi\|_{X_w} = \|\phi_1\|_{W^{1,\infty}(\Omega)} + \|\phi_2\|_{W^{1,\infty}(\Omega)} + \|\phi_3\|_{W^{1,\infty}(\Omega)}.$$

Define

$$\begin{aligned} &X_{w+} = \{\phi \in X_w : \phi_1 \geq 0, \phi_2 \geq 0, \phi_3 \geq 0\},\ X_{w+}^0 = \{\phi \in X_{w+} : \phi_2 \not\equiv 0\}, \\ &\partial X_{w+}^0 = X_{w+} \backslash X_{w+}^0 = \{\phi \in X_{w+} : \phi_2 \equiv 0\}. \end{aligned}$$

Let $\mathcal{P}$ be the set of ultimately uniformly bounded functions with initial values in X_{w+}. The following result shows that system (2.2) has a global attractor.

Theorem 2.12 *Let $(S(x,t), I(x,t), u(x,t))$ be the unique classical solution of system* (2.2) *with the initial value in X_{w+}. Then there exists an $\alpha > 0$ such that*

$$\|S(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})},\ \|I(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})},\ \|u(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} \in \mathcal{P}. \tag{2.42}$$

Moreover, system (2.2) *generates a continuous semiflow in X_{w+}, and the semiflow possesses a compact global attractor.*

Proof In light of Theorem 2.7, system (2.2) defines a semiflow $\pi(t)$, $t > 0$ in X_{w+}. According to (2.12), the solution $\|S(\cdot,t)\|_{L^\infty(\Omega)}$, $\|I(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ are in $\mathcal{P}$. Then it follows from Wu et al. (2016, Theorem 4.3) that (2.42) holds. Note that $C^{1+\alpha}(\overline{\Omega})$ $(\alpha > 0)$ compactly embeds in $C^1(\overline{\Omega})$, and $C^1(\overline{\Omega})$ continuously embeds in $W^{1,\infty}(\Omega)$, so $[C^{1+\alpha}(\overline{\Omega})]^3$ $(\alpha > 0)$ is compactly embedded in X_{w+}. Therefore, (2.42) shows that the semiflow in X_{w+} is point dissipative. By virtue of Amann (1989, Theorem 1), $\pi(t)$ is a continuous operator in $[C^2(\overline{\Omega})]^3$, thus, it is a compact operator in X_{w+} for any $t > 0$. As an application of Billotti and LaSalle (1971, Theorem 3.1) that $\pi(t)$, $t > 0$ has a global attractor. □

Let $\mathcal{A}$ be the global attractor in X_{w+} given in Theorem 2.12. Denote a small neighborhood of $\mathcal{A}$ by

$$\mathcal{B}(\mathcal{A}, \varepsilon) = \{y \in X_{w+} : \inf_{y' \in \mathcal{A}} \|y - y'\|_{X_w} < \varepsilon\}.$$

Take $t_2 > t_1 > 0$, and set

$$Y_1 = \overline{\pi(\mathcal{B}(\mathcal{A}, \varepsilon), t_1)}, \ Y = \pi(Y_1, t_2),$$

and $Z = Y \cap \partial X^0_{w+}$. Then it follows Cantrell and Cosner (2003, Theorem 4.1) that Y and Z are compact, and $Y, \ Z$ and $Y \backslash Z$ are forward invariant under π.

Lemma 2.13 *Set $\omega(Z) = \cup_{y \in Z} \omega(y)$ and $\omega(y)$ is the ω-limit set of the point y. Then*

$$\overline{\omega(Z)} = \{(\tilde{S}_0, 0, 0)\},$$

where $(\tilde{S}_0, 0, 0)$ is a constant steady state of system (2.2).

Proof For each initial value $P_0(x) \in Z$, we known that $I(x, t) \equiv 0$ for $t > 0$ and $x \in \overline{\Omega}$ from the strong maximum principal, and $S(x, t)$ and $u(x, t)$ satisfy

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S + \sigma d_S \nabla \cdot (S \nabla u), & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ S_0(x) \geq (\not\equiv) 0, & x \in \Omega, \\ \int_\Omega S(x, t) dx = N, & t \geq 0, \end{cases} \tag{2.43}$$

and

$$\begin{cases} \frac{\partial u}{\partial t} = d_I \Delta u - \frac{1}{\tau} u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = \frac{1}{\tau} \int_{-\infty}^0 \int_\Omega G(x, y, -s) e^{\frac{s}{\tau}} \eta(y, s) dy ds \geq 0, & x \in \Omega. \end{cases} \tag{2.44}$$

