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Bifurcations driven by generalist and specialist predation: mathematical interpretation of Fennoscandia phenomenon

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Abstract

In this paper, we revisit a predator–prey model with specialist and generalist predators proposed by Hanski et al. (J Anim Ecol 60:353–367, 1991) , where the density of generalist predators is assumed to be a constant. It is shown that the model admits a nilpotent cusp of codimension 4 or a nilpotent focus of codimension 3 for different parameter values. As the parameters vary, the model can undergo cusp type (or focus type) degenerate Bogdanov–Takens bifurcations of codimension 4 (or 3). Our results indicate that generalist predation can induce more complex dynamical behaviors and bifurcation phenomena, such as three small-amplitude limit cycles enclosing one equilibrium, one or two large-amplitude limit cycles enclosing one or three equilibria, three limit cycles appearing in a Hopf bifurcation of codimension 3 and dying in a homoclinic bifurcation of codimension 3. In addition, we show that generalist predation stabilizes the limit cycle driven by specialist predators to a stable equilibrium, which clearly explains the famous Fennoscandia phenomenon.

Keywords Predator–prey model · Specialist predator · Generalist predator · Nilpotent cusp of codimension 4 · Nilpotent focus of codimension 3 · Degenerate Bogdanov–Takens bifurcation

Mathematics Subject Classification 34C23 · 34C25 · 34C37 · 92D25

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1 Introduction

1.1 Motivation

Long period, large amplitude predator–prey cycles have attracted much attention for theoretical interpretation via mathematical models for a few decades. One famous phenomenon in the literature is the following: a quite regular population cycle of microtine rodents in northern Fennoscandia cannot be observed in southern Fennoscandia. To explain this striking Fennoscandia phenomenon in a mathematically rigorous way, we revisit the predator–prey model with specialist and generalist predators proposed in Hanski et al. (1991):

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{cxy}{a+x}-\frac{mx^2}{x^2+b^2},\\ \frac{dy}{dt} &= qy\left(1-\frac{y}{px}\right),\end{aligned}\tag{1.1}$$

where x is the small rodent density, y the specialist predator density, and m is the generalist predator density which is assumed to be a constant. The growth rate of the prey is logistic with carrying capacity k and intrinsic growth rate r. Model (1.1) contains two kinds of predation, the specialist predation: $\frac{cxy}{a+x}$, where the functional response of the specialist predators is Holling II (Michaelis–Menten kinetics) with half-saturation constant a, and the generalist predation: $\frac{mx^2}{x^2+b^2}$, where The functional response of the generalist predators is Holling III with half-saturation constant b. The intrinsic growth rate and the environmental carrying capacity for the specialist predators are q and px (directly proportional to the prey density), respectively.

There have been enormously many papers analyzing predator–prey models with only specialist predators, but little attention was paid to predator–prey models with generalist predators. Recently, we found that generalist predators can induce more complex dynamical behaviors and bifurcation phenomena (Lu et al. 2023; Xiang et al. 2022, 2020, 2023). To the best of our knowledge, there exists few rigorous analysis work for predator–prey models with both specialist and generalist predators due to their complexity of two different denominators. We make this effort in this paper.

1.2 Comparison to literature work

We first make the rescaling: $x = ku,\quad y = \frac{kr}{c}v,\quad t = \frac{1}{r}\tau$, then model (1.1) becomes (still denote τ by t)

$$\begin{aligned}\frac{du}{dt} &= u(1-u)-\frac{uv}{\alpha+u}-\frac{\gamma u^2}{u^2+\eta},\\ \frac{dv}{dt} &= v(\delta-\beta\frac{v}{u}),\end{aligned}\tag{1.2}$$

where $\alpha = \frac{a}{k},\ \gamma = \frac{m}{kr},\ \eta = \frac{b^2}{k^2},\ \delta = \frac{q}{r},\ \beta = \frac{q}{cp}$, and $\alpha, \beta, \gamma, \delta, \eta$ are all positive.

Unlike the traditional predator–prey models with only specialist predation, model (1.2) admits more mathematical challenges, such as the boundary equilibria and the positive equilibria depending on respectively a third-order algebraic equation and a fourth-order algebraic equation, which makes the qualitative and bifurcation analyses more difficult. Hanski et al. (1991) showed, by numerical simulation, that generalist predation is likely to be the main stabilizing factor in south Fennoscandia. Lindström (1993) analyzed the qualitative behavior and Hopf bifurcation of model (1.2) with two examples showing that there may exist two limit cycles enclosing one or multiple positive equilibria and complicated bifurcations by numerical simulation. Xiao and Zhang (2007) proved the existence of a nilpotent focus of codimension 3 under a set of specific parameter values, around which the existence of degenerate Bogdanov–Takens (BT) bifurcation of codimension 3 (focus case) was shown. However, for general parameter values, the theoretical exploration for the complex dynamical behaviors and bifurcation phenomena were not well understood, especially the complete analysis on bifurcations with high codimension. This is our main contribution in this paper. Furthermore, we will explicitly illustrate the effect of generalist predation on the dynamics of model (1.2), especially on the oscillatory dynamics, which will clearly explain the well-known Fennoscandia phenomenon.

For general parameter values in model (1.2), we will provide a useful formula on the fourth-order normal form of nilpotent singularities, and show theoretically the existence of a nilpotent focus of codimension 3, around which degenerate BT bifurcation of codimension 3 (focus case) can be fully unfolded in (1.2). Moreover, we will provide another useful formula on the fifth-order normal form of nilpotent cusp, and show the existence of a nilpotent cusp of codimension 4, around which the existence of degenerate BT bifurcation of codimension 4 (cusp case) will be shown. To the best of our knowledge, it is the first time to obtain these two bifurcations simultaneously in a predator–prey model. Some algebraic and symbolic computation methods (Chen and Zhang 2009; Yang 1999; Gelfand et al. 1994) are used to solve the semi-algebraic varieties of normal form coefficients or nondegeneracy conditions. Our theoretical results indicate that generalist predation can induce more complex bifurcations and dynamics, such as cusp type (or focus type) degenerate BT bifurcations of codimension 4 (or 3), three limit cycles appearing in a Hopf bifurcation of codimension 3 and dying in a homoclinic bifurcation of codimension 3, and for different parameter values, there exist three limit cycles enclosing one positive equilibrium, one or two limit cycles enclosing one or three positive equilibria, three positive equilibria but no limit cycles, which explains roughly the famous Fennoscandia phenomenon. On the other hand, when model (1.2) with $\gamma = 0$ has a stable cycle driven only by specialist predators, we will show, by numerical bifurcation diagrams, that the amplitude of the previous stable cycle will decrease as the density of generalist predators increases. Moreover, a sufficiently large density of generalist predators will stabilize the limit cycle to a stable equilibrium, which clearly answers the puzzling Fennoscandia phenomenon.

1.3 Statement of main results

In Sect. 3 we will see that the positive equilibria depend on a fourth-order algebraic Eq. (see (3.1)), which has at most three positive real roots. Denote the double positive equilibria E_{12} and E_{23} as $E_*(u_*, v_*)$ and the triple positive equilibrium E_{123} as $E^*(u^*, v^*)$ (see Lemma 3), where u_* and u^* are the double and triple positive real roots of the four-order algebraic Eq. (3.1), respectively.

First, we define the following sets

$$\begin{aligned}
\Omega_1 &= \{(u_*, \eta):\ 0 < u_* < \frac{1}{2},\ 0 < \eta < \eta_0\} \triangleq \Omega_{11} \cup \Omega_{12} \cup \Omega_{13},\\
\Omega_{11} &= \{(u_*, \eta):\ 0 < u_* < \frac{1}{3} \text{ and } 0 < \eta < \eta_1;\ \text{or } \frac{1}{3} \le u_* < \frac{1}{2} \text{ and } 0 < \eta < \eta_0\},\\
\Omega_{12} &= \{(u_*, \eta):\ 0 < u_* < \frac{1}{3},\ \eta = \eta_1\},\\
\Omega_{13} &= \{(u_*, \eta):\ 0 < u_* < \frac{1}{3},\ \eta_1 < \eta < \eta_0\},
\end{aligned} \tag{1.3}$$

where

$$\begin{aligned}
\eta_0 &= u_*^2(1 - 2u_*),\\
\eta_1 &= -\frac{u_*\sqrt{9\alpha^4 + 14\alpha^3 u_* + 12\alpha^2 u_*^3 + 3\alpha^2(2\alpha + 3)u_*^2 + 6\alpha u_*^5 + \alpha(\alpha + 14)u_*^4 + 9u_*^6}}{2\alpha}\\
&\quad + \frac{3\alpha^2 u_* + \alpha u_*^3 + 3\alpha u_*^2 + 3u_*^4}{2\alpha},
\end{aligned} \tag{1.4}$$

and define

$$\Omega_{25} = \Big\{(u_*, \alpha, \eta):\ (u_*,\ \alpha,\ \eta) = \big(u_{*3},\ \alpha_0(u_{*3}),\ \eta_{00}(u_{*3},\ \alpha_0(u_{*3}))\big)\Big\}, \tag{1.5}$$

where u_{*3}, $\alpha_0(u_{*3})$ and $\eta_{00}(u_*,\ \alpha)$ are shown in (A4), (A8) and (A15), respectively.

Then, define the following three sets of parameters

$$\text{(i) } \widetilde{\beta} \triangleq \frac{u_*}{\alpha+u_*},\ \widetilde{\gamma} \triangleq \frac{(\alpha+u_*^2)(\eta+u_*^2)^2}{u_*^2(-\eta+2\alpha u_*+u_*^2)},\ \widetilde{\delta} \triangleq -\frac{(\alpha+u_*)(\eta+(2u_*-1)u_*^2)}{u_*(-\eta+2\alpha u_*+u_*^2)}; \tag{1.6}$$

$$\begin{aligned}
\text{(ii) } \overline{\beta} &\triangleq \frac{\delta(u^*)^2\left(\eta-2\alpha u^*-(u^*)^2\right)}{(\alpha+u^*)^2\left(\eta+2(u^*)^3-(u^*)^2\right)},\quad \overline{\gamma} \triangleq \frac{\left(\alpha+(u^*)^2\right)\left(\eta+(u^*)^2\right)^2}{(u^*)^2\left(-\eta+2\alpha u^*+(u^*)^2\right)},\\
\overline{\delta} &\triangleq -\frac{\sqrt{\left(3\alpha^2+\alpha(u^*+3)u^*+3(u^*)^3\right)^2-4\alpha u^*\left(\alpha^2+3\alpha(u^*)^2+(u^*)^3\right)}}{2u^*(\alpha+u^*)}\\
&\quad -\frac{\alpha\left(u^*\right)^2-3\alpha u^*+3\left(u^*\right)^3-2\left(u^*\right)^2-3\alpha^2}{2u^*(\alpha+u^*)},\quad \overline{\eta} \triangleq \eta_1\ |_{u_*\to u^*};
\end{aligned} \tag{1.7}$$

$$\text{(iii) } \widehat{\beta} \triangleq \frac{u^*}{\alpha+u^*},\ \widehat{\gamma} \triangleq \frac{\left(\alpha+(u^*)^2\right)\left(\eta+(u^*)^2\right)^2}{(u^*)^2\left(-\eta+2\alpha u^*+(u^*)^2\right)},\ \widehat{\delta} \triangleq -\frac{(\alpha+u^*)\left(\eta+(2u^*-1)(u^*)^2\right)}{u^*\left(-\eta+2\alpha u^*+(u^*)^2\right)}, \tag{1.8}$$

and define

$$K_3 = \alpha^7(42u^* + 31) + 2\alpha^6(75(u^*)^2 + 401u^* + 88)u^* + \alpha^5(u^*)^2(-66(u^*)^3 \\ +2914(u^*)^2 + 3681u^* - 110) + \alpha^4(-30\,(u^*)^3 + 118\,(u^*)^2 + 15091u^* \\ +1716)(u^*)^4 + 5\alpha^3(-93\,(u^*)^3 + 1263\,(u^*)^2 + 3274u^* + 11)\,(u^*)^5 \\ +\alpha^2(-1335\,(u^*)^2 + 16236u^* + 5093)\,(u^*)^7 + 5\alpha(36u^* + 1985)\,(u^*)^9 \\ +4455\,(u^*)^{11}\,. \tag{1.9}$$

The following theorems are the main results of this paper.

Theorem 1 *When $(\beta, \gamma, \delta) = (\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta})$ and $(u_*, \eta) \in \Omega_1 \backslash \Omega_{12}$, the double positive equilibrium $E_*(u_*, v_*)$ (i.e., E_{12} or E_{23}) of system (1.2) is a cusp of codimension up to 4, where $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{\delta}$, Ω_1 and Ω_{12} are shown in (1.6) and (1.3), respectively.*

The proof of Theorem 1 is given in Sect. 3.1, where we provide a useful formula on the fifth-order normal form of nilpotent cusp (see Lemma 4) and a series of lengthy calculations of semi-algebraic varieties consisting of normal form coefficients and parameters conditions (see Appendix 1), some algebraic and symbolic computation methods (Chen and Zhang 2009; Yang 1999; Gelfand et al. 1994) are used. A cusp of codimension up to 4 has not been uncovered for system (1.2) in previous references.

System (1.2) with a cusp of codimension 4 is structurally unstable, and may undergo rich dynamics after parameters perturbation. In fact, the following theorem shows that system (1.2) can exhibit cusp type degenerate BT bifurcations of codimension 4.

Theorem 2 *If $(\beta, \gamma, \delta) = (\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta})$ and $(u_*, \alpha, \eta) \in \Omega_{25}$, then E_* is a cusp of codimension 4. System (1.2) can undergo cusp type BT bifurcation of codimension 4 around E_* as $(\beta, \gamma, \delta, \eta)$ varies near $(\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}, \eta_{00}(u_*, \alpha))$. There exists three limit cycles appearing in a Hopf bifurcation of codimension 3 and dying in a homoclinic bifurcation of codimension 3. Where $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{\delta}$ and Ω_{25} are shown in (1.6) and (1.5), respectively.*

The proof of Theorem 2 is given in Sect. 3.1. According to Chow et al. (1994); Li and Rousseau (1989), there exist a series of bifurcations with lower codimension within a cusp type BT bifurcation of codimension 4, such as

(i) codimension-1: saddle-node, Hopf, homoclinic bifurcations and double limit cycle bifurcation;
(ii) codimension-2: BT bifurcation, Hopf bifurcation, homoclinic bifurcation, bifurcation of a triple limit cycle, simultaneous occurrence of Hopf and homoclinic bifurcations, simultaneous occurrence of Hopf and double limit cycle bifurcations, simultaneous occurrence of homoclinic and double limit cycle bifurcations;
(iii) codimension-3: BT bifurcation, Hopf bifurcation, homoclinic bifurcation, simultaneous occurrence of degenerate Hopf and homoclinic bifurcations, simultaneous occurrence of Hopf and degenerate homoclinic bifurcations.

Theorem 3 *If $(\beta, \gamma, \eta) = (\overline{\beta}, \overline{\gamma}, \overline{\eta})$ and $0 < u^* < \frac{1}{3}$, then system (1.2) has a unique positive equilibrium E^* (i.e., E_{123}). Moreover,*

(I) *if* $\delta \neq \overline{\delta}$, *then* E^* *is a stable* (*or unstable*) *degenerate node of codimension 2 if* $0 < \delta < \overline{\delta}$ (*or* $\delta > \overline{\delta}$);

(II) *if* $\delta = \overline{\delta}$ *and* $K_3 \neq 0$, *then* E^* *is a nilpotent focus of codimension 3;*

where $\overline{\beta}$, $\overline{\gamma}$, $\overline{\eta}$, $\overline{\delta}$ *and* K_3 *are shown in* (1.7) *and (1.9), respectively.*

The proof of Theorem 3 is given in Sect. 3.2. Case (I) follows directly from the center manifold theorem and Theorem 7.1 in chapter 2 of Zhang et al. (1992). For Case (II), we provide a useful formula on the fourth-order normal form of nilpotent singularities (see Lemma 5) and use Lemma 3.1 in Cai et al. (2013). Xiao and Zhang (2007) proved the existence of a nilpotent focus of codimension 3 under a set of specific parameter values, while the exact codimension is unknown for general parameter values. Our result shows that, for general parameter values, the codimension of nilpotent focus is exact 3.

The following result shows that system (1.2) can exhibit focus type degenerate BT bifurcations of codimension 3.

Theorem 4 *If* $(\beta, \gamma, \delta, \eta) = (\widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \overline{\eta})$, $K_3 \neq 0$ *and* $0 < u^* < \frac{1}{3}$, *then* E^* *is a nilpotent focus of codimension 3. System* (1.2) *can undergo focus type BT bifurcation of codimension 3 around* E^* *as* (β, γ, δ) *varies near* $(\widehat{\beta}, \widehat{\gamma}, \widehat{\delta})$. *Where* $\widehat{\beta}$, $\widehat{\gamma}$, $\widehat{\delta}$, $\overline{\eta}$ *and* K_3 *are shown in* (1.8), (1.7) *and* (1.9), *respectively.*

The proof of Theorem 4 is given in Sect. 3.2. According to Dumortier et al. (1991), there exist a series of bifurcations with lower codimension within a focus type BT bifurcation of codimension 3, such as

(i) codimension-1: Hopf bifurcation, homoclinic bifurcation, saddle-node homoclinic bifurcation, saddle-node bifurcation of limit cycles, saddle-node bifurcation;

(ii) codimension-2: degenerate Hopf bifurcation, BT bifurcation, cuspidal bifurcation, degenerate homoclinic bifurcation, saddle-node homoclinic bifurcation.

The paper is organized as follows. In Sect. 2, we show asymptotic dynamics near the origin (0,0) and boundary equilibria and their types in system (1.2). In Sect. 3, the numbers and types of positive equilibria are considered, moreover, cusp type BT bifurcation of codimension 4 and focus type BT bifurcation of codimension 3 are shown in detail. We provide a series of numerical simulations to demonstrate the theoretical analysis and to interpret Fennoscandia phenomenon in Sect. 4. The paper ends with a brief discussion in Sect. 5.

2 Asymptotic dynamics near (0,0) and boundary equilibria

Define

$$R^2_+ := \{(u, v) : u > 0,\ v \geq 0\},\ \Omega := \left\{(u, v) : 0 < u < 1,\ 0 \leq v < \frac{\delta}{\beta}\right\}. \quad (2.1)$$

Notice that system (1.2) is not well defined at $u = 0$. We restrict our attention in R^2_+ for the biological meaning. We first consider the asymptotic dynamics near (0, 0) and boundedness of system (1.2) in the interior of R^2_+.

Lemma 1 *All trajectories near* (0, 0) *of system* (1.2) *with initial values located in* R_+^2 *will leave* (0, 0). *Moreover,* Ω *is a positive invariant and attracting set for the flows of system* (1.2) *in* R_+^2*, where* R_+^2 *and* Ω *are shown in* (2.1).

Proof First, we study the asymptotic dynamics near (0, 0). Let $t = u(\alpha+u)(u^2+\eta)\tau$, then system (1.2) becomes

$$\begin{aligned} \frac{du}{d\tau} &= u^2(1-u)(\alpha+u)(u^2+\eta) - u^2v(u^2+\eta) - \gamma u^3(\alpha+u), \\ \frac{dv}{d\tau} &= v(u\delta - \beta v)(\alpha+u)(u^2+\eta), \end{aligned} \tag{2.2}$$

which is topologically equivalent to system (1.2) in the interior of R_+^2. Notice that, the Jacobian matrix of system (2.2) at the equilibrium (0, 0) is a null matrix. By using the blow-up transformations: $u = r\cos\theta,\ v = r\sin\theta,\ t = r\tau$, system (2.2) becomes

$$\begin{aligned} \frac{dr}{dt} &= r\left(\alpha\eta(-\beta\sin^3\theta + \delta\sin^2\theta\cos\theta + \cos^3\theta) + O(r)\right), \\ \frac{d\theta}{dt} &= \alpha\eta\sin\theta\cos\theta((\delta-1)\cos\theta - \beta\sin\theta) + O(r), \end{aligned} \tag{2.3}$$

where $(r,\theta) \in [0,+\infty) \times [0, \frac{\pi}{2}]$.

When $\delta > 1$, system (2.3) has three equilibria: a hyperbolic unstable node (0, 0), two hyperbolic saddles $(0, \arctan\frac{\delta-1}{\beta})$ and $(0, \frac{\pi}{2})$ (see Fig. 1a); when $\delta < 1$, it has two equilibria: (0, 0) and $(0, \frac{\pi}{2})$, which are hyperbolic saddles (see Fig. 1b). Then the corresponding trajectory structures of system (2.2) at degenerate equilibrium (0, 0) are shown numerically in Fig. 1d, e, respectively.

Notice that, when $\delta = 1$, system (2.3) has two equilibria: a degenerate equilibrium (0, 0) and a hyperbolic saddle $(0, \frac{\pi}{2})$. By using the blow-up transformations along u and v directions, and letting $u = r,\ v = ry,\ t = r\tau$ in coordinate chart $x = 1$, then system (2.2) becomes

$$\frac{dr}{dt} = \alpha\eta r + O(r^2), \quad \frac{dy}{dt} = -\alpha\beta\eta y^2 + O(r), \tag{2.4}$$

where $r \geq 0,\ y \geq 0$. System (2.4) has one equilibrium (0, 0) which is a saddle-node. Similarly, let $u = rx,\ v = r,\ t = r\tau$ in coordinate chart $y = 1$, then system (2.2) becomes

$$\frac{dr}{dt} = (-\alpha\beta\eta + \alpha\delta\eta x)r + O(r^2), \quad \frac{dx}{dt} = \alpha\beta\eta x + O(r), \tag{2.5}$$

where $r \geq 0,\ x \geq 0$. System (2.5) has one equilibrium (0, 0) which is a hyperbolic saddle. The trajectory structure in coordinate charts is shown in Fig. 1c, and the corresponding trajectory structure of system (2.2) at degenerate equilibrium (0, 0) is shown in Fig. 1e.

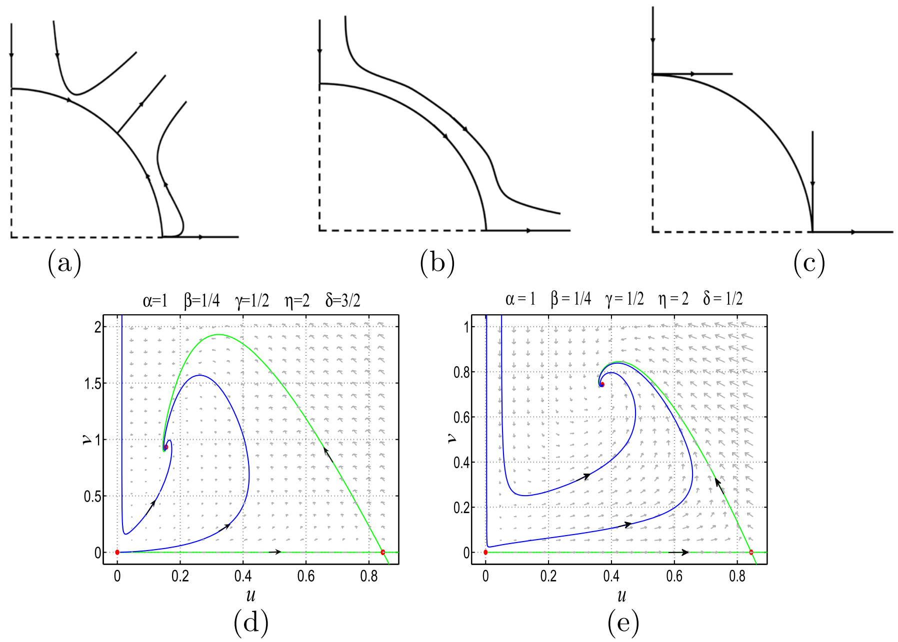

Fig. 1 **a** and **b** Represent the trajectory structure of system (2.3) outside of unit circle when $\delta > 1$ and $\delta < 1$, respectively. **c** Trajectory structure of systems (2.4) and (2.5) outside of unit circle when $\delta = 1$. **d** Phase portrait of system (2.2) near (0, 0) in R_+^2 when $\delta > 1$. **e** Phase portrait of system (2.2) near (0, 0) in R_+^2 when $\delta \leq 1$

Therefore, we know that all trajectories near (0, 0) of system (1.2) with initial values located in R_+^2 will leave (0, 0).

Second, we consider the boundedness. It is easy to know that the u-axis is invariant. From the first equation of system (1.2), we have $\frac{du}{dt} < 0$ when $u \geq 1$ and $v \geq 0$, then there exists $T_0 > 0$ such that $0 < u(t) < 1$ when $t > T_0$. From the second equation of system (1.2), we have $\frac{dv}{dt} < v(\delta - \beta v) \leq 0$ when $0 < u < 1$ and $v \geq \frac{\delta}{\beta}$, then there exists $T > T_0$, such that all trajectories with initial values located in R_+^2 will enter and remain in Ω when $t > T$, that is Ω is a positive invariant and attracting set for the flows of (1.2) in R_+^2. □

We next consider the possible boundary equilibrium $E_b(u_b, 0)$ of system (1.2), it is easy to see that u_b is the positive real root of the following third-order algebraic equation

$$F_b(u) \triangleq -\eta + (\gamma + \eta)u - u^2 + u^3 = 0, \tag{2.6}$$

which has at least one and at most three positive real roots. Define

$$A \triangleq 1 - 3(\gamma + \eta), \quad B \triangleq 9\gamma - 18\eta - 2, \quad \Delta \triangleq B^2 - 4A^3. \tag{2.7}$$

We have the following results by the Cardan formula and Theorems 7.1–7.3 in Zhang et al. (1992).

Lemma 2 *System (1.2) has at least one and at most three boundary equilibria.*

(I) If $\Delta > 0$, then system (1.2) has a unique boundary equilibrium $E_{b1}(u_{b1}, 0)$ (or $E_{b3}(u_{b3}, 0)$).
(II) If $\Delta = 0$ and

(i) $B < 0$, then system (1.2) has two different boundary equilibria: $E_{b12}(\frac{1-\sqrt{A}}{3}, 0)$, $E_{b3}(\frac{1+2\sqrt{A}}{3}, 0)$;
(ii) $B > 0$, then system (1.2) has two different boundary equilibria: $E_{b1}(\frac{1-2\sqrt{A}}{3}, 0)$, $E_{b23}(\frac{1+\sqrt{A}}{3}, 0)$;
(iii) $B = 0$, then system (1.2) has a unique boundary equilibrium $E_b^*(\frac{1}{3}, 0)$.

(III) If $\Delta < 0$, then system (1.2) has three different boundary equilibria $E_{bi}(u_{bi}, 0)$ (i=1, 2, 3).
(IV) E_{b1} and E_{b3} are elementary saddle, E_{b2} is an unstable node, E_{b12} and E_{b23} are saddle-node of codimension 1, E_b^* is a degenerate saddle of codimension 2.

Where A B and Δ are shown in (2.7) and $0 < u_{b1} < u_{b2} < u_{b3} < 1$.

The stability and types of boundary equilibria of system (1.2) are also shown in Theorem 2.2 of Xiao and Zhang (2007).

3 Positive equilibria and bifurcation

In this section, we consider the positive equilibria of system (1.2).

At the possible positive equilibrium $E(u, v)$, we can see that $v = \frac{\delta u}{\beta}$ and u is the positive real root of the following fourth-order algebraic equation

$$\begin{aligned} F(u) = &\ \alpha\beta\eta + (\eta(\beta-\delta) - \alpha\beta(\gamma+\eta))u + \beta(\alpha-\gamma-\eta)u^2 \\ &+(-\alpha\beta+\beta-\delta)u^3 - \beta u^4 \\ \triangleq &\ a_4 + a_3u + a_2u^2 + a_1u^3 + a_0u^4 = 0, \end{aligned} \tag{3.1}$$

which has at least one negative real root and at most three positive real roots. Moreover, through simple calculation, we have $\frac{d^i F(u)}{du^i}\ |_{u=1} < 0\ (i = 0, 1, 2, 3)$, which imply that the positive roots of Eq. (3.1) must satisfy $u < 1$.

The Jacobian matrix of system (1.2) at any positive equilibrium $E(u, v)$ has the form

$$J(E(u,v)) = \begin{pmatrix} \frac{2\gamma u^3}{(u^2+\eta)^2} + \frac{vu}{(u+\alpha)^2} - \frac{2\gamma u}{u^2+\eta} - 2u - \frac{v}{u+\alpha} + 1 & -\frac{u}{u+\alpha} \\ \frac{v^2\beta}{u^2} & \delta - \frac{2v\beta}{u} \end{pmatrix}.$$

From $F(u) = 0$, we have

$$\beta = \frac{-\delta u^3 - \delta\eta u}{(\alpha+u)\left(-\eta+u^3-u^2+\gamma u+\eta u\right)}, \tag{3.2}$$

where $-\eta + u^3 - u^2 + \gamma u + \eta u < 0$, then we can get that

$$\det J(E(u, v)) = \frac{-\eta + u^3 - u^2 + u(\gamma + \eta)}{\left(\eta + u^2\right)^2} F'(u). \tag{3.3}$$

Define

$$\begin{aligned} D_2 &= 3a_1^2 - 8a_0a_2, \\ D_3 &= 14a_0a_2a_3a_1 - 3a_3a_1^3 + (a_2^2 - 6a_0a_4)a_1^2 + 2a_0(-2a_2^3 + 8a_0a_4a_2 - 9a_0a_3^2), \\ D_4 &= a_0(16a_4a_2^4 - 4a_3^2a_2^3 - 128a_0a_4^2a_2^2 + 144a_0a_3^2a_4a_2 + a_0(256a_0a_4^3 - 27a_3^4)) \\ &\quad - 27a_4^2a_1^4 + (18a_2a_3a_4 - 4a_3^3)a_1^3 + (-4a_4a_2^3 + a_3^2a_2^2 + 144a_0a_4^2a_2 \\ &\quad - 6a_0a_3^2a_4)a_1^2 - 2a_0a_3(40a_4a_2^2 - 9a_3^2a_2 + 96a_0a_4^2)a_1, \end{aligned} \tag{3.4}$$

where a_0, ..., a_4 are given in (3.1), and notice that $a_0 < 0$ and $a_4 > 0$. According to the *Complete Discrimination System* of polynomials (Yang 1999), we classify the numbers and types of positive equilibria and provide some representative numerical simulations of $F(u)$ in Fig. 2.

Lemma 3 *System* (1.2) *has at least one and at most three positive equilibria.*

- *(i) If $D_i > 0$ $(i = 2, 3, 4)$, then system* (1.2) *has a unique positive equilibrium $E_3(u_3, v_3)$ or three positive equilibria $E_i(u_i, v_i)$ $(i = 1, 2, 3)$ (see Fig. 2e);*
- *(ii) If $D_4 < 0$, then system* (1.2) *has a unique positive equilibrium $E_1(u_1, v_1)$ (or $E_3(u_3, v_3)$) (see Fig. 2d);*
- *(iii) If $D_4 = 0$ and $D_3 > 0$, then system* (1.2) *has a unique positive equilibrium $E_3(u_3, v_3)$, or two positive equilibria: $E_{12}(u_{12}, v_{12})$ and $E_3(u_3, v_3)$ (or $E_1(u_1, v_1)$ and $E_{23}(u_{23}, v_{23})$) (see Fig. 2a–b);*
- *(iv) If $D_4 = D_3 = 0$ and $D_2 > 0$, then system* (1.2) *has a unique positive equilibrium $E_3(u_3, v_3)$ (or $E_{123}(u_{123}, v_{123})$) (see Fig. 2c);*
- *(v) E_1 and E_3 are elementary anti-saddle, E_2 is an elementary saddle, E_{12}, E_{23} and E_{123} are degenerate equilibria.*

Where D_i (i=2, 3, 4) are shown in (3.4) *and $u_1 < u_2 < u_3$.*

Next, we further explore the detailed types of the double positive equilibria E_{12} and E_{23} and the triple positive equilibrium E_{123}.

3.1 A double positive equilibrium

For simplicity, we denote the double positive equilibrium (E_{12} or E_{23}) as $E_*(u_*, v_*)$. From $F(u_*) = 0$, $F'(u_*) = 0$ and $\mathrm{tr}(J(E_*)) = 0$, we obtain $(\beta, \gamma, \delta) = (\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta})$, where $\widetilde{\beta}$, $\widetilde{\gamma}$ and $\widetilde{\delta}$ are shown in (1.6).

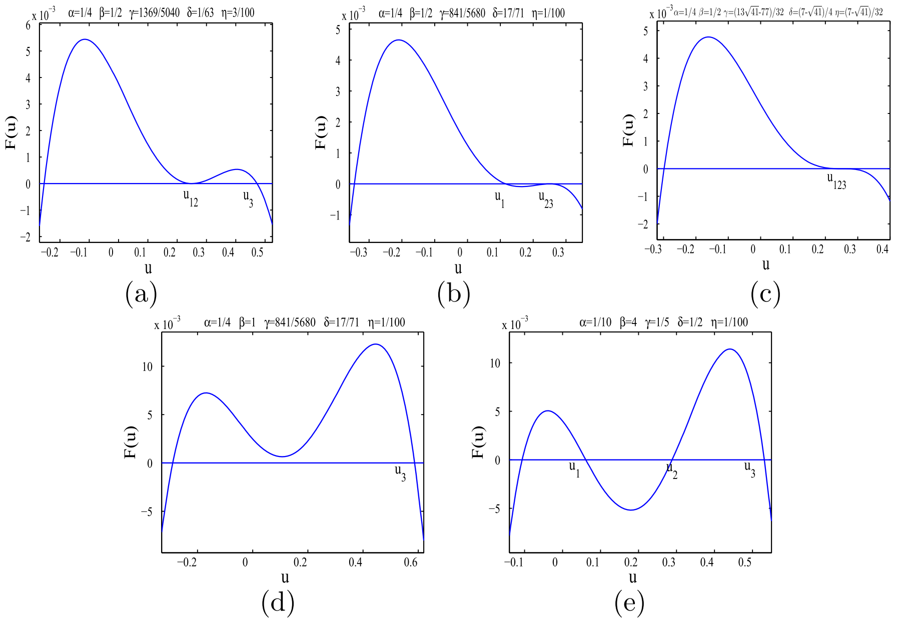

Fig. 2 The positive roots of $F(u) = 0$. **a** A double positive root u_{12} and a single positive root u_3. **b** A double positive root u_{23} and a single positive root u_1. **c** A triple positive root u_{123}. **d** A single positive root u_3. **e** Three single positive roots u_i $(i = 1, 2, 3)$

From $\widetilde{\gamma} > 0$ and $\widetilde{\delta} > 0$ we have

$$0 < u_* < \tfrac{1}{2}, \; 0 < \eta < \eta_0, \tag{3.5}$$

where η_0 is shown in (1.4).

If (β, γ, δ) satisfies the condition (1.6), we have

$$F''(u_*) = -\frac{2H}{u_*(\alpha+u_*)\left(-\eta+2\alpha u_*+u_*^2\right)}, \tag{3.6}$$

where

$$H = \alpha^2 u_*^3 + 3\alpha u_*^5 + u_*^6 + \left(-3\alpha^2 u_* - \alpha u_*^3 - 3\alpha u_*^2 - 3u_*^4\right)\eta + \alpha\eta^2,$$

which is a quadratic function of η, and $H = 0$ has two positive real roots and η_1 is the smaller one, where η_1 is shown in (1.4).

Through straightforward calculation, we have $\operatorname{sign} H(\eta_0) = \operatorname{sign}(3u_* - 1)$ and $\operatorname{sign} H'(\eta_0) = -1$. Combining the above analysis, we obtain

$$H > 0 \; (= 0, \text{ or } < 0) \quad \text{if} \quad (u_*, \eta) \in \Omega_{11} \; (\Omega_{12}, \text{ or } \Omega_{13}), \tag{3.7}$$

where Ω_{1i} $(i = 1, 2, 3)$ are shown in (1.3). Notice that E_* becomes a triple equilibrium when $H = 0$.

We first provide a useful formula on the fifth-order normal form of nilpotent cusp, which is a generalization of Proposition 5.3 in Lamontagne et al. (2008).