Again using strong maximum principle, we have $S(x, t) > 0$ and $u(x, t) \geq 0$ for $t > 0$ and $x \in \overline{\Omega}$. Let $(\tilde{\mu}_0, \tilde{\psi}_0)$ be the principal eigenpair of the

$$\begin{cases} d_I \Delta \psi - \frac{1}{\tau} \psi + \mu \psi = 0, & x \in \Omega, \\ \frac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Then $\tilde{\mu}_0 = \frac{1}{\tau} > 0$, and by the comparison principle, $u(x, t) \leq \tilde{M} e^{-\tilde{\mu}_0 t} \tilde{\psi}_0$ for a large $\tilde{M} > 0$. This and the non-negativity of $u(x, t)$ gives $\lim_{t \to \infty} u(x, t) = 0$.

Multiplying both sides of (2.44) by u, and integrating over Ω, we get

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx = -d_I \int_\Omega |\nabla u|^2 dx - \frac{1}{\tau} \int_\Omega u^2 dx.$$

Integrating the above equation on $(0, \infty)$ leads to

$$\begin{aligned} d_I \int_0^\infty \int_\Omega |\nabla u|^2 dxdt &= \frac{1}{2}\int_\Omega u^2(x,0)dx - \frac{1}{2}\int_\Omega u^2(x,\infty)dx \\ &\quad - \frac{1}{\tau}\int_0^\infty \int_\Omega u^2 dxdt \\ &\le \frac{1}{2}\int_\Omega u^2(x,0)dx < \infty. \end{aligned} \tag{2.45}$$

From (2.43) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega S^2 dx &= -d_S \int_\Omega |\nabla S|^2 dx - \sigma d_S \int_\Omega S\nabla u \cdot \nabla S dx \\ &\le -d_S \int_\Omega |\nabla S|^2 dx + \frac{d_S}{2}\int_\Omega |\nabla S|^2 dx + \frac{\sigma^2 d_S}{2}\int_\Omega S^2 |\nabla u|^2 dx \\ &\le -\frac{d_S}{2}\int_\Omega |\nabla S|^2 dx + \frac{\sigma^2 d_S}{2}\|S(\cdot,t)\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla u|^2 dx. \end{aligned}$$

Integrate the above equation on $(0, \infty)$ with Theorem 2.7 and (2.45), then

$$\int_0^\infty \int_\Omega |\nabla S|^2 dxdt < \infty. \tag{2.46}$$

For any test function $\varphi \in H^1(\Omega)$, using Hölder inequality leads to

$$\begin{aligned} \int_\Omega S_t \varphi &= -d_S \int_\Omega \nabla\varphi \cdot \nabla S dx - \sigma d_S \int_\Omega \nabla\varphi \cdot S\nabla u dx \\ &\le d_S \|\nabla S(\cdot,t)\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)} + \sigma d_S \|S(\cdot,t)\|_{L^\infty(\Omega)} \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)} \\ &\le \left[d_S \|\nabla S(\cdot,t)\|_{L^2(\Omega)} + \sigma d_S \|S(\cdot,t)\|_{L^\infty(\Omega)} \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \right] \|\varphi\|_{H^1(\Omega)}. \end{aligned}$$

Therefore, it holds that

$$\|S_t(\cdot,t)\|_{(H^1(\Omega))^*} \le d_S \|\nabla S(\cdot,t)\|_{L^2(\Omega)} + \sigma d_S \|S(\cdot,t)\|_{L^\infty(\Omega)} \|\nabla u(\cdot,t)\|_{L^2(\Omega)}.$$

Based on Theorem 2.7, (2.45) and (2.46), we have $\int_0^\infty \|S_t(\cdot,t)\|^2_{(H^1(\Omega))^*} dt < \infty$, where $(H^1(\Omega))^*$ and the associated norm are given as those in Lemma 2.10. Now, using the similar analysis as in the proof of in Li et al. (2020a, Theorem 3.5), we have $S(x,t) \to \tilde{S}_0$ as $t \to \infty$ uniformly on $\overline{\Omega}$.