Lemma 4 *When $a_{20} \neq 0$, the system*

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \sum_{i+j=2}^{5} a_{ij} x^i y^j \tag{3.8}$$

is equivalent to the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = b_{20}x^2 + b_{11}xy + b_{31}x^3y + b_{41}x^4y \tag{3.9}$$

in a small neighborhood of the origin, where $b_{20} = a_{20},\ b_{11} = a_{11}$,

$$\begin{aligned}
b_{31} =& \frac{1}{40a_{20}^2}\big(6a_{30}a_{11}^3 - 8a_{20}a_{21}a_{11}^2 + (35a_{30}^2 + 4a_{20}(3a_{12}a_{20} - 8a_{40}))a_{11} \\
&+ 16a_{02}^2a_{20}^2a_{11} - 4a_{02}a_{20}(a_{11}^3 + 9a_{30}a_{11} - 10a_{20}a_{21}) - 40a_{20}(3a_{03}a_{20}^2 \\
&- a_{31}a_{20} + a_{21}a_{30})\big), \\
b_{41} =& \frac{1}{16a_{20}^3}\big(8a_{12}a_{21}a_{20}^3 - 8a_{11}a_{22}a_{20}^3 + 4a_{03}(4a_{11}^2 + 3(a_{30} - 4a_{02}a_{20}))a_{20}^3 \\
&- 24a_{13}a_{20}^4 + 20a_{21}a_{30}^2a_{20} + 8a_{02}^2(2a_{20}a_{21} - a_{11}a_{30})a_{20}^2 - 20a_{30}a_{31}a_{20}^2 \\
&- 16a_{21}a_{40}a_{20}^2 + 56a_{11}a_{50}a_{20}^2 - 72a_{11}a_{30}a_{40}a_{20} - 2a_{02}a_{20}(2a_{20}(7a_{21}a_{30} \\
&- 6a_{20}a_{31}) + a_{11}(4a_{20}(a_{12}a_{20} - 6a_{40}) + 13a_{30}^2)) + 16a_{41}a_{20}^3 + 25a_{11}a_{30}^3\big).
\end{aligned}$$

Proof Make the following time and near-identity transformations

$$t = \left(1 - \tfrac{a_{30}}{2a_{20}}\, x\right)\tau, \quad x = x_1 + \sum_{i+j=2}^{5} c_{1ij}\, x_1^i y_1^j, \quad y = y_1 + \sum_{i+j=2}^{5} c_{2ij}\, x_1^i y_1^j,$$

where c_{1ij} and c_{2ij} are functions of a_{ij} and we omit their detailed expressions, system (3.8) becomes

$$\frac{dx_1}{d\tau} = y_1, \quad \frac{dy_1}{d\tau} = b_{20}x_1^2 + b_{11}x_1y_1 + b_{31}x_1^3y_1 + b_{41}x_1^4y_1, \tag{3.10}$$

where b_{ij} are shown in (3.9). □

Remark 1 Lemma 4 can be used directly to investigate the codimension of a cusp. When $a_{11}=0$, we can simplify $b_{31},\ b_{41}$ as

$$\begin{aligned} b_{31} =& \frac{40a_{02}a_{20}^2a_{21}-40a_{20}(3a_{03}a_{20}^2-a_{31}a_{20}+a_{21}a_{30})}{40a_{20}^2}, \\ b_{41} =& \frac{1}{16a_{20}^3}(16a_{02}^2a_{21}a_{20}^3-24a_{13}a_{20}^4+8a_{12}a_{21}a_{20}^3+12a_{03}\left(a_{30}-4a_{02}a_{20}\right)a_{20}^3 \\ &+16a_{41}a_{20}^3-20a_{30}a_{31}a_{20}^2-4a_{02}\left(7a_{21}a_{30}-6a_{20}a_{31}\right)a_{20}^2-16a_{21}a_{40}a_{20}^2 \\ &+20a_{21}a_{30}^2a_{20}). \end{aligned}$$

Moreover, when $a_{20}=1,\ a_{11}=a_{02}=a_{03}=a_{13}=a_{04}=0$ we have

$$b_{31}=a_{31}-a_{21}a_{30},$$

which shows that Lemma 4 is a generalization of Proposition 5.3 in Lamontagne et al. (2008).

Proof of Theorem 1 (I) Computing the fifth-order normal form. Under the conditions (3.3) and (β,γ,δ) satisfies (1.6), we know that $\operatorname{tr}J(E_*)=\det J(E_*)=0$. Make the following transformations successively

$$X=u-u_*,\ Y=v-\tfrac{\delta u_*}{\beta};\quad x=X,\ y=\frac{dX}{dt},$$

system (1.2) becomes (up to the fifth-order expansion)

$$\frac{dx}{dt}=y,\quad \frac{dy}{dt}=\sum_{i+j=2}^{5}c_{ij}x^iy^j, \tag{3.11}$$

where c_{ij} are expressions of $(\alpha,\ \eta,\ u_*)$, and we omit the detailed expressions.

Next, by Lemma 4 we know that in a small neighborhood of the origin, system (3.11) is topologically equivalent to

$$\frac{dx}{dt}=y,\quad \frac{dy}{dt}=d_{20}x^2+d_{11}xy+d_{31}x^3y+d_{41}x^4y, \tag{3.12}$$

where

$$d_{20}=\frac{(\eta+2u_*^3-u_*^2)H}{u_*^2(\eta+u_*^2)(-\eta+2\alpha u_*+u_*^2)^2},\quad d_{11}=-\frac{M}{u_*^2(\alpha+u_*)(\eta+u_*^2)(-\eta+2\alpha u_*+u_*^2)}, \tag{3.13}$$

and

$$\begin{aligned} M =& u_*^4(2u_*^3+(8u_*-1)u_*\alpha+(2u_*+1)\alpha^2)-2u_*^2(3u_*^3+3u_*\alpha \\ &+(3-u_*)\alpha^2)\eta+\alpha(\alpha+3u_*)\eta^2, \end{aligned} \tag{3.14}$$

and H is shown in (3.6). We omit the lengthy expressions of d_{31} and d_{41}.

It is easy to know that

$$\text{sign} d_{20} = -\text{sign} H \neq 0, \quad \text{sign} d_{11} = -\text{sign} M. \tag{3.15}$$

(II) Computing sign$M$. Notice that M is a quadratic polynomial of η, the coefficients are functions of α. Through simple calculation, we have sign$M(\eta_0) = \text{sign}(3u_* - 1)$ and sign$M'(\eta_0) = -1$, which show that M has at most one positive real root when $(u_*, \eta) \in \Omega_1 \backslash \Omega_{12}$. Define

$$\begin{aligned}
\Omega_2 &= \{(u_*, \alpha) : \ 0 < u_* < \frac{1}{2} \text{ and } \alpha > 0\} \triangleq \Omega_{21} \cup \Omega_{22} \cup \Omega_{23}, \\
\Omega_{21} &= \Big\{(u_*, \alpha) : \ 0 < u_* < \frac{3-\sqrt{6}}{12} \text{ and } 0 < \alpha < \alpha_1 \text{ or } \alpha > \alpha_2; \ \text{or } u_* = \frac{3-\sqrt{6}}{12}, \\
&\qquad \alpha > 0 \text{ and } \alpha \neq \alpha_1; \text{ or } \frac{3-\sqrt{6}}{12} < u_* < \frac{1}{3} \text{ and } \alpha > 0\Big\}, \\
\Omega_{22} &= \Big\{(u_*, \alpha) : \ 0 < u_* < \frac{3-\sqrt{6}}{12} \text{ and } \alpha_1 \leq \alpha \leq \alpha_2; \ \text{or } u_* = \frac{3-\sqrt{6}}{12} \text{ and } \alpha = \alpha_1\Big\}, \\
\Omega_{23} &= \{(u_*, \alpha) : \ \frac{1}{3} \leq u_* < \frac{1}{2} \text{ and } \alpha > 0\},
\end{aligned} \tag{3.16}$$

where α_1 and α_2 are positive real roots of $M(0)$, i.e.,

$$\alpha_1 = \frac{u_*(1-8u_*-\sqrt{48u_*^2-24u_*+1})}{4u_*+2}, \quad \alpha_2 = \frac{u_*(1-8u_*+\sqrt{48u_*^2-24u_*+1})}{4u_*+2}. \tag{3.17}$$

When $(u_*, \alpha) \in \Omega_{21}$, it is easy to know that $M(0) > 0$ and $M(\eta_0) < 0$, M has exactly one positive real root $\eta = \eta_* \in (0, \ \eta_0)$, where

$$\begin{aligned}
\eta_* = &-\frac{u_*^2\sqrt{8\alpha^4-8(\alpha-2)\alpha^3u_*-8\alpha^2u_*^3+\alpha^2((\alpha-20)\alpha+12)u_*^2-6(\alpha-2)\alpha u_*^4+9u_*^6}}{\alpha(\alpha+3u_*)} \\
&+\frac{3\alpha^2u_*^2-(\alpha-3)\alpha u_*^3+3u_*^5}{\alpha(\alpha+3u_*)},
\end{aligned} \tag{3.18}$$

otherwise, M has no root in $(0, \ \eta_0)$. Moreover, we can get that

$$M \begin{cases} > 0, \text{ if } \ (u_*, \ \alpha, \ \eta) \in (\Omega_{21} \times (0, \ \eta_*)) \cup (\Omega_{23} \times (0, \ \eta_0)), \\ = 0, \text{ if } \ (u_*, \ \alpha, \ \eta) \in \Omega_{21} \times \{\eta_*\}, \\ < 0, \text{ if } \ (u_*, \ \alpha, \ \eta) \in (\Omega_{21} \times (\eta_*, \ \eta_0)) \cup (\Omega_{22} \times (0, \ \eta_0)). \end{cases} \tag{3.19}$$

Therefore, E_* is a cusp of codimension at least 3 when $(u_*, \ \alpha, \ \eta) \in \Omega_{21} \times \{\eta_*\}$, otherwise, E_* is a cusp of codimension 2.

(III) Proving E_* **is a cusp of codimension exactly 4.** Notice that, when $M = 0$ we have

$$
\begin{aligned}
d_{31} =& \frac{M_1}{Hu_*^3\,(\alpha + u_*)^{\,3}\left(\eta + u_*^2\right)^{\,3}\left(-\eta + 2\alpha u_* + u_*^2\right)}, \\
d_{41} =& \frac{M_2}{4H^2u_*^4\,(\alpha + u_*)^{\,4}\left(\eta + u_*^2\right)^{\,4}\left(\eta + 2u_*^3 - u_*^2\right)\left(-\eta + 2\alpha u_* + u_*^2\right)},
\end{aligned}
\tag{3.20}
$$

where H is shown in (3.6). We omit the lengthy expressions of M_1 and M_2.

Define

$$
\Omega_{24} \triangleq \{(u_*, \alpha, \eta) :\ (u_*,\ \alpha) \in \Omega_{21} \text{ and } \eta = \eta_*\}, \tag{3.21}
$$

where Ω_{21} and η_* are shown in (3.16) and (3.18), respectively. By using the algebraic methods of resultant elimination (Chen and Zhang 2009) and pseudo-division (Gelfand et al. 1994), we can show that $\mathrm{V}(M,\ M_1) \cap \Omega_{24} \neq \emptyset$ and $\mathrm{V}(M,\ M_1,\ M_2) \cap \Omega_{24} = \emptyset$, i.e., E_* is a cusp of codimension exactly 4. The complete analysis is very lengthy, we only give the key steps in Appendix 1.

Moreover, in Appendix 1, we show $M = M_1 = 0$ and $M_2 \neq 0$ in $\Omega_{25} \subseteq \Omega_{24}$, that is $d_{11} = d_{31} = 0$ and $d_{41} \neq 0$ in Ω_{25}, where Ω_{25} is shown in (1.5). Therefore, E_* is a cusp of codimension exactly 4 in Ω_{25}. □

From Theorem 1, we know that system (1.2) has a cusp E_* of codimension 4, and in the following we will show that there exists a cusp type degenerate BT bifurcation of codimension 4 around the double equilibrium E_*.

***Proof of Theorem* 2** We choose $(\beta,\ \gamma,\ \delta,\ \eta)$ as bifurcation parameters, and consider the following unfolding system of system (1.2)

$$
\begin{aligned}
\frac{du}{dt} &= u(1-u) - \frac{uv}{\alpha + u} - \frac{(\gamma + \mu_1)u^2}{u^2 + \eta + \mu_2}, \\
\frac{dv}{dt} &= v(\delta + \mu_3 - (\beta + \mu_4)\frac{v}{u}),
\end{aligned}
\tag{3.22}
$$

where $(\beta,\ \gamma,\ \delta,\ \eta) = \left(\widetilde{\beta},\ \widetilde{\gamma},\ \widetilde{\delta},\ \eta_{00}(u_*,\ \alpha_0(u_*))\right)$, $(u_*, \alpha, \eta) \in \Omega_{25}$ and $\mu = (\mu_1,\ \mu_2,\ \mu_3,\ \mu_4)$ is a parameter vector in a small neighborhood of $(0,\ 0,\ 0,\ 0)$.

For simplicity, we only use the substitution $(\beta,\ \gamma,\ \delta) = (\widetilde{\beta},\ \widetilde{\gamma},\ \widetilde{\delta})$ in the following calculations. We next use the following steps to transform system (3.22) to its universal unfolding (Xiang et al. 2022).

Step 1. Translation and near-identity transformations. Make the following transformations successively

$$
X = u - u_*,\ Y = v - \tfrac{\delta u_*}{\beta}; \quad x = X,\ y = \tfrac{dX}{dt},
$$

system (3.22) can be rewritten as

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \sum_{i+j=0}^{5} \overline{a}_{ij} x^i y^j + o(|\, x, y\,|^5), \tag{3.23}$$

where $\overline{a}_{03} = 0$ and the other $\overline{a}_{ij}$ are high-order polynomials of $(u_*, \ \alpha, \ \eta)$, the detailed expressions are omitted for brevity.

Step 2. Removing y^2, xy^2, x^2y^2 and x^3y^2-terms of system (3.23). Let

$$\begin{aligned} x &= x_1 + \tfrac{\bar{a}_{02}}{2} x_1^2 + \tfrac{2\bar{a}_{02}^2+\bar{a}_{12}}{6} x_1^3 + \tfrac{6\bar{a}_{02}^3+7\bar{a}_{12}\bar{a}_{02}+2\bar{a}_{22}}{24} x_1^4 \\ &\quad + \tfrac{24\bar{a}_{02}^4+46\bar{a}_{12}\bar{a}_{02}^2+22\bar{a}_{22}\bar{a}_{02}+7\bar{a}_{12}^2+6\bar{a}_{32}}{120} x_1^5, \\ y &= y_1 + \bar{a}_{02} x_1 y_1 + \tfrac{2\bar{a}_{02}^2+\bar{a}_{12}}{2} x_1^2 y_1 + \tfrac{6\bar{a}_{02}^3+7\bar{a}_{12}\bar{a}_{02}+2\bar{a}_{22}}{6} x_1^3 y_1 \\ &\quad + \tfrac{24\bar{a}_{02}^4+46\bar{a}_{12}\bar{a}_{02}^2+22\bar{a}_{22}\bar{a}_{02}+7\bar{a}_{12}^2+6\bar{a}_{32}}{24} x_1^4 y_1, \end{aligned}$$

system (3.23) becomes

$$\frac{dx_1}{dt} = y_1, \quad \frac{dy_1}{dt} = \sum_{i+j=0}^{5} \overline{b}_{ij} x_1^i y_1^j + o(|\, x_1, \ y_1\,|^5), \tag{3.24}$$

where $\overline{b}_{02} = \overline{b}_{12} = \overline{b}_{03} = \overline{b}_{22} = \overline{b}_{32} = 0$, and the other $\overline{b}_{ij}$ can be expressed by $\overline{a}_{ij}$.

*Step 3. Removing $x_1^3, \ x_1^4$ and x_1^5***-terms of system** (3.24). Let

$$\begin{aligned} dt &= \big(1 - \tfrac{\bar{b}_{30}}{2\bar{b}_{20}} x_2 - \tfrac{316\bar{b}_{20}\bar{b}_{40}-15\bar{b}_{30}^2}{80\bar{b}_{20}^2} x_2^2 + \tfrac{-175\bar{b}_{30}^3+336\bar{b}_{20}\bar{b}_{40}\bar{b}_{30}-160\bar{b}_{20}^2\bar{b}_{50}}{240\bar{b}_{20}^3} x_2^3\big) d\tau, \\ x_1 &= x_2 - \tfrac{\bar{b}_{30}}{4\bar{b}_{20}} x_2^2 - \tfrac{16\bar{b}_{20}\bar{b}_{40}-15\bar{b}_{30}^2}{80\bar{b}_{20}^2} x_2^3 + \tfrac{-175\bar{b}_{30}^3+336\bar{b}_{20}\bar{b}_{40}\bar{b}_{30}-160\bar{b}_{20}^2\bar{b}_{50}}{960\bar{b}_{20}^3} x_2^4, \ \ y_1 = y_2, \end{aligned}$$

system (3.24) becomes (still denote τ by t)

$$\frac{dx_2}{dt} = y_2, \quad \frac{dy_2}{dt} = \sum_{i+j=0}^{5} \overline{c}_{ij} x_2^i y_2^j + o(|\, x_2, \ y_2\,|^5), \tag{3.25}$$

where $\overline{c}_{02} = \overline{c}_{12} = \overline{c}_{03} = \overline{c}_{22} = \overline{c}_{32} = 0$, and $\overline{c}_{30} = \overline{c}_{40} = \overline{c}_{50} = 0$ when $\mu = 0$, the other $\overline{c}_{ij}$ can be expressed by $\overline{b}_{ij}$.

Step 4. Removing $x_2^2 y_2$-term of system (3.25). Let

$$d\tau = \big(1 + \tfrac{\bar{c}_{21}}{3\bar{c}_{20}} y_3 + \tfrac{\bar{c}_{21}^2}{36\bar{c}_{20}^2} y_3^2\big) dt, \quad x_2 = x_3, \quad y_2 = y_3 + \tfrac{\bar{c}_{21}}{3\bar{c}_{20}} y_3^2 + \tfrac{\bar{c}_{21}^2}{36\bar{c}_{20}^2} y_3^3;$$

then system (3.25) becomes (still denote τ by t)

$$\begin{aligned}\frac{dx_3}{dt} &= y_3,\\ \frac{dy_3}{dt} &= \overline{d}_{00} + \overline{d}_{10}x_3 + \overline{d}_{01}y_3 + \overline{d}_{20}x_3^2 + \overline{d}_{11}x_3y_3\\ &\quad + \overline{d}_{31}x_3^3y_3 + \overline{d}_{41}x_3^4y_3 + R(x_3, y_3, \mu),\end{aligned} \tag{3.26}$$

where $\overline{d}_{ij}$ can be expressed by $\overline{c}_{ij}$, and $R(x_3, y_3, \mu)$ satisfies the following property

$$\begin{aligned}R(x, y, \mu) &= y^2 O(\mid x, y \mid^2) + O(\mid x, y \mid^6) + O(\mu)(O(y^2) + O(\mid x, y \mid^3))\\ &\quad + O(\mu^2)O(\mid x, y \mid).\end{aligned} \tag{3.27}$$

Step 5. Removing x_3-term and transforming the coefficients of x_3^2 and $x_3^4y_3$ to 1 and -1 in (3.26), respectively. Notice that, under the conditions of Theorem 2, we can get that $\overline{d}_{20} < 0$ and $\overline{d}_{41} < 0$ for small μ, the detailed analysis is shown in Appendix 1. Let

$$dt = -\bar{d}_{41}^{1/7}\bar{d}_{20}^{-4/7}d\tau, \quad x_3 = \bar{d}_{20}^{1/7}\bar{d}_{41}^{-2/7}x_4 - \frac{\bar{d}_{10}}{2\bar{d}_{20}}, \quad y_3 = -\bar{d}_{20}^{5/7}\bar{d}_{41}^{-3/7}y_4,$$

then system (3.26) becomes (still denote τ by t)

$$\begin{aligned}\frac{dx_4}{dt} &= y_4,\\ \frac{dy_4}{dt} &= \overline{e}_1 + \overline{e}_2y_4 + \overline{e}_3x_4y_4 + \overline{e}_4x_4^3y_4 + x_4^2 - x_4^4y_4 + R(x_4, y_4, \mu),\end{aligned} \tag{3.28}$$

where $\overline{e}_i$ can be expressed by $\overline{d}_{ij}$, and $R(x_4, y_4, \mu)$ satisfies the property (3.27).

Further calculation shows that (using $(u_*, \alpha, \eta) \in \Omega_{25}$)

$$\begin{aligned}&\left|\frac{\partial(\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_4)}{\partial(\mu_1, \mu_2, \mu_3, \mu_4)}\right|_{\mu=0}\\ &= \frac{-u_*^{4/7}\left(-\eta + 2\alpha u_* + u_*^2\right)^2\overline{f}_1}{70 * 2^{3/7} * 3^{5/7}(\alpha + u_*)^{17/7}\left(\eta + u_*^2\right)^3\left(\eta + 2u_*^3 - u_*^2\right)^{23/7}\overline{f}_{21}^{6}\overline{f}_{22}^{8/7}} \neq 0,\end{aligned} \tag{3.29}$$

where

$$\overline{f}_{21} = \alpha\eta^2 - u_*(3\alpha^2 + \alpha u_*^2 + 3\alpha u_* + 3u_*^3)\eta + u_*^3(\alpha^2 + 3\alpha u_*^2 + u_*^3),$$

$\overline{f}_1$ and $\overline{f}_{22}$ are high-order polynomials of $(u_*, \ \alpha, \ \eta)$, we omit their expressions. The complete analysis of (3.29) is shown in Appendix 1.

Therefore, by the results in Chow et al. (1994); Li and Rousseau (1989), we know that system (3.28) is the versal unfolding of cusp type BT bifurcation of codimension

4, and system (1.2) undergoes cusp type BT bifurcation of codimension 4 in a small neighborhood of E_* as $(\beta, \gamma, \delta, \eta)$ varies near $(\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}, \eta_{00}(u_*, \alpha))$. □

3.2 A triple positive equilibrium

For simplicity, we denote the triple positive equilibrium $E_{123}(u_{123}, v_{123})$ as $E^*(u^*, v^*)$, and consider the type of E^*, where $v^* = \frac{\delta u^*}{\beta}$ and u^* is a triple positive root of Eq. (3.1), i.e., $F(u^*) = F'(u^*) = F''(u^*) = 0$.

From $F(u^*) = F'(u^*) = 0$, we can get that $(\beta, \gamma) = (\overline{\beta}, \overline{\gamma})$, where $\overline{\beta}$ and $\overline{\gamma}$ are shown in (1.7). Moreover, from $\overline{\beta} > 0$ and $\overline{\gamma} > 0$, we have

$$0 < \eta < \overline{\eta}_0 \triangleq (u^*)^2 (1 - 2u^*), \quad 0 < u^* < \tfrac{1}{2}. \tag{3.30}$$

When $(\beta, \gamma) = (\overline{\beta}, \overline{\gamma})$, we have

$$F''(u^*) = \frac{2\delta\overline{H}}{(\alpha+u^*)^2\left(\eta+2(u^*)^3-(u^*)^2\right)}, \tag{3.31}$$

where $\overline{H} = H\mid_{u_*\to u^*}$ and H is shown in (3.6). From $F''(u^*) = 0$, we have $\eta = \overline{\eta}$ where $\overline{\eta}$ is shown in (1.7), moreover we have $0 < u^* < \frac{1}{3}$ and $0 < \overline{\eta} < \overline{\eta}_0$.

Moreover, if $(\beta, \gamma) = (\overline{\beta}, \overline{\gamma})$, we have

$$\begin{aligned} \operatorname{tr}(J(E^*)) &= \frac{\delta u^*(-\eta + 2\alpha u^* + (u^*)^2) + (\alpha + u^*)\left(\eta + (2u^* - 1)(u^*)^2\right)}{u^*(\eta - 2\alpha u^* - (u^*)^2)} \\ &\triangleq \frac{t_1}{u^*(\eta - 2\alpha u^* - (u^*)^2)}, \end{aligned} \tag{3.32}$$

and letting $\operatorname{tr}(J(E^*)) = 0$, we have $\delta = \overline{\delta}$ where $\overline{\delta}$ is shown in (1.7), moreover we have

$$\operatorname{tr}(J(E^*)) > 0 \text{ (or } < 0) \text{ if } 0 < \delta < \overline{\delta} \text{ (or } \delta > \overline{\delta}). \tag{3.33}$$

Before exploring the type of $E^*(u^*, v^*)$ (i.e., E_{123}), we provide another useful formula on the fourth-order normal form of nilpotent singularities.

Lemma 5 *The system*

$$\frac{dx}{dt} = y + \sum_{i+j=2}^{4} e_{ij}x^i y^j, \quad \frac{dy}{dt} = \sum_{i+j=2}^{4} f_{ij}x^i y^j \tag{3.34}$$

is equivalent to the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = A_{20}x^2 + A_{11}xy + A_{30}x^3 + A_{21}x^2y + A_{40}x^4 + A_{31}x^3y \tag{3.35}$$

in a small neighborhood of the origin, where

$$
\begin{aligned}
A_{20} &= f_{20}, \quad A_{11} = 2e_{20} + f_{11}, \quad A_{30} = -e_{20} f_{11} + e_{11} f_{20} + f_{30}, \\
A_{21} &= \frac{1}{2}\left(-2e_{20} f_{02} + e_{11} f_{11} - 2e_{02} f_{20} + 6e_{30} + f_{02} f_{11} + 2f_{21}\right), \\
A_{40} &= \frac{1}{12}(7e_{11}^2 f_{20} + 6e_{11} f_{02} f_{20} + 18e_{11} f_{30} + 12e_{20}^2 f_{02} - 12e_{30} f_{11} + 8e_{21} f_{20} \\
&\quad + 2e_{02} f_{11} f_{20} - 2e_{20}\left(3e_{11} f_{11} + 8e_{02} f_{20} + 3f_{02} f_{11} + 6f_{21}\right) - f_{02}^2 f_{20} \\
&\quad - 2f_{12} f_{20} + 6f_{02} f_{30} + 12f_{40}), \\
A_{31} &= \frac{1}{6}(e_{11}^2\left(2e_{20} + f_{11}\right) + 3e_{11}\left(4e_{30} + f_{02} f_{11} + 2f_{21}\right) - 8e_{20} f_{02}^2 + 6e_{30} f_{02} \\
&\quad + 2e_{21} f_{11} - 10e_{20} f_{12} - 6e_{12} f_{20} - e_{02}(8e_{20}^2 + f_{11}^2 - 12f_{02} f_{20} + 6f_{30}) \\
&\quad - 8e_{20}e_{21} + 24e_{40} + 2f_{02}^2 f_{11} + f_{11} f_{12} - 18f_{03} f_{20} + 6f_{02} f_{21} + 6f_{31}).
\end{aligned}
\tag{3.36}
$$

Proof Make the following near-identity transformation

$$x = x_1 + \sum_{i+j=2}^{4} e_{1ij}\, x_1^i y_1^j, \quad y = y_1 + \sum_{i+j=2}^{4} e_{2ij}\, x_1^i y_1^j,$$

where e_{1ij} and e_{2ij} are functions of e_{ij} and f_{ij} and we omit their detailed expressions, system (3.34) becomes

$$\frac{dx_1}{dt} = y_1, \quad \frac{dy_1}{dt} = A_{20}x_1^2 + A_{11}x_1y_1 + A_{30}x_1^3 + A_{21}x_1^2y_1 + A_{40}x_1^4 + A_{31}x_1^3y_1, \tag{3.37}$$

where A_{ij} are given in (3.36). □

Proof of Theorem 3 **(I)** Under the conditions (3.3) and $(\beta, \gamma) = (\overline{\beta}, \overline{\gamma})$, we know that E^* is a degenerate equilibrium, and $\mathrm{tr}(J(E^*)) \neq 0$ when $\delta \neq \overline{\delta}$. Make the following transformations successively:

$$X = u - u^*,\ Y = v - v^*; \quad X = \tfrac{u^*}{(\alpha+u^*)^2}\left(\tfrac{u^*(\eta-u^*(2\alpha+u^*))x}{\eta+(2u^*-1)(u^*)^2} + \tfrac{y(\alpha+u^*)}{\delta}\right),$$
$$Y = x + y,\ t = \tfrac{1}{\mathrm{tr}(J(E^*))}\tau,$$

where $\mathrm{tr}(J(E^*))$ is shown in (3.32), system (1.2) becomes (still denote τ by t)

$$\frac{dx}{dt} = \sum_{i+j=2}^{3} g_{ij}x^i y^j + o(|\, x, y\,|^3), \quad \frac{dy}{dt} = y + \sum_{i+j=2}^{3} h_{ij}x^i y^j + o(|\, x, y\,|^3), \tag{3.38}$$

where $g_{20} = h_{20} = 0$ when $\eta = \overline{\eta}$, and

$$g_{30} = \frac{\delta(u^*)^4(-\eta + 2\alpha u^* + (u^*)^2)^3 L}{t_1^2(\alpha + u^*)^6(\eta + (u^*)^2)^2(\eta + (2u^* - 1)(u^*)^2)^2},$$
$$L = \big(\alpha^2\eta^3 + \alpha^2(2u^* - 1)(u^*)^6 - \eta^2 u^*(4\alpha^3 + 7\alpha^2 u^* + 8\alpha(u^*)^3 + 2\alpha(\alpha + 2)(u^*)^2 + 4(u^*)^4) + \eta(u^*)^3(4\alpha^3 + \alpha^2(8u^* + 7)u^* + 4\alpha(2u^* + 1)(u^*)^2 + 4(u^*)^4)\big),$$

and t_1 is shown in (3.32). The other coefficients are omitted here for simplicity.

According to Theorem 7.1 in chapter 2 of Zhang et al. (1992), we obtain the center manifold as $y = -h_{30}x^3 + o(|\ x\ |^3)$, and system (3.38) restricted to the center manifold takes the form

$$\frac{dx}{dt} = g_{30}x^3 + o(|\ x\ |^3). \tag{3.39}$$

Notice that, signg_{30} = sign$L$, we next consider sign$L$. Under the condition (1.7), we have

$$L = \frac{L_1 + L_2\sqrt{(u^*)^2((3\alpha^2+\alpha(u^*+3)u^*+3(u^*)^3)^2 - 4\alpha u^*(\alpha^2+3\alpha(u^*)^2+(u^*)^3))}}{8\alpha^2},$$
$$\begin{aligned} L_1 = &-36\alpha^7(u^*)^3 - 60\alpha^6(u^*)^5 - 172\alpha^6(u^*)^4 - 28\alpha^5(u^*)^7 - 340\alpha^5(u^*)^6 \\ &-316\alpha^5(u^*)^5 - 4\alpha^4(u^*)^9 - 212\alpha^4(u^*)^8 - 724\alpha^4(u^*)^7 - 276\alpha^4(u^*)^6 \\ &-44\alpha^3(u^*)^{10} - 580\alpha^3(u^*)^9 - 748\alpha^3(u^*)^8 - 96\alpha^3(u^*)^7 - 172\alpha^2(u^*)^{11} \\ &-748\alpha^2(u^*)^{10} - 304\alpha^2(u^*)^9 - 276\alpha(u^*)^{12} - 352\alpha(u^*)^{11} - 144(u^*)^{13}, \end{aligned}$$
$$\begin{aligned} L_2 = &\ 12\alpha^5(u^*)^2 + 16\alpha^4(u^*)^4 + 48\alpha^4(u^*)^3 + 4\alpha^3(u^*)^6 \\ &+80\alpha^3(u^*)^5 + 68\alpha^3(u^*)^4 + 32\alpha^2(u^*)^7 + 144\alpha^2(u^*)^6 + 32\alpha^2(u^*)^5 \\ &+76\alpha(u^*)^8 + 80\alpha(u^*)^7 + 48(u^*)^9, \end{aligned}$$

and

$$\begin{aligned} &L_1^2 - L_2^2(u^*)^2((3\alpha^2 + \alpha(u^* + 3)u^* + 3(u^*)^3)^2 - 4\alpha u^*(\alpha^2 + 3\alpha(u^*)^2 + (u^*)^3)) \\ &= -512\alpha^2(u^*)^8(\alpha + u^*)^5(\alpha + (u^*)^2)^2(\alpha^3 + 6\alpha^2(u^*)^2 \\ &\quad +\alpha(u^*)^4 + 4\alpha(u^*)^3 + 4(u^*)^5) < 0, \end{aligned}$$

which shows that sign$L$ = signL_2 = 1, that is $g_{30} > 0$.

Combining the time scale transformation with (3.33), we know that E^* is an unstable (or stable) degenerate node if $0 < \delta < \overline{\delta}$ (or $\delta > \overline{\delta}$).

(II) From $F(u^*) = F'(u^*) = \text{tr}(J(E^*)) = 0$, we have $(\beta, \gamma, \delta) = (\widehat{\beta}, \widehat{\gamma}, \widehat{\delta})$ and η satisfies (3.30), where $\widehat{\beta}$, $\widehat{\gamma}$ and $\widehat{\delta}$ are shown in (1.8).

When $(\beta, \gamma, \delta) = (\widehat{\beta}, \widehat{\gamma}, \widehat{\delta})$, from $F''(u^*) = 0$, we have

$$\eta = \overline{\eta}, \tag{3.40}$$

where $\overline{\eta}$ is shown in (1.7).

We will use the parameter condition $(\beta, \gamma, \delta, \eta) = (\widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \overline{\eta})$ in the following calculations instead of the parameter condition $(\beta, \gamma, \delta, \eta) = (\overline{\beta}, \overline{\gamma}, \overline{\delta}, \overline{\eta})$ to simplify the analysis. In fact, it is easy to know that these two parameter conditions are equivalent.

First, make the following transformations successively:

$$\widehat{u} = u - u^*,\ \widehat{v} = v - v^*; \quad \widehat{u} = x + \frac{y}{u^*\left(\frac{\delta u^*}{\beta(\alpha+u^*)^2} + \frac{2\gamma(u^*)^2}{\left(\eta+(u^*)^2\right)^2} - \frac{\gamma}{\eta+(u^*)^2} - 1\right)},$$
$$\widehat{v} = x\,(\alpha + u^*)\left(\frac{\delta u^*}{\beta(\alpha+u^*)^2} + \frac{2\gamma(u^*)^2}{\left(\eta+(u^*)^2\right)^2} - \frac{\gamma}{\eta+(u^*)^2} - 1\right),$$

then system (1.2) becomes (up to the fourth-order expansion)

$$\frac{dx}{dt} = y + \sum_{i+j=2}^{4} k_{ij}x^i y^j, \qquad \frac{dy}{dt} = \sum_{i+j=2}^{4} l_{ij}x^i y^j, \tag{3.41}$$

where k_{ij} and l_{ij} are expressions of $(\eta,\ \alpha,\ u^*)$, we omit their detailed expressions.

According to Lemma 5, system (3.41) is equivalent to

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \widehat{A}_{20}x^2 + \widehat{A}_{11}xy + \widehat{A}_{30}x^3 + \widehat{A}_{21}x^2y + \widehat{A}_{40}x^4 + \widehat{A}_{31}x^3y \tag{3.42}$$

in a small neighborhood of the origin, where the coefficients $\widehat{A}_{20}$, ..., $\widehat{A}_{31}$ are expressions of k_{ij} and l_{ij}, see (3.36) for details.

Notice that,

$$\begin{aligned} \widehat{A}_{20} &= \frac{(\eta + 2(u^*)^3 - (u^*)^2)\widehat{H}}{(u^*)^2(\eta + (u^*)^2)(2\alpha u^* - \eta + (u^*)^2)^2} = 0, \\ \widehat{A}_{11} &= \frac{-\widehat{M}}{(u^*)^2(\alpha + u^*)(\eta + (u^*)^2)(2\alpha u^* - \eta + (u^*)^2)}, \end{aligned} \tag{3.43}$$

where $(\widehat{H},\ \widehat{M}) = (H\ |_{u_* \to u^*},\ M\ |_{u_* \to u^*})$, H and M are shown in (3.6) and (3.14), respectively. Moreover, we can see that $\widehat{A}_{11} \neq 0$. In fact, by using the method of resultant elimination (Chen and Zhang 2009), we have $\mathrm{res}(\widehat{H},\ \widehat{M},\ \eta) = 8\alpha^2(u^*)^6(\alpha + u^*)^4(\alpha + (u^*)^2)^2(3u^* - 1)$, and then we have

$$\mathrm{V}(\widehat{H},\ \widehat{M}) = \mathrm{V}(\widehat{H},\ \widehat{M},\ 3u^* - 1),$$

by the condition in Theorem 3, we have $3u^* - 1 \neq 0$, which shows that $\mathrm{V}(\widehat{H},\ \widehat{M}) = \emptyset$, that is $\widehat{A}_{11} \neq 0$ since $\widehat{H} = 0$ when $\eta = \overline{\eta}$.