Consequently, for any initial value $P_0(x) \in Z$, the solution $(S(x,t), I(x,t), u(x,t))$ of system (2.2) satisfies

$$\lim_{t\to\infty} \|(S(x,t), I(x,t), u(x,t)) - (\tilde{S}_0, 0, 0)\|_{X_w} = 0.$$

$\square$

Lemma 2.14 *If $R_0 > 1$, then there exists an $\eta_1 > 0$ such that for any initial value $P_0(x) \in X^0_{w+}$, the solution $(S(x,t), I(x,t), u(x,t))$ of system (2.2) satisfies*

$$\limsup_{t\to\infty} \|(S(x,t), I(x,t), u(x,t)) - (\tilde{S}_0, 0, 0)\|_{X_w} \geq \eta_1,$$

where $(\tilde{S}_0, 0, 0)$ is a constant steady state of system (2.2).

Proof Observe that the eigenvalue problem (2.32) has a principal eigenvalue $\mu_0 < 0$ when $R_0 > 1$. By the continuity of μ_0 with respect to β, we take a small constant $\eta_1 > 0$ such that $\tilde{S}_0 - \eta_1 > 0$ and $\mu_0(\eta_1) < 0$, where $\mu_0(\eta_1)$ is the principle eigenvalue of the following problem:

$$\begin{cases} d_I \Delta\psi - \left(\gamma - \beta\frac{\tilde{S}_0-\eta_1}{\tilde{S}_0+2\eta_1}\right)\psi + \mu\psi = 0, & x \in \Omega, \\ \frac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.47}$$

Denote the eigenfunction corresponding to $\mu_0(\eta_1)$ as ψ_{η_1}.

Suppose that there exists an initial value $(S_0(x), I(x,0), u(x,0)) \in X^0_{w+}$ such that

$$\limsup_{t\to\infty} \|(S(x,t), I(x,t), u(x,t)) - (\tilde{S}_0, 0, 0)\|_{X_w} < \eta_1.$$

Then there is a large t_1 such that

$$\tilde{S}_0 - \eta_1 < S(x,t) < \tilde{S}_0 + \eta_1,\ I(x,t) < \eta_1,\ u(x,t) < \eta_1,$$

for $t > t_1$ and $x \in \overline{\Omega}$. Since $\eta(x,0) \geq (\not\equiv)0$, then by the strong maximum principle, $I(x,t) > 0$ for $t > 0$ and $x \in \overline{\Omega}$. Thus, we can choose a small constant $c_0 > 0$ such that $I(x,t_1) > c_0\psi_{\eta_1}$ for $x \in \overline{\Omega}$. One easily checks that $c_0\psi_{\eta_1}e^{-\mu_0(\eta_1)(t-t_1)}$ is the solution of

$$\begin{cases} \frac{\partial z}{\partial t} = d_I \Delta z - (\gamma - \beta\frac{\tilde{S}_0-\eta_1}{\tilde{S}_0+2\eta_1})z, & x \in \Omega,\ t > t_1, \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > t_1, \\ z(x,t_1) = c_0\psi_{\eta_1}, & x \in \Omega. \end{cases}$$

Given the I-equation in system (2.2), we have

$$\begin{cases} \frac{\partial I}{\partial t} \geq d_I \Delta I - (\gamma - \beta\frac{\tilde{S}_0-\eta_1}{\tilde{S}_0+2\eta_1})I, & x \in \Omega,\ t > t_1 \\ \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > t_1. \end{cases} \tag{2.48}$$

Thus, the comparison principle implies that $I(x,t) \geq c_0\psi_{\eta_1}e^{-\mu_0(\eta_1)(t-t_1)}$ for $t \geq t_1$ and $x \in \overline{\Omega}$. Recall that $\mu_0(\eta_1) < 0$, which leads to $\lim_{t\to\infty} c_0\psi_{\eta_1}e^{-\mu_0(\eta_1)(t-t_1)} = \infty$. Consequently,

$$\lim_{t\to\infty} I(x,t) = \infty.$$

It is a contradiction to Theorem 2.7. □

Theorem 2.15 *Suppose that $R_0 > 1$. Then the solution semiflow $\pi(t)$, $t > 0$, of system (2.2) is uniformly persistent in X_w. Namely, there exists a constant $\eta > 0$, independent of the initial value $P_0(x)$, such that*

$$\liminf_{t\to\infty} S(x,t) \geq \eta, \ \ \liminf_{t\to\infty} I(x,t) \geq \eta, \ \ \liminf_{t\to\infty} u(x,t) \geq \eta$$

uniformly for $x \in \overline{\Omega}$.