Next, using the same method as in the analysis of $\mathrm{sign}(L)$ in the proof (I) of Theorem 3, we have

$$\widehat{A}_{30} < 0 \tag{3.44}$$

under the condition (3.40).

Finally, we need to determine the signs of $5\widehat{A}_{30}\widehat{A}_{21} - 3\widehat{A}_{11}\widehat{A}_{40}$ and $\widehat{A}_{11}^2 + 8\widehat{A}_{30}$ to check the nondegeneracy. By a series of calculations, we have

$$\begin{aligned} &5\widehat{A}_{30}\widehat{A}_{21} - 3\widehat{A}_{11}\widehat{A}_{40} \\ &\quad = \frac{K}{4(u^*)^6(\alpha+u^*)^5\left(\eta+(u^*)^2\right)^4\left(\eta+2(u^*)^3-(u^*)^2\right)\left(-\eta+2\alpha u^*+(u^*)^2\right)^3}, \\ &\widehat{A}_{11}^2 + 8\widehat{A}_{30} = \frac{J}{(u^*)^4(\alpha+u^*)^2\left(\eta+(u^*)^2\right)^2\left(-\eta+2\alpha u^*+(u^*)^2\right)^2}, \end{aligned} \tag{3.45}$$

where K and J are polynomials of $(u^*,\ \alpha,\ \eta)$, we omit their complicated expressions.

First, we consider the sign of K. Substituting $\eta = \overline{\eta}$ (where $\overline{\eta}$ is shown in (3.40)) into K, we have

$$\begin{aligned} K = &\frac{2(u^*)^7(\alpha+u^*)^6(\alpha+(u^*)^2)^2}{-\alpha^6}\times \\ &\left(K_1+K_2u^*\sqrt{(3\alpha^2+\alpha(u^*+3)u^*+3(u^*)^3)^2-4\alpha u^*(\alpha^2+3\alpha(u^*)^2+(u^*)^3)}\right), \end{aligned}$$

where K_1 and K_2 are polynomials of $(u^*,\ \alpha)$ and we omit their expressions. Notice that, we have $K_1 > 0$ (all terms positive) and $K_2 < 0$ (all terms negative) when $u^* > 0$ and $\alpha > 0$. By further calculation, we have

$$\begin{aligned} &K_1^2 - K_2^2(u^*)^2((3\alpha^2+\alpha(u^*+3)u^*+3(u^*)^3)^2 - 4\alpha u^*(\alpha^2+3\alpha(u^*)^2+(u^*)^3)) \\ &\quad = 1024\alpha^6(1-3u^*)^2(u^*)^9(\alpha+u^*)^4(\alpha+(u^*)^2)^2K_3, \end{aligned}$$

where K_3 is given in (1.9).

Notice that, $K_3 = 0$ is equivalent to $K = 0$ because of $K_1K_2 < 0$. We can use some symbolic calculations to show that K_3 has zeros when $0 < u^* < \frac{1}{3}$ and $\alpha > 0$. Let $u^* = \frac{1}{1000}$, by using "realroot" isolation algorithm, there are two intervals of real roots for K_3, that is from $K_3 = 0$ we have $\alpha \doteq 0.000052$ or $\alpha \doteq 0.000532$. Therefore, $5\widehat{A}_{30}\widehat{A}_{21} - 3\widehat{A}_{11}\widehat{A}_{40}$ can be zero.

Moreover, we can provide a sufficient condition such that $K_3 \neq 0$. In fact, from (1.9) we know that K_3 is a 7-order polynomial of α with coefficients are functions of u^*, and all coefficients are positive when $\frac{1}{3} > u^* \geq \frac{34483039494027383649}{1180591620717411303424} \doteq 0.029208$, that is $K_3 > 0$.

Second, we consider the sign of J. Substituting $\eta = \overline{\eta}$ into J, we have

$$\begin{aligned} J = &-\frac{(u^*)^3(\alpha+u^*)^3(\alpha+(u^*)^2)}{2\alpha^3}\times \\ &\left(J_1+J_2u^*\sqrt{(3\alpha^2+\alpha(u^*+3)u^*+3(u^*)^3)^2-4\alpha u^*(\alpha^2+3\alpha(u^*)^2+(u^*)^3)}\right), \end{aligned}$$

where

$$\begin{aligned} J_1 = &-81\alpha^7u^* + 9\alpha^6(1-15u^*)(u^*)^2 + \alpha^5(-63(u^*)^2 - 57u^* + 529)(u^*)^3 \\ &+ \alpha^4(-9(u^*)^3 - 105(u^*)^2 + 1079u^* + 791)(u^*)^4 + \alpha^3(-39(u^*)^3 \end{aligned}$$

$$
\begin{aligned}
&+ 655(u^*)^2 + 2153u^* + 384)(u^*)^5 + \alpha^2(105(u^*)^2 + 1945u^* \\
&+ 1376)(u^*)^7 + 9\alpha(79u^* + 192)(u^*)^9 + 864(u^*)^{11}, \\
J_2 =& 27\alpha^5 + 12\alpha^4(3u^* - 2)u^* + \alpha^3(9(u^*)^2 - 12u^* - 163)(u^*)^2 + 4\alpha^2 \\
&\times (3(u^*)^2 - 76u^* - 32)(u^*)^3 - \alpha(141u^* + 352)(u^*)^5 - 288(u^*)^7. \quad (3.46)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&J_1^2 - J_2^2(u^*)^2((3\alpha^2 + \alpha(u^* + 3)u^* + 3(u^*)^3)^2 - 4\alpha u^*(\alpha^2 + 3\alpha(u^*)^2 + (u^*)^3)) \\
&= -256\alpha^3(u^*)^6(3u^* - 1)(\alpha + u^*)(\alpha + (u^*)^2)J_3,
\end{aligned}
$$

where

$$
\begin{aligned}
J_3 =& \alpha^5(-3u^* - 5) + \alpha^4(-3(u^*)^2 - 30u^* + 13)u^* + \alpha^3(-(u^*)^2 + 81u^* \\
&+10)(u^*)^2 + 4\alpha^2(9u^* + 35)(u^*)^4 + 2\alpha(37u^* + 32)(u^*)^5 + 64(u^*)^7. \quad (3.47)
\end{aligned}
$$

From (3.46), we know that J_1 is a 7-order polynomial of α with coefficients are functions of u^*, and the coefficients of (0–5)-order terms are positive when $0 < u^* < \frac{1}{3}$, the coefficient of 7-order term is negative, and the coefficient of 6-order term is uncertain. By using the Descartes' rule of signs, we know that J_1 has a unique positive real root $\overline{\alpha}_1$, and

$$
J_1 > 0\ (= 0,\ \text{or}\ < 0) \quad \text{if} \quad 0 < \alpha < \overline{\alpha}_1\ (\alpha = \overline{\alpha}_1,\ \text{or}\ \alpha > \overline{\alpha}_1). \qquad (3.48)
$$

Similarly, J_2 (or J_3) also has a unique positive real root $\overline{\alpha}_2$ (or $\overline{\alpha}_3$), and

$$
\begin{aligned}
J_2 < 0\ (= 0,\ \text{or}\ > 0) \quad \text{if} \quad 0 < \alpha < \overline{\alpha}_2\ (\alpha = \overline{\alpha}_2,\ \text{or}\ \alpha > \overline{\alpha}_2); \\
J_3 > 0\ (= 0,\ \text{or}\ < 0) \quad \text{if} \quad 0 < \alpha < \overline{\alpha}_3\ (\alpha = \overline{\alpha}_3,\ \text{or}\ \alpha > \overline{\alpha}_3).
\end{aligned} \qquad (3.49)
$$

Moreover, we can get that

$$
-\text{sign} J = \text{sign} J_1\ (\tfrac{\text{sign} J_1 + \text{sign} J_2}{2},\ \text{or sign} J_2) \quad \text{if} \quad 0 < \alpha < \overline{\alpha}_3\ (\alpha = \overline{\alpha}_3,\ \text{or}\ \alpha > \overline{\alpha}_3), \quad (3.50)
$$

and $\alpha = \overline{\alpha}_3$ is the necessary condition of $J = 0$, that is

$$
J = 0 \quad \text{onlyif} \quad \alpha = \overline{\alpha}_3. \qquad (3.51)
$$

Next, we can prove $\overline{\alpha}_i \neq \overline{\alpha}_j$ $(i, j = 1, 2, 3, i \neq j)$. By using the method of resultant elimination again, we have $\text{res}(J_1,\ J_2,\ \alpha) = 110805439331916840\,96(u^*)^{46}(3u^* - 1)^6(255(u^*)^2 - 5u^* - 24) = 0$ if $u^* = \frac{5+13\sqrt{145}}{510} \doteq 0.316747$. When $u^* = \frac{5+13\sqrt{145}}{510}$, from $J_1 = J_2 = 0$ we have $\alpha = \frac{43\sqrt{145}-925}{1530} < 0$, which shows that $\text{V}(J_1,\ J_2) = \emptyset$ when $0 < u^* < \frac{1}{3}$ and $\alpha > 0$, that is $\overline{\alpha}_1 \neq \overline{\alpha}_2$. Similarly, we can get that $\overline{\alpha}_i$ (i=1, 2, 3) are unequal to each other.

In fact, we can prove that $\overline{\alpha}_2 < \overline{\alpha}_3 < \overline{\alpha}_1$ by contradiction. Assume that $\overline{\alpha}_3 < \overline{\alpha}_1 < \overline{\alpha}_2$, according to (3.48), (3.49) and (3.50), we have $J < 0$ if $\alpha \in (0,\ \overline{\alpha}_3) \cup (\overline{\alpha}_2,\ +\infty)$;

$J = 0$ if $\alpha = \overline{\alpha}_i$ (i=2, 3); and $J > 0$ if $\alpha \in (\overline{\alpha}_3, \overline{\alpha}_2)$, which contradicts with (3.51). Similarly, we can eliminate the other cases.

Therefore, from $\overline{\alpha}_2 < \overline{\alpha}_3 < \overline{\alpha}_1$ and (3.50), we have that $J < 0$ when $0 < u^* < \frac{1}{3}$ and $\alpha > 0$. By Lemma 3.1 in Cai et al. (2013), we know that E^* is a degenerate nilpotent focus of codimension 3 under the conditions of Theorem 3(II). □

From Theorem 3, system (1.2) has a nilpotent focus E^* of codimension 3, in the following we will show that the focus type BT bifurcation of codimension 3 around the triple equilibrium E^* can be fully unfolded in system (1.2).

***Proof of Theorem* 4** We choose (β, γ, δ) as bifurcation parameters, and consider the following unfolding system of system (1.2)

$$\begin{aligned} \frac{du}{dt} &= u(1-u) - \tfrac{uv}{\alpha+u} - \tfrac{u^2(\gamma+\lambda_1)}{\eta+u^2}, \\ \frac{dv}{dt} &= v(\delta+\lambda_2 - \tfrac{v(\beta+\lambda_3)}{u}), \end{aligned} \tag{3.52}$$

where $(\beta, \gamma, \delta, \eta) = (\widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \overline{\eta})$, and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a parameter vector in a small neighborhood of $(0, 0, 0)$.

We next use the following two steps to transform system (3.52) to its versal unfolding (Xiao and Zhang (2007); Lu and Huang (2021)).

Step 1. Using the similar transformations as Step 1 in the proof of Theorem 2. Therefore, system (3.52) can be rewritten as

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \sum_{i+j=0}^{3} \hat{a}_{ij} x^i y^j + o(|\, x, \; y\,|^3), \tag{3.53}$$

where $\hat{a}_{ij}$ are high-order polynomials of (u^*, α), we omit the detailed expressions.

Step 2. Removing x^2-term and transforming the coefficients of x^3 and x^2y to -1 in system (3.53). Notice that, when $\lambda = 0$ we have $\hat{a}_{30} = \widehat{A}_{30} < 0$ ($\widehat{A}_{30}$ is shown in (3.44)) and

$$\begin{aligned} \hat{a}_{21} = &[3\alpha^3 u^*(\eta^2 - 6\eta(u^*)^2 + (u^*)^4) + \alpha^2(\eta^3 + 5\eta^2(u^*)^3 + 10\eta^2(u^*)^2 - 14\eta(u^*)^5 \\ &- 35\eta(u^*)^4 + 5(u^*)^7 + 4(u^*)^6) + \alpha(3\eta^3 u^* + 15\eta^2(u^*)^4 + 9\eta^2(u^*)^3 \\ &- 22\eta(u^*)^6 - 19\eta(u^*)^5 + 11(u^*)^8 - (u^*)^7) + 2(3\eta^2(u^*)^5 - 8\eta(u^*)^7 \\ &+ (u^*)^9)]/[(u^*)^2(\alpha+u^*)^2(\eta+(u^*)^2)^2(-\eta+2\alpha u^* + (u^*)^2)], \end{aligned}$$

under the condition $\eta = \overline{\eta}$, we have $\hat{a}_{21}\,|_{\lambda=0} < 0$. Let

$$t = \frac{\hat{a}_{21}}{\hat{a}_{30}}\tau, \; x = -\frac{\sqrt{-\hat{a}_{30}}x_1}{\hat{a}_{21}} - \frac{\hat{a}_{20}}{3\hat{a}_{30}}, \quad y = \frac{(-\hat{a}_{30})^{3/2} y_1}{\hat{a}_{21}^2},$$

then system (3.53) becomes (still denote τ by t)

$$\begin{aligned}\frac{dx_1}{dt} &= y_1,\\ \frac{dy_1}{dt} &= \hat{b}_{00} + \hat{b}_{10}x_1 + \hat{b}_{01}y_1 + \hat{b}_{11}x_1y_1 + \hat{b}_{02}y_1^2 - x_1^3 - x_1^2y_1 + \hat{b}_{12}x_1y_1^2\\ &\quad + \hat{b}_{03}y_1^3 + o(|\ x_1,\ y_1\ |^3),\end{aligned} \tag{3.54}$$

where

$$\begin{aligned}\hat{b}_{00} &= -\frac{2\hat{a}_{20}^3\hat{a}_{21}^3+27\hat{a}_{00}\hat{a}_{30}^2\hat{a}_{21}^3-9\hat{a}_{10}\hat{a}_{20}\hat{a}_{30}\hat{a}_{21}^3}{27(-\hat{a}_{30})^{9/2}},\ \hat{b}_{10} = \frac{\hat{a}_{21}^2(3\hat{a}_{10}\hat{a}_{30}-\hat{a}_{20}^2)}{3\hat{a}_{30}^3},\\ \hat{b}_{01} &= \frac{\hat{a}_{21}(\hat{a}_{21}\hat{a}_{20}^2-3\hat{a}_{11}\hat{a}_{30}\hat{a}_{20}+9\hat{a}_{01}\hat{a}_{30}^2)}{9\hat{a}_{30}^3},\ \hat{b}_{11} = -\frac{3\hat{a}_{11}\hat{a}_{30}-2\hat{a}_{20}\hat{a}_{21}}{3(-\hat{a}_{30})^{3/2}},\\ \hat{b}_{02} &= \frac{3\hat{a}_{02}\hat{a}_{30}-\hat{a}_{12}\hat{a}_{20}}{3\hat{a}_{21}\sqrt{-\hat{a}_{30}}},\ \hat{b}_{12} = -\frac{\hat{a}_{12}\hat{a}_{30}}{\hat{a}_{21}^2},\ \hat{b}_{03} = -\frac{\hat{a}_{03}\hat{a}_{30}^2}{\hat{a}_{21}^3}.\end{aligned}$$

Further calculation shows that

$$\left|\frac{\partial(\hat{b}_{00},\ \hat{b}_{10},\ \hat{b}_{01})}{\partial(\lambda_1,\ \lambda_2,\ \lambda_3)}\right|_{\lambda=0} = \frac{\hat{c}_1^5\hat{c}_2((u^*)^2-\eta-2(u^*)^3)\left(\frac{(u^*)^3(\alpha+u^*)(\eta+(u^*)^2)^2(-\eta+2\alpha u^*+(u^*)^2)^2}{-\hat{c}_3(\eta+2(u^*)^3-(u^*)^2)}\right)^{\frac{11}{2}}}{9\hat{c}_3^3(u^*)^{14}(\alpha+u^*)^{10}(\eta+(u^*)^2)^{14}(2\alpha u^*+(u^*)^2-\eta)^9},$$

where

$$\begin{aligned}\hat{c}_1 &= \eta^2u^*(3\alpha^3+10\alpha^2u^*+15\alpha(u^*)^3+\alpha(5\alpha+9)(u^*)^2+6(u^*)^4)-\eta(u^*)^3(18\alpha^3\\ &\quad +35\alpha^2u^*+22\alpha(u^*)^3+\alpha(14\alpha+19)(u^*)^2+16(u^*)^4)+(u^*)^5(3\alpha^3+4\alpha^2u^*\\ &\quad +11\alpha(u^*)^3+\alpha(5\alpha-1)(u^*)^2+2(u^*)^4)+\alpha\eta^3(\alpha+3u^*),\\ \hat{c}_3 &= \alpha^2\eta^3+\alpha^2(2u^*-1)(u^*)^6-\eta^2u^*(4\alpha^3+7\alpha^2u^*+8\alpha(u^*)^3+2\alpha(\alpha+2)(u^*)^4)\\ &\quad +\eta(u^*)^3(4\alpha^3+7\alpha^2u^*+8\alpha(u^*)^3+4\alpha(2\alpha+1)(u^*)^2+4(u^*)^4),\end{aligned}$$

and $\hat{c}_2$ is a lengthy polynomial of $(u^*,\ \alpha,\ \eta)$, we omit its expression.

We next prove the non-degeneracy condition: $|\ \frac{\partial(\hat{b}_{00},\hat{b}_{10},\hat{b}_{01})}{\partial(\lambda_1,\lambda_2,\lambda_3)}\ |_{\lambda=0}\neq 0$ when $\eta=\overline{\eta}$ and $0<u^*<\frac{1}{3}$. From (3.31), we know that $\eta=\overline{\eta}$ is the positive real root of $\overline{H}=0$. Thus, we calculate the resultant "res$(\overline{H},\ \hat{c}_1\hat{c}_2\hat{c}_3,\ \eta)$" as follows

$$\begin{aligned}\text{res}(\overline{H},\ \hat{c}_1\hat{c}_2\hat{c}_3,\ \eta) &= -169869312\alpha^6(u^*)^{58}(3u^*-1)^2(\alpha+u^*)^{34}(\alpha+(u^*)^2)^{15}(\alpha^3\\ &\quad +6\alpha^2(u^*)^2+\alpha(u^*)^4+4\alpha(u^*)^3+4(u^*)^5)^4\widehat{r}_{11}^2,\end{aligned}$$

where

$$\begin{aligned}\widehat{r}_{11} &= 18\alpha^4+84\alpha^3(u^*)^2+19\alpha^3u^*-6\alpha^2(u^*)^4+106\alpha^2(u^*)^3+13\alpha^2(u^*)^2\\ &\quad +3\alpha(u^*)^5+70\alpha(u^*)^4+45(u^*)^6.\end{aligned}$$

Since $\widehat{r}_{11}\mid_{u^*=0}=18\alpha^4>0, \widehat{r}_{11}\mid_{u^*=\frac{1}{3}}=\frac{1}{81}(1458\alpha^4+1269\alpha^3+429\alpha^2+71\alpha+5)>0$, and

$$\begin{aligned}\widehat{r}_{11}\mid_{u^*=\frac{1}{3(1+u^*)}}=&\frac{1}{81(1+u^*)^6}(1458\alpha^4(u^*+1)^6+27\alpha^3(19u^*+47)(u^*+1)^4\\&+3\alpha^2(39(u^*)^2+184u^*+143)(u^*+1)^2+\alpha(70(u^*)^2+141u^*\\&+71)+5)>0,\end{aligned}$$

by Lemma 3.1 in Yang (1999), we know that $\widehat{r}_{11}>0$, that is res$(\overline{H},\ \hat{c}_1\hat{c}_2\hat{c}_3,\ \eta)<0$. Hence, we can get that V$(\overline{H},\ \hat{c}_1\hat{c}_2\hat{c}_3)=\emptyset$, which shows that $\hat{c}_1\hat{c}_2\hat{c}_3\neq 0$, that is, $\mid\frac{\partial(\hat{b}_{00},\hat{b}_{10},\hat{b}_{01})}{\partial(\lambda_1,\lambda_2,\lambda_3)}\mid_{\lambda=0}\neq 0$ when $\eta=\overline{\eta}$ and $0<u^*<\frac{1}{3}$.

Moreover, by simple calculation, we have

$$\hat{b}_{11}\mid_{\lambda=0}=\frac{\hat{d}_2(\eta+2(u^*)^3-(u^*)^2)(\frac{(u^*)^3(\alpha+u^*)(\eta+(u^*)^2)^2(2\alpha u^*+(u^*)^2-\eta)^2}{-\hat{d}_1(\eta+2(u^*)^3-(u^*)^2)})^{3/2}}{3(u^*)^5(\alpha+u^*)^2(\eta+(u^*)^2)^2(2\alpha u^*+(u^*)^2-\eta)^3},\quad(3.55)$$

where $\hat{d}_1$ and $\hat{d}_2$ are shown in (D1). We can prove $0<\hat{b}_{11}\mid_{\lambda=0}<2\sqrt{2}$, the detailed proof is shown in Appendix 1.

Therefore by the results in Dumortier et al. (1991), we know that system (3.54) is the universal unfolding of focus type degenerate BT bifurcation of codimension 3, that is system (3.54) (or system (1.2)) can undergo a focus type degenerate BT bifurcation of codimension 3 in a small neighborhood of E^*. □

4 Numerical simulations

In this section, by *Matcont* software, we provide some numerical simulations to illustrate the existence of three limit cycles in Fig. 3, the existence of cusp type BT bifurcation in Figs. 4 and 5, the existence of focus type BT bifurcation of codimension 3 in Figs. 6 and 7, and explore the effect of generalist predation in model (1.2) with $\gamma\neq 0$ on the dynamics of model (1.2) with $\gamma=0$, especially on the oscillations dynamics driven by specialist predation in Fig. 8.

4.1 Bifurcation diagrams and phase portraits

In Fig. 3, we simulate the bifurcation diagram and phase portrait of three limit cycles of system (1.2). From Fig. 3 we can get that there are three limit cycles for system (1.2) when $\beta\in(0.5082,\ 0.5093)$, the middle limit cycle is stable and the other two limit cycles are unstable, and there is a stable positive equilibrium E_1 near the origin.

In Fig. 4, we show the bifurcation diagram of cusp type BT bifurcation for system (1.2). In details, in Fig. 4a, we fix $(\alpha,\ \eta,\ \delta)=(0.022424,\ 0.000264,\ 0.607066)$ and show the biparametric bifurcation diagram in (γ,β) plane. In Fig. 4b, we further fix $\beta=0.736$ and show the existence of saddle-node bifurcation of limit cycles in (γ,u) plane. These bifurcation curves divide the (γ,β) plane into several regions,

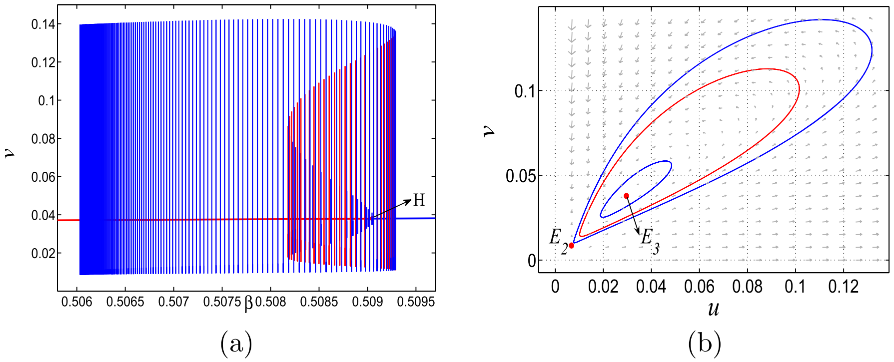

Fig. 3 **a** Bifurcation diagram of system (1.2) in $\beta - v$ plane with $(\alpha, \gamma, \delta, \eta) = (0.016, 0.004167, 0.65, 0.0000003)$. **b** Three limit cycles where $\beta = 0.5086$ and the other parameters are same as (**a**). The red (or blue) lines represent stable (or unstable) equilibrium or limit cycle (color figure online)

system (1.2) will undergo a series of bifurcations and exhibit complex dynamics when parameters vary in these regions.

In Fig. 5, we give the corresponding phase portraits for system (1.2) when (γ, β) locate in different regions of Fig. 4a. Notice that, there is always an unstable boundary equilibrium E_b. In Fig. 5a–f, $\beta = 0.736$ and γ increases from 0.0185 to 0.024; while in (g)-(j), $\beta = 0.755$ and γ increases from 0.0185 to 0.024. The other parameters are the same as those in Fig. 4a. Table 1 shows the detailed dynamics and bifurcation phenomena in Fig. 5. From Fig. 5 and Table 1, as parameters vary, we observe the following results:

(I) in case (a)-(f), system (1.2) undergoes successively saddle-node, repelling homoclinic cycle, attracting Hopf, degenerate Hopf and saddle-node bifurcations;

(II) in case (g)-(j), system (1.2) undergoes successively saddle-node, repelling homoclinic cycle and repelling Hopf bifurcations, which actually correspond to the repelling BT bifurcation of codimension 2.

In Fig. 6, we show the biparametric bifurcation diagram of focus type BT bifurcation of codimension 3 for system (3.52) in (λ_3, λ_1) plane by using *Matcont* program. In details, in Fig. 6a, we fix $(\alpha, \eta, \beta, \gamma, \delta, \lambda_2) = (\frac{1}{5}, \frac{17-2\sqrt{61}}{125}, \frac{1}{2}, \frac{8(4\sqrt{61}-29)}{125}, \frac{9-\sqrt{61}}{5}, -\frac{1}{50})$. Similarly, in Fig. 6b, we fix $(\alpha, \eta, \beta, \gamma, \delta, \lambda_2) = (\frac{3}{10}, \frac{199-2\sqrt{8689}}{1620}, \frac{5}{14}, \frac{167\sqrt{8689}-15064}{4050}, \frac{102-\sqrt{8689}}{35}, \frac{1}{100})$. These bifurcation curves divide the (λ_3, λ_1) plane into several regions, system (3.52) will undergo a series of bifurcations and exhibit complex dynamical behaviors when parameters vary in these regions.

In Fig. 7, we give the corresponding phase portraits for system (3.52) when (λ_3, λ_1) locate in different regions of Fig. 6a, b. Notice that, there is always an unstable boundary equilibrium E_b. In Fig. 7a–g, $\lambda_1 = 0.001$ and λ_3 increases from -0.04 to -0.0337, the other parameters are the same as Fig. 6a. In Fig. 7h–j, $(\lambda_1, \lambda_3) = (0.0003, 0.01626)$, $(\lambda_1, \lambda_3) = (0.0003, 0.0164)$ and $(\lambda_1, \lambda_3) = (0.00059, 0.0184)$, respectively, the other parameters are the same as Fig. 6b. Table 2 shows the detailed dynamical behaviors and bifurcation phenomena in Fig. 7. From Fig. 7 and Table 2, as parameters vary, we observe the following results:

Table 1 Dynamics and bifurcations in Fig. 5, where ⓢ (or ⓤ) represents stable (or unstable) equilibria and limit cycles

Fig. 5	Equilibria	Limit cycles	Bifurcation
(a)	E_3 ⓢ	0	No
(b)	E_1 ⓢ E_2 ⓤ E_3 ⓢ	0	Saddle-node
(c)	E_1 ⓢ E_2 ⓤ E_3 ⓢ	1: ⓤ	Homoclinic
(d)	E_1 ⓢ E_2 ⓤ E_3 ⓤ	2: outer ⓤ inner ⓢ	Hopf
(e)	E_1 ⓢ E_2 ⓤ E_3 ⓤ	0	Degenerate Hopf
(f)	E_1 ⓢ	0	Saddle-node
(g)	E_3 ⓢ	0	No
(h)	E_1 ⓢ E_2 ⓤ E_3 ⓢ	0	Saddle-node
(i)	E_1 ⓢ E_2 ⓤ E_3 ⓢ	1: ⓤ	Homoclinic
(j)	E_1 ⓢ E_2 ⓤ E_3 ⓤ	0	Hopf

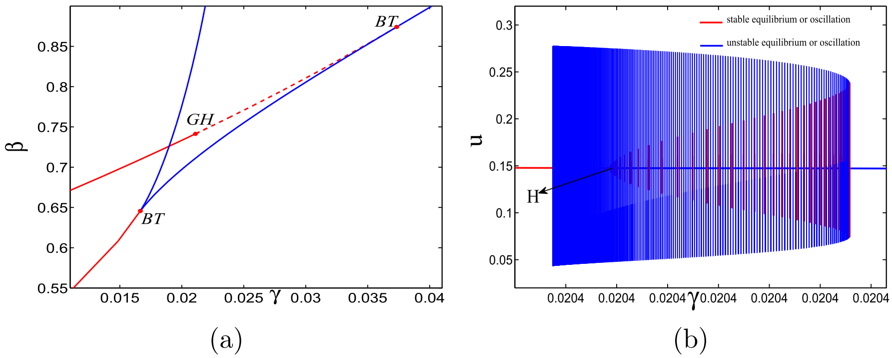

Fig. 4 Bifurcation diagrams of cusp type BT bifurcation for system (1.2) in (γ, β) and (γ, u) plane with $(\alpha, \eta, \delta) = (0.022424, 0.000264, 0.607066)$. **a** The red solid and dashed lines denote the supercritical and subcritical Hopf bifurcation, respectively; the blue lines denote the saddle-node bifurcation; BT and GH are the BT point and degenerate Hopf bifurcation point, respectively. **b** $\beta = 0.736$, the red and blue lines denote the stable and unstable equilibrium or oscillation, respectively (color figure online)

(I) In case (a)–(g), system (3.52) undergoes successively degenerate Hopf, saddle-node, repelling big homoclinic cycle, repelling small homoclinic cycle, repelling Hopf and saddle-node bifurcations;

(II) In case (h)–(j), system (3.52) undergoes successively attracting Hopf and attracting homoclinic bifurcations.

Moreover, it is easy to know that Fig. 7d–g actually implies the existence of the repelling BT bifurcation of codimension 2; and Fig. 7h–j implies the existence of the attracting BT bifurcation of codimension 2.

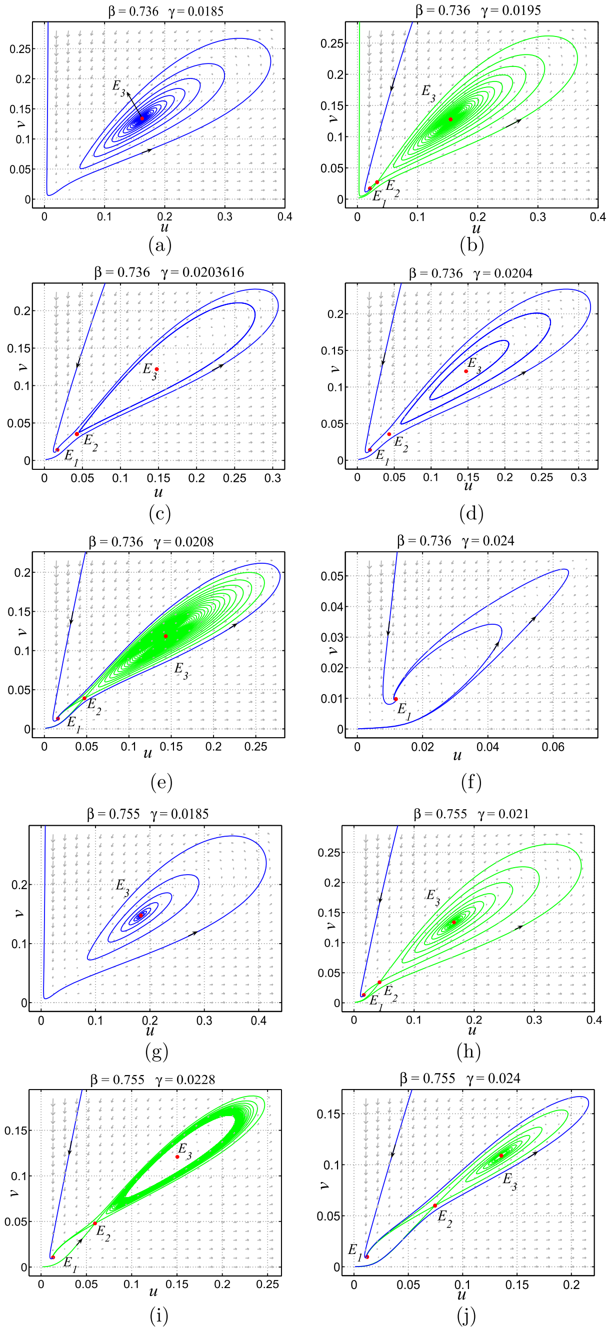

Fig. 5 Phase portraits of system (1.2) corresponding to Fig. 4. **a**–**f** $\beta = 0.736$ and γ increases from 0.0185 to 0.024; **g**–**j** $\beta = 0.755$ and γ increases from 0.0185 to 0.024. The detailed dynamics are shown in Table 1

Table 2 Dynamics and bifurcations in Fig. 7, where ⓢ (or ⓤ) represents stable (or unstable) equilibria and limit cycles

Fig. 7	Equilibria	Limit cycles	Bifurcation
(a)	E_1 ⓢ	0	No
(b)	E_1 ⓢ	2: outer ⓢ inner ⓤ	Degenerate Hopf
(c)	E_1 ⓢ E_2 ⓤ E_3 ⓤ	2: outer ⓢ inner ⓤ	Saddle-node
(d)	E_1 ⓢ E_2 ⓤ E_3 ⓤ	1: ⓢ	Homoclinic
(e)	E_1 ⓢ E_2 ⓤ E_3 ⓤ	2: outer ⓢ inner ⓤ	Homoclinic
(f)	E_1 ⓤ E_2 ⓤ E_3 ⓤ	1: ⓢ	Hopf
(g)	E_3 ⓤ	1: ⓢ	Saddle-node
(h)	E_1 ⓢ E_2 ⓤ E_3 ⓢ	0	No
(i)	E_1 ⓢ E_2 ⓤ E_3 ⓤ	1: ⓢ	Hopf
(j)	E_1 ⓢ E_2 ⓤ E_3 ⓤ	0	Homoclinic

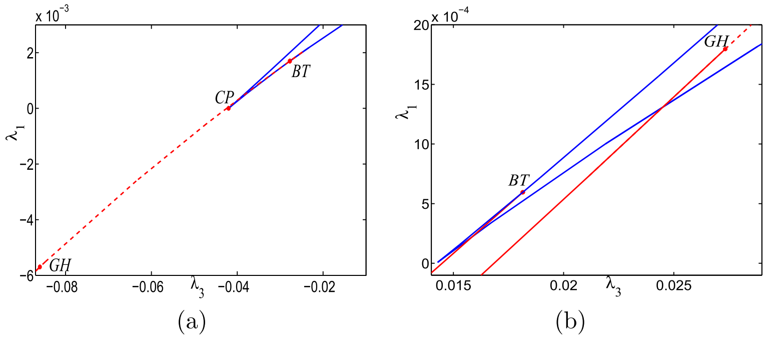

Fig. 6 Bifurcation diagrams of focus type BT bifurcation of codimension 3 for system (3.52) in (λ_3, λ_1) plane. **a** $(\alpha, \eta, \beta, \gamma, \delta, \lambda_2) = (\frac{1}{5}, \frac{17-2\sqrt{61}}{125}, \frac{1}{2}, \frac{8(4\sqrt{61}-29)}{125}, \frac{9-\sqrt{61}}{5}, -\frac{1}{50})$; **b** $(\alpha, \eta, \beta, \gamma, \delta, \lambda_2) = (\frac{3}{10}, \frac{199-2\sqrt{8689}}{1620}, \frac{5}{14}, \frac{167\sqrt{8689}-15064}{4050}, \frac{102-\sqrt{8689}}{35}, \frac{1}{100})$. The red solid and dashed lines denote the supercritical and subcritical Hopf bifurcation, respectively; the blue lines denote the saddle-node bifurcation; BT, CP and GH are the BT point, cusp point, and degenerate Hopf bifurcation point, respectively (color figure online)

4.2 The impact of generalist predation on oscillations

In Fig. 8a, for model (1.2) with $\gamma = 0$, i.e., no generalist predation, there exists a stable limit cycle enclosing an unstable positive equilibrium E_0; for model (1.2) with $\gamma \geq 0$, we plot the bifurcation diagram in $(\gamma - u)$ plane in Fig. 8c, from which we can see that the amplitude of prey cycle density decreases as γ (the density of generalist predators) increases. Moreover, a sufficiently large density of generalist predators will stabilize the stable limit cycle to a stable equilibrium, which answers the famous Fennoscandia phenomenon (Hanski et al. 1991): why can a quite regular population cycle of microtine rodents in northern Fennoscandia not be observed in southern Fennoscandia?