Proof Lemma 2.13 suggests that $M = \{(\tilde{S}_0, 0, 0)\}$ is an isolated covering of $\omega(Z)$. Note that $Y\backslash Z = Y \cap X^0_{w+} \subset X^0_{w+}$, then it follows from Lemma 2.14 that $W^s(M) \cap (Y\backslash Z) = \emptyset$, where $W^s(M)$ is the stable set of M. As a consequence, the conclusion directly follows from Theorem 4.8 in Wu et al. (2016) □

For strong kernel, for $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4) \in X_s$, where X_s is defined as (2.6), let the norm in X_s be $\|\tilde{\phi}\|_{X_s} = \sum_{i=1}^4 \|\tilde{\phi}_i\|_{W^{1,\infty}(\Omega)}$. Let

$$\begin{aligned} X_{s+} &= \{\tilde{\phi} \in X_s : \tilde{\phi}_i \geq 0, \ i = 1,2,3,4\}, \ \ X^0_{s+} = \{\tilde{\phi} \in X_{s+} : \tilde{\phi}_2 \not\equiv 0\}, \\ \partial X^0_{s+} &= X_{s+}\backslash X^0_{s+} = \{\tilde{\phi} \in \tilde{Y}_+ : \tilde{\phi}_2 \equiv 0\}. \end{aligned}$$

Define $\tilde{\mathcal{P}}$ as the set of ultimately uniformly bounded functions with initial values in X_{s+}.

Based on the global existence and boundedness of solutions to system (2.4) given in Theorem 2.7, and applying the similar argument in Theorem 2.12, we have the following result.

Theorem 2.16 *Let $(S(x,t), I(x,t), u(x,t), v(x,t))$ be the unique classical solution of system (2.4) with initial value $Z_0(x) \in X_{s+}$. Then there exists an $\alpha > 0$ such that*

$$\|S(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})}, \ \|I(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})}, \ \|u(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})}, \ \|v(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} \in \tilde{\mathcal{P}}.$$

Moreover, system (2.4) generates a continuous semiflow $\tilde{\pi}(t)$, $t > 0$ in X_{s+}, and the semiflow possesses a compact global attractor.

Let $\widetilde{\mathcal{A}}$ be the global attractor of $\tilde{\pi}$ on X_{s+}, define

$$\tilde{Y}_1 = \overline{\tilde{\pi}(\mathcal{B}(\tilde{\mathcal{A}}, \varepsilon), t_1)}, \ \tilde{Y} = \tilde{\pi}(\tilde{Y}_1, t_2),$$

for $t_2 > t_1 > 0$, where $\mathcal{B}(\tilde{\mathcal{A}}, \varepsilon) = \{y \in X_{s+} : \inf_{y' \in \tilde{\mathcal{A}}} \|y - y'\|_{X_s} < \varepsilon\}$, and denote $\tilde{Z} = \tilde{Y} \cap \partial X^0_{s+}$.

Similar to Theorem 2.15, we give the persistence for system (2.4) below.

Theorem 2.17 *Suppose that $R_0 > 1$. Then the solution semiflow $\tilde{\pi}(t)$, $t > 0$, of system (2.4) is uniformly persistent in X_{s+}. Namely, there exists a constant $\tilde{\eta} > 0$,* *independent of the initial value $Z_0(x)$, such that*

$$\liminf_{t\to\infty} S(x,t) \geq \tilde{\eta}, \quad \liminf_{t\to\infty} I(x,t) \geq \tilde{\eta},$$
$$\liminf_{t\to\infty} u(x,t) \geq \tilde{\eta}, \quad \liminf_{t\to\infty} v(x,t) \geq \tilde{\eta}$$

uniformly for $x \in \overline{\Omega}$.

By Lemmas 2.1 and 2.2, and Theorems 2.15 and 2.17, the dynamics of the original system (1.9) for $R_0 > 1$ is concluded as follows.

Corollary 2.18 *Suppose that $R_0 > 1$, the initial value satisfies (A1) and N is fixed, and let $(S(x,t), I(x,t))$ be the unique solution of system (1.9) with weak kernel or strong kernel, then there exists a constant $\hat{\eta} > 0$ independent of the initial value such that*

$$\liminf_{t\to\infty} S(x,t) \geq \hat{\eta}, \quad \liminf_{t\to\infty} I(x,t) \geq \hat{\eta}$$

uniformly for $x \in \overline{\Omega}$.

3 Numerical examples

In this section, we perform some numerical simulations to verify the theoretical results and provide biological insights into the control of diseases.

3.1 Simulations for weak kernel

For the weak kernel delay, take $d_S = 0.9$, $d_I = 0.3$, $\tau = 1$, $N = 12$ and $\Omega = (0, \pi)$. When $\beta = 0.8$ and $\gamma = 1.2$, we have $R_0 = 2/3 < 1$. By Theorem 2.11, all solutions of system (1.9) with initial values satisfying (A1) converge to DFE, see Fig. 1. When $\beta = 1.8$ and $\gamma = 0.6$, then $R_0 = 3 > 1$. Thus, system (1.9) has an endemic equilibrium EE (1.2732, 2.5465). Moreover, Corollary 2.18 shows that system (1.9) is uniformly persistent, see Fig. 2, where the solution with an initial value satisfying (A1) converges to EE for small σ, and it tends to a stable periodic solution near EE for a relatively large σ.

In view of Figs. 1 and 2, we see that the extinction/persistence is determined by the basic reproduction number R_0. Fast memory-based movement cannot alter the extinction of diseases when $R_0 < 1$. This is intuitively obvious. When $R_0 > 1$, neither slow nor fast memory-based movement can eliminate diseases, but the number of infected population is stable at EE under slow memory-based movement and is stable at periodic oscillations under fast memory-based movement. This observation is intriguing and not intuitively obvious.

Due to the joint effect of memory-based movement and spatiotemporal delay, the global stability of EE is challenging and left as an open problem. Note that EE is globally asymptotically stable when $\sigma = 0$ or σ is small and there is no delay. Further,

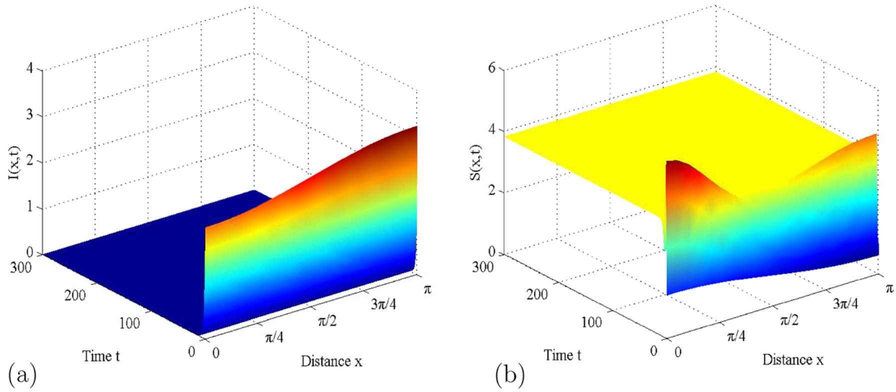

Fig. 1 When $R_0 < 1$ and $\sigma = 16$, the solution with initial value $S_0(x) = 1 + 0.4 \cos x$, $I(x, 0) = \frac{12}{\pi} - S_0(x)$ tends to DFE for (1.9) with weak kernel