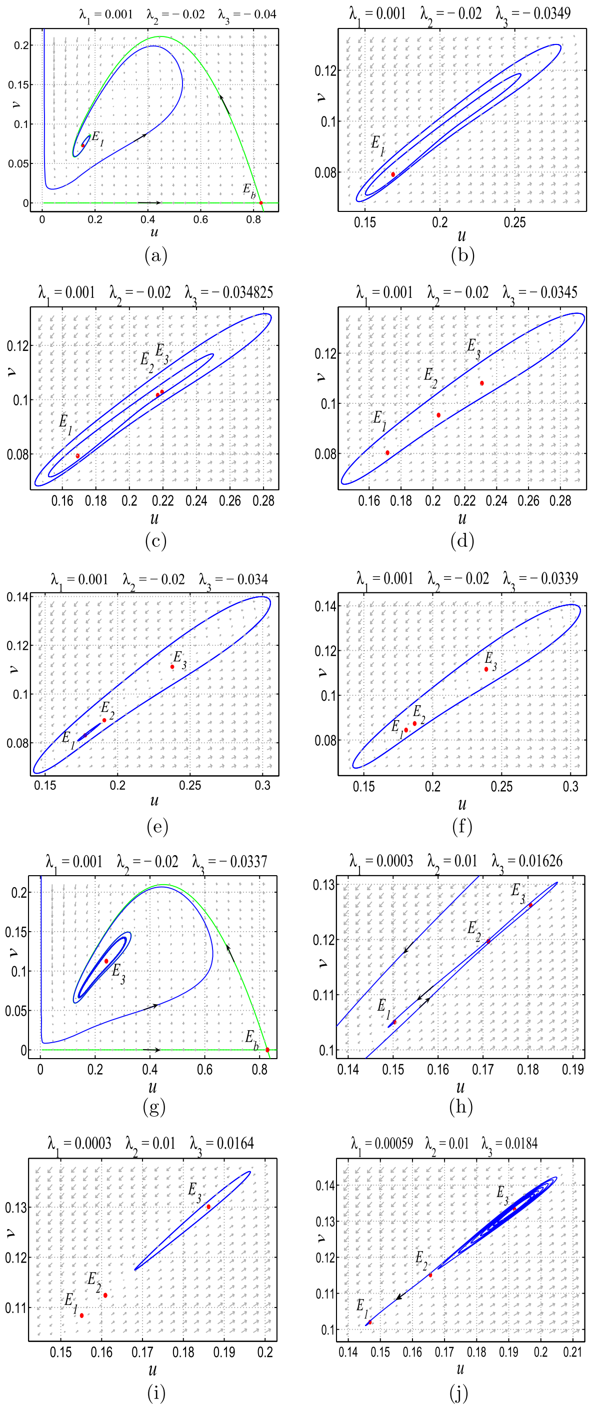

Fig. 7 Phase portraits of system (3.52) corresponding to Fig. 6. **a**–**g** $\lambda_1 = 0.001$ and λ_3 increases from -0.04 to -0.0337; **h** $(\lambda_1, \lambda_3) = (0.0003, 0.01626)$; **i** $(\lambda_1, \lambda_3) = (0.0003, 0.0164)$; **j** $(\lambda_1, \lambda_3) = (0.00059, 0.0184)$. The detailed dynamics are shown in Table 2

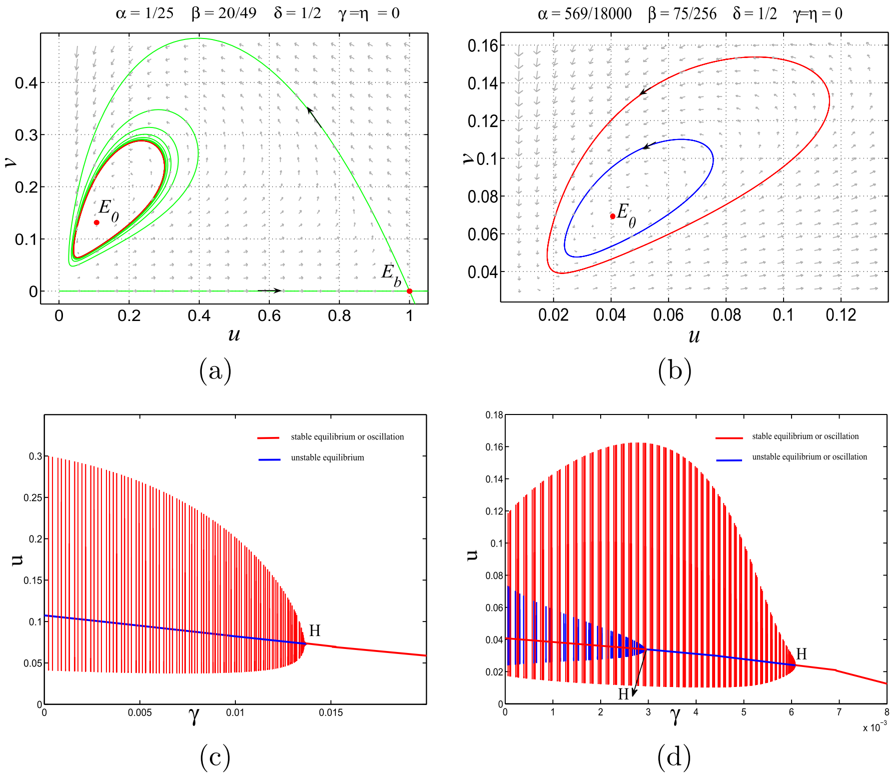

Fig. 8 **a** One stable limit cycle of system (1.2) with $\gamma = 0$ (i.e., no generalist predators). **b** Two limit cycles (the outer one is stable) of system (1.2) with $\gamma = 0$. **c** Bifurcation diagram of system (1.2) with $\gamma \geq 0$ in $(\gamma - u)$ plane, where $\eta = 0.002$ and the other parameters are the same as (**a**). **d** Bifurcation diagram of system (1.2) with $\gamma \geq 0$ in $(\gamma - u)$ plane, where $\eta = 0.0000435$ and the other parameters are the same as (**b**)

In Fig. 8b, for model (1.2) with $\gamma = 0$, there exists two nested limit cycles, the outer one is stable, enclosing a stable positive equilibrium E_0; for model (1.2) with $\gamma \geq 0$, we plot the bifurcation diagram in $(\gamma - u)$ plane in Fig. 8d, from which we observe that the amplitude of the unstable prey cycle decreases, while the amplitude of the stable prey cycle increases first and then decreases as γ (the proxy for the constant density of generalist predators) increases. Moreover, a sufficiently large density of generalist predators will stabilize these two limit cycles to a stable equilibrium, which further interprets Fennoscandia phenomenon.

5 Discussion

In this paper, we revisited a predator–prey model with two kinds of predations: specialist and generalist, where the density of generalist predators was assumed to be a constant. This model was proposed by Hanski et al. (1991) and studied by Lindström (1993) and Xiao and Zhang (2007). By using the normal form theory and some algebraic and symbolic computation methods (Chen and Zhang 2009; Yang 1999; Gelfand et al. 1994), such as resultant elimination, pseudo-division, Complete Discrimination System of polynomials and Sturm's theorem, etc., we showed the existence of a nilpotent cusp of codimension 4 or a nilpotent focus of codimension 3 for different parameter values, and model (1.2) can undergo cusp type (or focus type) degenerate BT bifurcations of codimension 4 (or 3) as the parameters vary. Our results provide a rigorous mathematical confirmation of numerical observations and a mathematical generalization of special cases in Hanski et al. (1991); Lindström (1993); Xiao and Zhang (2007).

Compared to model (1.2) with only specialist predator, i.e., (1.2) with $m = 0$, where there exist at most one positive equilibrium, Hopf bifurcation with codimension at most 2, and no BT bifurcation (Gasull et al. 1997; Hsu and Huang 1999; Sáez and González-Olivares 1999). Our results about (1.2) with $m > 0$ indicate that generalist predation can induce more complex dynamical behaviors and bifurcation phenomena, such as three small-amplitude limit cycles enclosing one positive steady state, one or two large-amplitude limit cycles enclosing one or three positive steady states, three limit cycles appearing in a Hopf bifurcation of codimension 3 and dying in a homoclinic bifurcation of codimension 3, cusp type (or focus type) degenerate BT bifurcations of codimension 4 (or 3).

Our results imply that generalist predation can stabilize the limit cycle driven by specialist predators to a stable equilibrium, which answers Fennoscandia phenomenon (Hanski et al. 1991). The scientific value of this work is to interpret the puzzling Fennoscandia observation in the mathematical perspective and to rigorously show all possibilities of predator–prey limit cycles in the presence of both generalist and specialist predators.

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Data Availability This paper has no associated data.

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A: The proof of Theorem 1

In this Appendix, we first show that E_* is a cusp of codimension at most 4 in Theorem 1. We need to show that

$$\mathrm{V}(M,\ M_1,\ M_2) \cap \Omega_{24} = \emptyset, \tag{A1}$$

where Ω_{24} is shown in (3.21), and the other notations are shown in Theorem 1.

We denote "$\mathrm{V}(f_1,\ f_2, ..., f_n)$" as the set of common zeros of $f_1,\ f_2, ..., f_n$, "$\mathrm{res}(f_1,\ f_2,\ x)$" as the resultant of f_1 and f_2 with respect to x, "$\mathrm{prem}(f_1,\ f_2,\ x)$" as the pseudo-remainder of f_1 divided by f_2 with respect to x, and "$\mathrm{lcoeff}(f_1,\ x)$" as the leading coefficient of f_1 with respect to x.

Step 1. Simplify the algebraic variety $\mathrm{V}(M, M_1, M_2) \cap \Omega_{24}$.

By eliminating variables in the order $\eta \prec \alpha$, we have

$$\begin{aligned}
\mathrm{res}(M,\ M_1,\ \eta) &= 1024\alpha^2 u_*^{20}(\alpha + u_*)^8(\alpha + u_*^2)^4 r_{11},\\
\mathrm{res}(M,\ M_2,\ \eta) &= -262144\alpha^3 u_*^{34}(3u_* - 1)(\alpha + u_*)^{13}(\alpha + u_*^2)^7 r_{12},\\
\mathrm{res}(r_{11},\ r_{12},\ \alpha) &= 36762585439786041826344960000(u_* - 3)^8(u_* - 1)^7 u_*^{175}\\
&\quad \times (3u_* - 1)^9 r_{21} r_{22},
\end{aligned} \tag{A2}$$

where r_{11} and r_{12} are polynomials of (u_*, α), r_{21} and r_{22} are 17- and 54-order polynomials of u_*, and we omit their complicated expressions.

Notice that, $\mathrm{lcoeff}(M, \eta) = \alpha(\alpha + 3u_*) > 0$ and $\mathrm{lcoeff}(r_{11}, \alpha) = 27u_*^4 + 86u_*^3 - 112u_*^2 - 8u_* - 8 < 0$ in Ω_{24}. Similarly, from Theorem 1 in Chen and Zhang (2009) we can get that

$$\mathrm{V}(M,\ M_1,\ M_2) \cap \Omega_{24} = \mathrm{V}(M,\ M_1,\ M_2,\ r_{11},\ r_{12},\ r_{21}r_{22}) \cap \Omega_{24}. \tag{A3}$$

Step 2. Simplify the algebraic variety $\mathrm{V}(r_{11},\ r_{12},\ r_{21}r_{22}) \cap \Omega_{24}$.

First, by using the Maple command "realroot″", we know that there exist one root $u_{*0} \in I_0$ for $\mathrm{realroot}(r_{21}, 1/10^{10})$, and four roots $u_{*i} \in I_i$ (i=1,...,4) for $\mathrm{realroot}(r_{22}, 1/10^{10})$ in Ω_{24}, such that $r_{21}\ |_{u_*=u_{*0}} = r_{22}\ |_{u_*=u_{*i}} = 0$ (i=1,...,4), where

$$\begin{aligned}
&u_{*0} \doteq 0.005234,\ \ u_{*1} \doteq 0.017194,\ \ u_{*2} \doteq 0.022588,\\
&u_{*3} \doteq 0.070624,\ \ u_{*4} \doteq 0.2989611526,
\end{aligned} \tag{A4}$$

and

$$\begin{aligned}
I_0 &= [\tfrac{1581734929692734191289}{302231454903657293676544}, \tfrac{395433732423183547829}{75557863725914323419136}],\\
I_1 &= [\tfrac{20786838110638649144145}{1208925819614629174706176}, \tfrac{41573676221277298288317}{2417851639229258349412352}],\\
I_2 &= [\tfrac{101727566390447}{4503599627370496}, \tfrac{406910265561815}{18014398509481984}],\\
I_3 &= [\tfrac{5336187978172332291869}{75557863725914323419136}, \tfrac{10672375956344664583765}{151115727451828646838272}],\\
I_4 &= [\tfrac{2629690108339}{8796093022208}, \tfrac{5259380216687}{17592186044416}].
\end{aligned} \tag{A5}$$

Second, by using pseudo-division we can get nine pseudo-remainders

$$\begin{aligned}
&w_1 = \mathrm{prem}(r_{12},\ r_{11},\ \alpha), \quad w_2 = \mathrm{prem}(r_{11},\ w_1,\ \alpha),\\
&w_i = \mathrm{prem}(w_{i-2},\ w_{i-1},\ \alpha),\ \ (i = 3, ..., 9),
\end{aligned} \tag{A6}$$

where w_i (i=1,...,9) are (10-i)-order functions of α, and the coefficients are functions of u_*.

Notice that, $\mathrm{lcoeff}(r_{11},\ \alpha) \neq 0$, and by using Sturm's theorem we can get that $\mathrm{lcoeff}(w_i,\ \alpha) \neq 0$, (i=1,...,9) when $u_* \in \cup_{j=0}^4 I_j$.

Combining above analysis, we can get that in $\cup_{j=0}^4 I_j$, $\mathrm{V}(r_{12},\ r_{11}) \neq \emptyset$ and

$$\mathrm{V}(r_{12},\ r_{11}) = \mathrm{V}(r_{11},\ w_1) = \mathrm{V}(w_1,\ w_2) = ... = \mathrm{V}(w_8,\ w_9) \subseteq \mathrm{V}(w_9). \tag{A7}$$

From (A6), we know that w_9 is a linear function of α which has a unique root, and from $w_9 = 0$ we can get that $\alpha = \alpha_0(u_*)$. By using Sturm's theorem again, we can show that $\alpha_0(u_*)$ is a well defined monotone function and has no roots in I_i (i=0,...,4). Therefore, we can obtain that

$$\begin{aligned} &\alpha_0(u_{*0}) \doteq 0.002379,\ \alpha_0(u_{*1}) \doteq 0.022203,\ \alpha_0(u_{*2}) \doteq 0.015347, \\ &\alpha_0(u_{*3}) \doteq 0.048550,\ \alpha_0(u_{*4}) \doteq -0.199313, \end{aligned} \tag{A8}$$

which show that when $u_* = u_{*4}$, we have $\alpha_0(u_*) < 0$, that is $\mathrm{V}(w_9,\ r_{22}) \cap \Omega_{24} = \emptyset$. So that, we only need to consider the cases $u_* = u_{*i}$ (i=0,...,3).

Notice that, $(u_*, \alpha_0(u_*)) \in \Omega_{21}$. In the following, we first consider the monotonicity of α_1 and α_2 respect to u_* where $u_* \in (0, \frac{3-\sqrt{6}}{12}]$. Through simple calculation, we can get that

$$\frac{d\alpha_1}{du_*} = \frac{36u_*-1-96u_*^3-72u_*^2+(1-16u_*^2-16u_*)\sqrt{48u_*^2-24u_*+1}}{2(2u_*+1)^2\sqrt{48u_*^2-24u_*+1}} \triangleq \frac{\alpha_{11}+\alpha_{12}\sqrt{48u_*^2-24u_*+1}}{2(2u_*+1)^2\sqrt{48u_*^2-24u_*+1}},$$

moreover, we have $\alpha_{11}^2 - \alpha_{12}^2(48u_*^2 - 24u_* + 1) < 0$ ($= 0$, or > 0) if $0 < u_* < 0.035607$ ($u_* = 0.035607$, or $0.035607 < u_* \le \frac{3-\sqrt{6}}{12}$), that is $\mathrm{sign}\frac{d\alpha_1}{du_*} = \mathrm{sign}\alpha_{12}$ ($\frac{\mathrm{sign}\alpha_{11}+\mathrm{sign}\alpha_{12}}{2}$, or $\mathrm{sign}\alpha_{11}$)if$0 < u_* < 0.035607$ ($u_* = 0.035607$, or $0.035607 < u_* \le \frac{3-\sqrt{6}}{12}$). Further, we have $\alpha_{11} > 0$ when $0.035607 \le u_* \le \frac{3-\sqrt{6}}{12}$, and $\alpha_{12} > 0$ when $0 < u_* \le 0.035607$. Therefore, we can get that $\frac{d\alpha_1}{du_*} > 0$ when $u_* \in (0, \frac{3-\sqrt{6}}{12}]$. Similarly, we can show that $\frac{d\alpha_2}{du_*} > 0$ ($= 0$, < 0) when $0 < u_* < 0.035607$ ($u_* = 0.035607$, or $0.035607 < u_* \le \frac{3-\sqrt{6}}{12}$).