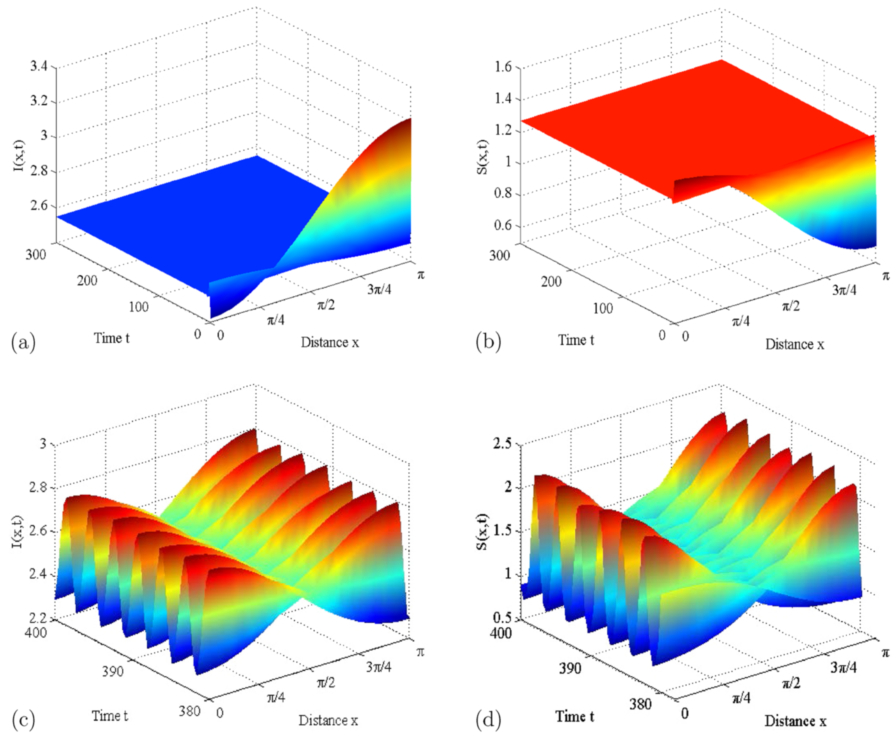

Fig. 2 For $R_0 > 1$, (1.9) with weak kernel has a sable EE when $\sigma = 2$; there is a stable periodic solution when $\sigma = 16$, where the initial value is $S_0(x) = 1 + 0.4 \cos x$, $I(x, 0) = \frac{12}{\pi} - S_0(x)$

the simulation in Fig. 3 shows that even if the initial point is far away from EE, the solution converges to EE under small σ. Thus, we conjecture that EE is globally asymptotically stable under small σ for system (1.9).

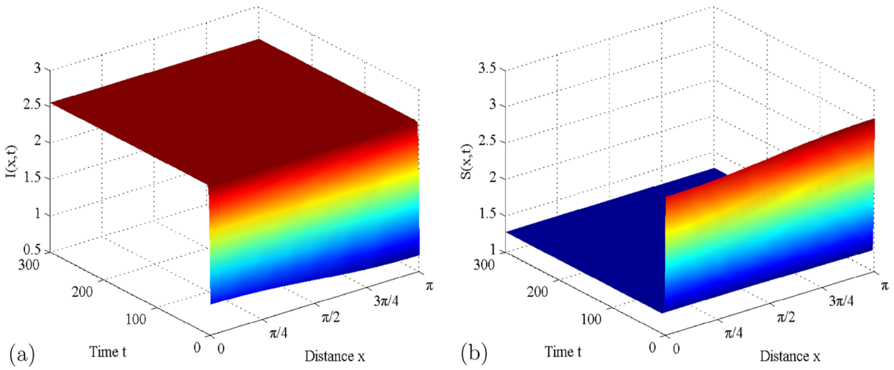

Fig. 3 For (1.9) with weak kernel, the solution with initial value away from EE finally goes to EE, where $S_0(x) = 3 - 0.1\cos x,\ I(x,0) = \frac{12}{\pi} - S_0(x)$ and the parameters are the same as those in Fig. 2a, b

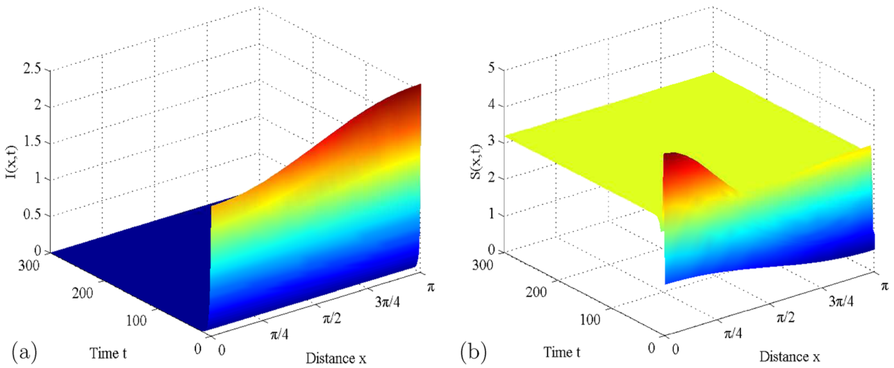

Fig. 4 When $R_0 < 1$ and $\sigma = 11$, the solution converges to DFE for (1.9) with strong kernel, where the initial value is $S_0(x) = 1 + 0.4\cos x,\ I(x,0) = \frac{10}{\pi} - S_0(x)$

3.2 Simulations for strong kernel

Similar dynamics appear for strong kernel delay. For example, we take $d_S = 0.8$, $d_I = 0.2$, $\tau = 1$, $N = 10$ and $\Omega = (0, \pi)$. If $\beta = 0.7$ and $\gamma = 1$, then DFE attracts any solutions since $R_0 = 0.7 < 1$, which is shown in Fig. 4. If $\beta = 1.7$ and $\gamma = 1$, then $R_0 = 1.7 > 1$, which implies the existence of EE $(1.8724, 1.3107)$. Moreover, for any initial values satisfying (A1), the solution finally goes to EE when σ is small, and the solution converges to a stable periodic solution when σ is relatively large, see Fig. 5. Let the initial point be far away from EE, we find the solution still converges to EE under small σ, see Fig. 6.

4 Discussion

Recently, much progress has been made to investigate the role of animal perception in movement and spatiotemporal dynamics. In this paper, we study the effect of

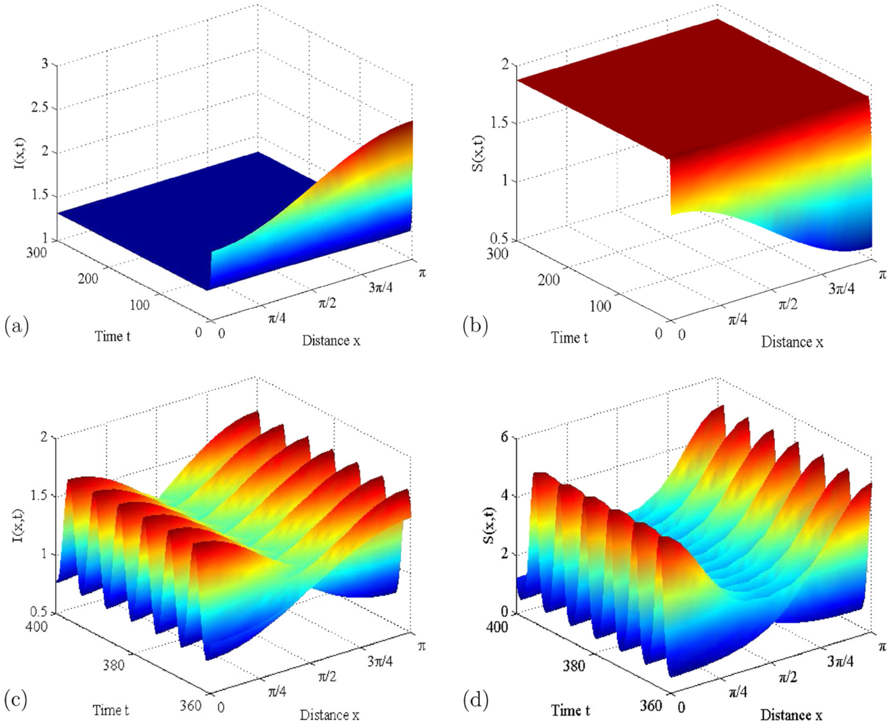

Fig. 5 When $R_0 > 1$, (1.9) with strong kernel has a sable EE for $\sigma = 1$: (**a**, **b**); it has a stable periodic solution near EE for $\sigma = 11$: (**c**, **d**). Here the initial value is $S_0(x) = 1 + 0.4\cos x,\ I(x,0) = \frac{10}{\pi} - S_0(x)$

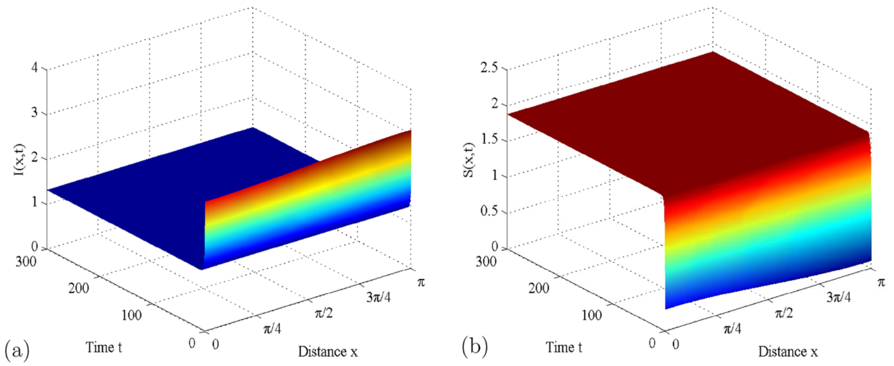

Fig. 6 For (1.9) with strong kernel, the solution initiating from $S_0(x) = 0.2 + 0.1\cos x,\ I(x,0) = \frac{10}{\pi} - S_0(x)$ converges to EE, where the parameters are the same as those in Fig. 5a, b

memory-based movement on the spread of infectious diseases by an SIS reaction-diffusion model proposed in Allen et al. (2008). For simplification, the rates of disease transmission and recovery are assumed to be positive constants. The memory-based movement incorporates the memory growth or decay over time and spatial locations, which is different from the predator-taxis (e.g., see Ahn and Yoon 2021; Wang et al. 2021) and the cross-diffusion in Li et al. (2020a). By choosing specific spatial weighting function and temporal delay kernel function, our system can be transformed to chemotaxis systems, see Lemmas 2.1 and 2.2, in which the memory-based movement is similar to the predator-taxis phenomenon. The chemotaxis systems without delay are helpful in the analysis of the original system with distributed delayed diffusion.

For system (1.9) without directed avoidance movement, it is known that the unique DFE is globally asymptotically stable when $R_0 < 1$, while if $R_0 > 1$, DFE is unstable, and there is a unique EE that is globally asymptotically stable, see Allen et al. (2008), Peng and Liu (2009). For system (1.9) with directed avoidance movement: (i) the cross-diffusion in Li et al. (2020a) where the directed movement only depends on the spatial gradient of current infected population density; (ii) memory-based movement in this paper where the directed movement depends on the spatial gradient of infected population density in the past time and certain spatial range. For case (i), it is shown in Li et al. (2020a) that the system still has threshold dynamics: for $R_0 < 1$, the unique DFE is globally asymptotically stable; for $R_0 > 1$, there is a unique EE that is globally asymptotically stable for small strength of the cross-diffusion. In addition, the disease is uniformly persistent for any strength of the cross-diffusion. For case (ii), the unique DFE has similar dynamics as that in case (i); when $R_0 > 1$, there is a constant EE and system (1.9) is uniformly persistent. Due to the presence of memory-based movement of susceptible population, system (1.9) is a quasilinear parabolic system. The global existence and uniform boundedness in time of solutions are proved in this paper, but it is challenging to study the global dynamics of EE.

In terms of epidemiology, we find that the population densities of susceptible and infected individuals have upper boundedness regardless of memory-based movement. The epidemic threshold for disease persistence is not altered by such self-initiated movement of susceptible individuals, it is still determined by the basic reproduction number R_0. However, in the presence of memory-based perceptive movement, the susceptible and infected populations are stable at EE for slow memory-based movement, and the susceptible and infected populations oscillate for fast memory-based movement. It is worth mentioning that the oscillation behaviours in disease models have been observed in Brauer et al. (2008).

Human reactions to a disease outbreak are recognized to be significant for the disease transmission. Recently this has been made evident in the modeling efforts on COVID-19 in which many works have estimated and predicted the effect of human behaviours (such as wearing masks, quarantine and vaccination) on the disease spread (Li et al. 2020b; Lin et al. 2020; Chernozhukov et al. 2021; Feng et al. 2022; Hou et al. 2020). In this paper, the use of memorized information during all the past time and the spatial distribution of infected population influences is explored in the modeling perspective. Our finding points out that even if susceptible individuals are cautious and smartly move away from memorized infected individuals, the threshold dynamics in terms of basic reproduction number is almost unchanged. However, when the disease has pervaded the system ($R_0 > 1$), a fast memory-based movement can induce oscillation patterns: the number of infected population decreases to a minimum level, then it bounces back and increases to an upper bound, and then it decreases again. Meanwhile, the number of susceptible population changes in the opposite way. It has been proved that keeping social distance enables to better control and contain the COVID-19, but our results show that the perceptive movement alters the asymptotic dynamics only when the disease is persistent. Our model exploits the effectiveness of memory-based movement to eradicate or slow down the spreading process of diseases. This improves the understanding on the impact of directed human mobility responses to diseases, and provides insights for public health agencies to design control strategies and improve the use of health resources.

The perceptive movement can potentially lead to spatial segregation of susceptible population and infected population as shown in Wang et al. (2022), and thus effectively reduces the contact rate between them. We believe this natural movement mechanism of susceptible population is key to exhibiting more realistic disease dynamics. Any model ignoring this perception usually exaggerates the severity of disease spread. Of course, this departure mainly appears in transient dynamics instead of asymptotic dynamics. Therefore, the next pivotal step is to explore the dependence of transient dynamics on perceptive movement with spatial memory. This effort is mathematically challenging but numerically feasible.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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