Therefore, from the above analysis, we know that α_1 and α_2 are monotone when $u_* \in \cup_{j=0}^{2} I_j$, and we have

$$\begin{aligned} &\alpha_1(u_{*0}) \doteq 0.000058,\ \alpha_2(u_{*0}) \doteq 0.004905,\ \alpha_1(u_{*1}) \doteq 0.000722, \\ &\alpha_2(u_{*1}) \doteq 0.013614,\ \alpha_1(u_{*2}) \doteq 0.001348,\ \alpha_2(u_{*2}) \doteq 0.016358. \end{aligned} \tag{A9}$$

Combining (A4), (A8) and (A9), when $u_* = u_{*i}$ (i=0, 2, 4), we have $(u_{*i}, \alpha_0(u_{*i})) \notin \Omega_{21}$, that is $\mathrm{V}(w_9,\ r_{21}r_{22}) \cap \Omega_{24} = \emptyset$, i.e., $\mathrm{V}(r_{11},\ r_{12},\ r_{21}r_{22}) \cap \Omega_{24} = \emptyset$. When $u_* = u_{*i}$ (i=1, 3), we have $(u_{*i}, \alpha_0(u_{*i})) \in \Omega_{21}$, that is $\mathrm{V}(w_9,\ r_{21}r_{22}) \cap \Omega_{24} = \mathrm{V}(w_9,\ r_{22}) \cap \Omega_{24} = \{(u_*, \alpha) :\ u_* = u_{*i}$ and $\alpha = \alpha_0(u_{*i}),\ i = 1,\ 3\} \neq \emptyset$, then we can get that

$$\begin{aligned} \mathrm{V}(r_{11},\ r_{12},\ r_{21}r_{22}) \cap \Omega_{24} &= \mathrm{V}(r_{11},\ r_{12},\ r_{22}) \cap \Omega_{24} \\ &= \cup_{i=1,3}\{(u_*, \alpha) :\ u_* = u_{*i} \text{ and } \alpha = \alpha_0(u_{*i})\}, \end{aligned} \tag{A10}$$

where u_{*i} and $\alpha_0(u_{*i})$ are shown in (A4) and (A8), respectively.

Summarizing the above analysis and from (A3) we can get that

$$\mathrm{V}(M,\ M_1,\ M_2,\ r_{11},\ r_{12},\ r_{21}r_{22}) \cap \Omega_{24}$$

$$
\begin{aligned}
&= \mathrm{V}(M,\ M_1,\ M_2,\ r_{11},\ r_{12},\ r_{22}) \cap \Omega_{24} \\
&= \mathrm{V}(M,\ M_1,\ M_2) \cap \left(\cup_{i=1,3} \left\{ (u_*, \alpha) :\ (u_*,\ \alpha) = (u_{*i},\ \alpha_0(u_{*i})) \right\} \right) \cap \Omega_{24} \\
&\triangleq \cup_{i=1,3} \mathrm{V_i}, \qquad \text{(A11)}
\end{aligned}
$$

where

$$
\mathrm{V_i} = \mathrm{V}(M,\ M_1,\ M_2) \cap \left\{ (u_*, \alpha) :\ (u_*,\ \alpha) = (u_{*i},\ \alpha_0(u_{*i})) \right\} \cap \Omega_{24}. \qquad \text{(A12)}
$$

Step 3. Prove that $\mathrm{V}_1 = \emptyset$ *in* (A11).

First, we simplify the algebraic variety $\mathrm{V}(M,\ M_1)$ in (A11). By using pseudo-division again, we can get one pseudo-remainder

$$
w_{10} = \mathrm{prem}(M_1,\ M,\ \eta), \qquad \text{(A13)}
$$

where w_{10} is 1-order function of η, and the coefficients are functions of $(u_*,\ \alpha)$. Then we can get that

$$
\mathrm{V}(M_1,\ M) = \mathrm{V}(M,\ w_{10}) \subseteq \mathrm{V}(w_{10}). \qquad \text{(A14)}
$$

From (A13), we know that w_{10} is a linear function of η which has a unique root, and from $w_{10} = 0$ we can get that $\eta = \eta_{00}(u_*,\ \alpha)$, where

$$
\begin{aligned}
&\eta_{00}(u_*,\ \alpha) \\
&= [u_*^2(-\alpha^{10}(27u_*^3 - 130u_*^2 + 64u_* + 68) - \alpha^9(231u_*^3 - 1358u_*^2 + 882u_* \\
&\quad +276)u_* - 2\alpha^8(252u_*^3 - 2567u_*^2 + 1931u_* + 180)u_*^2 + 2\alpha^7(438u_*^3 + 3625u_*^2 \\
&\quad -3981u_* + 128)u_*^3 + 2\alpha^6(2385u_*^3 - 2701u_*^2 - 2579u_* + 618)u_*^4 + 2\alpha^5 \\
&\quad \times(1311u_*^3 - 14171u_*^2 + 5013u_* + 402)u_*^5 - 6\alpha^4(1956u_*^3 + 3817u_*^2 - 3453u_* \\
&\quad +132)u_*^6 - 6\alpha^3(2406u_*^3 - 1897u_*^2 - 1827u_* + 144)u_*^7 + 9\alpha^2(777u_*^2 + 2032u_* \\
&\quad -18)u_*^9 + 27\alpha(547u_* + 80)u_*^{11} + 3240u_*^{13})]\big/[\alpha^{10}(27u_*^3 - 238u_*^2 + 584u_* \\
&\quad -396) + \alpha^9(123u_*^3 - 1438u_*^2 + 3850u_* - 2396)u_* - 2\alpha^8(48u_*^3 + 1063u_*^2 \\
&\quad -4771u_* + 3060)u_*^2 + 2\alpha^7(-654u_*^3 + 2495u_*^2 + 2993u_* - 3384)u_*^3 - 2\alpha^6 \\
&\quad \times(477u_*^3 - 8865u_*^2 + 8669u_* - 1022)u_*^4 + 2\alpha^5(2565u_*^3 + 2883u_*^2 - 16321u_* \\
&\quad +7254)u_*^5 + 18\alpha^4(360u_*^3 - 1805u_*^2 - 295u_* + 852)u_*^6 - 54\alpha^3(162u_*^3 \\
&\quad +553u_*^2 - 489u_* - 104)u_*^7 - 27\alpha^2(507u_*^2 - 540u_* - 622)u_*^9 + 81\alpha(67u_* \\
&\quad +240)u_*^{11} + 9720u_*^{13}],
\end{aligned}
\qquad \text{(A15)}
$$

and notice that $\left(u_{*1},\ \alpha_0(u_{*1}),\ \eta_{00}\left(u_{*1},\ \alpha_0(u_{*1})\right)\right) \in \Omega_{24}$.

Denote $\eta_{00}(u_*,\ \alpha) \triangleq \frac{\eta_{001}}{\eta_{002}}$, where η_{001} and η_{002} are polynomials of $(u_*,\ \alpha)$, and we know that $(u_{*1},\ \alpha_0(u_{*1})) \in I_1 \times [\frac{2220321857376}{10^{14}}, \frac{2220332345838}{10^{14}}] \triangleq [u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$, where I_1 is shown in (A5).

We first prove that $\eta_{002} \neq 0$ in the region $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$ by the following 2 substeps.

Step 3.1. Prove that η_{002} *has no critical point in the interior of the region.*

The corresponding first-order partial derivatives of η_{002} as $\frac{\partial \eta_{002}}{\partial u_*}$ and $\frac{\partial \eta_{002}}{\partial \alpha}$, where we omit the detailed expressions. By using Sturm's theorem, we can get that $\mathrm{res}(\frac{\partial \eta_{002}}{\partial u_*}, \frac{\partial \eta_{002}}{\partial \alpha}, \alpha)$ has no root in $[u_{00}, u_{01}]$, and we have $\mathrm{lcoeff}(\frac{\partial \eta_{002}}{\partial u_*}, \alpha) = 81{u_*}^2 - 476u_* + 584 \neq 0$. Then, we can get that

$$\mathrm{V}\Big(\frac{\partial \eta_{002}}{\partial u_*}, \frac{\partial \eta_{002}}{\partial \alpha}\Big) = \mathrm{V}\Big(\frac{\partial \eta_{002}}{\partial u_*}, \frac{\partial \eta_{002}}{\partial \alpha}, \mathrm{res}(\frac{\partial \eta_{002}}{\partial u_*}, \frac{\partial \eta_{002}}{\partial \alpha}, \alpha)\Big) = \emptyset$$

in $[u_{00}, u_{01}]$, which shows that η_{002} has no critical point in the interior of $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$, moreover, the maximum and minimum of η_{002} can only be achieved on the boundary of this region.

Step 3.2. Prove that η_{002} is monotone and rootless at the boundary of the region.

By using Sturm's theorem again, we can get that

$$\eta_{002}\ |_{u_*=u_{00}},\quad \eta_{002}\ |_{u_*=u_{01}},\quad \frac{d(\eta_{002}\ |_{u_*=u_{00}})}{d\alpha},\quad \frac{d(\eta_{002}\ |_{u_*=u_{01}})}{d\alpha}$$

have no roots in $[\alpha_{00}, \alpha_{01}]$, and

$$\eta_{002}\ |_{\alpha=\alpha_{00}},\quad \eta_{002}\ |_{\alpha=\alpha_{01}},\quad \frac{d(\eta_{002}\ |_{\alpha=\alpha_{00}})}{du_*},\quad \frac{d(\eta_{002}\ |_{\alpha=\alpha_{01}})}{du_*}$$

have no roots in $[u_{00}, u_{01}]$.

Hence, by calculating the values of four vertices of rectangular field $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$, we can get that $\eta_{002} \in [-1.719412 \times 10^{-16}, -1.676935 \times 10^{-16}]$ in $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$, which shows that η_{002} is well defined in this region.

Second, by using the same techniques, we analyse the value range of $\eta_{00}(u_*, \alpha)$ in $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$. Through a series of calculations, we have

$$\frac{\partial \eta_{00}(u_*, \alpha)}{\partial \alpha} = \frac{4u_*^3 \eta_{003}}{\eta_{002}^2},\quad \frac{\partial \eta_{00}(u_*, \alpha)}{\partial u_*} = \frac{-2u_*(\alpha+u_*)\eta_{004}}{\eta_{002}^2},$$

where η_{003} and η_{004} are polynomials of (u_*, α), and $\mathrm{lcoeff}(\eta_{003}, \alpha) = 729u^6 - 9207u^5 + 45790u^4 - 109538u^3 + 131768u^2 - 74136u + 13408 \neq 0$ in $[u_{00}, u_{01}]$. By using Sturm's theorem again, we can get that $\mathrm{res}(\eta_{003}, \eta_{004}, \alpha)$ has no real roots in $[u_{00}, u_{01}]$.

Combining the above analysis, we have

$$\mathrm{V}(\eta_{003}, \eta_{004}) = \mathrm{V}(\eta_{003}, \eta_{004}, \mathrm{res}(\eta_{003}, \eta_{004}, \alpha)) = \emptyset$$

in $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$, which shows that $\eta_{00}(u_*, \alpha)$ has no critical point in the interior of $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$. Therefore, the maximum and minimum of $\eta_{00}(u_*, \alpha)$ can only be achieved on the boundary of this region.

By using Sturm's theorem, we can get that

$$\eta_{00}(u_{00}, \alpha),\quad \eta_{00}(u_{01}, \alpha),\quad \frac{d(\eta_{00}(u_{00}, \alpha))}{d\alpha},\quad \frac{d(\eta_{00}(u_{01}, \alpha))}{d\alpha}$$

have no roots in $[\alpha_{00}, \alpha_{01}]$, moreover, $\frac{d(\eta_{00}(u_{00},\alpha))}{d\alpha} < 0$ and $\frac{d(\eta_{00}(u_{01},\alpha))}{d\alpha} < 0$ in $[\alpha_{00}, \alpha_{01}]$. Similarly,

$$\eta_{00}(u_*,\ \alpha_{00}),\quad \eta_{00}(u_*,\ \alpha_{01}),\quad \frac{d(\eta_{00}(u_*,\ \alpha_{00}))}{du_*},\quad \frac{d(\eta_{00}(u_*,\ \alpha_{01}))}{du_*}$$

have no roots in $[u_{00}, u_{01}]$, moreover, $\frac{d(\eta_{00}(u_*,\alpha_{00}))}{du_*} > 0$ and $\frac{d(\eta_{00}(u_*,\alpha_{01}))}{du_*} > 0$ in $[u_{00}, u_{01}]$.

Hence, the minimum and maximum of $\eta_{00}(u_*,\ \alpha)$ in $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$ are $\eta_{00}(u_{00},\ \alpha_{01}) \doteq 0.000920$ and $\eta_{00}(u_{01},\ \alpha_{00}) \doteq 0.000943$, respectively. Therefore, we have $\eta_{00}(u_{*1},\ \alpha_0(u_{*1})) \in [0.000920, 0.000943]$. Similarly, we have $\eta_0 \doteq 0.000285$ in $[u_{00}, u_{01}]$. Therefore, we can get that $\eta_{00}(u_{*1},\ \alpha_0(u_{*1})) > \eta_0$. That is, $\big(u_{*1},\ \alpha_0(u_{*1}),\ \eta_{00}\big(u_{*1},\ \alpha_0(u_{*1})\big)\big) \notin \Omega_{24}$.

Summarizing the above analysis, we can obtain that

$$\begin{aligned}
\mathrm{V}_1 &= \mathrm{V}(M,\ M_1,\ M_2) \cap \big\{(u_*, \alpha):\ (u_*,\ \alpha) = (u_{*1},\ \alpha_0(u_{*1}))\big\} \cap \Omega_{24} \\
&\subseteq \mathrm{V}(w_{10},\ M_2) \cap \big\{(u_*, \alpha):\ (u_*,\ \alpha) = (u_{*1},\ \alpha_0(u_{*1}))\big\} \cap \Omega_{24} \\
&= \mathrm{V}(M_2) \cap \big\{(u_*, \alpha, \eta):\ (u_*,\ \alpha,\ \eta) = \big(u_{*1},\ \alpha_0(u_{*1}),\ \eta_{00}(u_{*1},\ \alpha_0(u_{*1}))\big)\big\} \\
&\quad \cap \Omega_{24} \\
&= \emptyset.
\end{aligned}$$

Step 4. Prove that $\mathrm{V}_3 = \emptyset$ *in* (A11).

In the following, we use the same method as Step 3 to analyse V_3. We first simplify the algebraic variety $\mathrm{V}(M,\ M_1)$ in (A11), from (A13) and (A14) we know that

$$\begin{aligned}
&\mathrm{V}(M,\ M_1) \cap \big\{(u_*, \alpha):\ (u_*,\ \alpha) = (u_{*3},\ \alpha_0(u_{*3}))\big\} \cap \Omega_{24} \\
&\quad \subseteq \mathrm{V}(w_{10}) \cap \big\{(u_*, \alpha):\ (u_*,\ \alpha) = (u_{*3},\ \alpha_0(u_{*3}))\big\} \cap \Omega_{24} \\
&= \big\{(u_*, \alpha, \eta):\ (u_*,\ \alpha,\ \eta) = \big(u_{*3},\ \alpha_0(u_{*3}),\ \eta_{00}(u_{*3},\ \alpha_0(u_{*3}))\big)\big\} \cap \Omega_{24}.
\end{aligned} \tag{A16}$$

We next prove that $\big(u_{*3},\ \alpha_0(u_{*3}),\ \eta_{00}\big(u_{*3},\ \alpha_0(u_{*3})\big)\big) \in \Omega_{24}$. In fact, we have $(u_{*3},\ \alpha_0(u_{*3})) \in I_3 \times [\frac{48549816647411}{10^{15}}, \frac{48549816647417}{10^{15}}] \triangleq [u_{10}, u_{11}] \times [\alpha_{10}, \alpha_{11}]$, where I_3 is shown in (A5).

Similarly, we can also show that $\eta_{00}(u_*,\ \alpha)$ is well defined and

$$\eta_{00}(u_{*3},\ \alpha_0(u_{*3})) \in [0.00026494773853172, 0.00026494773853175] \tag{A17}$$

in $[u_{10}, u_{11}] \times [\alpha_{10}, \alpha_{11}]$, and $\eta_0 \doteq 0.004283$ in $[u_{10}, u_{11}]$. Hence, we can get that $\eta_{00}(u_{*3},\ \alpha_0(u_{*3})) < \eta_0$. That is, $\big(u_{*3},\ \alpha_0(u_{*3}),\ \eta_{00}\big(u_{*3},\ \alpha_0(u_{*3})\big)\big) \in \Omega_{24}$.

Therefore, we can obtain that

$$\begin{aligned}
\mathrm{V}(M,\ M_1) \cap \Omega_{24} &\supseteq \big\{(u_*, \alpha, \eta):\ (u_*,\ \alpha,\ \eta) \\
&= \big(u_{*3},\ \alpha_0(u_{*3}),\ \eta_{00}(u_{*3},\ \alpha_0(u_{*3}))\big)\big\} \cap \Omega_{24} \neq \emptyset,
\end{aligned} \tag{A18}$$

then we can get that E_* is a cusp of codimension at least 4.

Second, we continue to analyze the algebraic variety $\mathrm{V}(M,\ M_2)$ in (A11), by using pseudo-division once again, we can get one pseudo-remainder,

$$w_{11} = \mathrm{prem}(M_2,\ M,\ \eta), \tag{A19}$$

where w_{11} is 1-order function of η, and the coefficients are functions of $(u_*,\ \alpha)$. Then we have

$$\mathrm{V}(M_2,\ M) = \mathrm{V}(M,\ w_{11}) \subseteq \mathrm{V}(w_{11}), \tag{A20}$$

and from $w_{11} = 0$ we can get a unique root $\eta = \eta_{01}(u_*,\ \alpha)$.

Similarly, we can also show that $\eta_{01}(u_*,\ \alpha)$ is well defined and $\eta_{01}(u_{*3},\ \alpha_0(u_{*3})) \in [0.014132345, 0.014132346]$ in $[u_{10}, u_{11}] \times [\alpha_{10}, \alpha_{11}]$. Hence, we can get that $\eta_{01}(u_{*3},\ \alpha_0(u_{*3})) > \eta_0$, that is, $\big(u_{*3},\ \alpha_0(u_{*3}),\ \eta_{01}\big(u_{*3},\ \alpha_0(u_{*3})\big)\big) \notin \Omega_{24}$.

Therefore, we can obtain that

$$\begin{aligned}
\mathrm{V}_3 &= \mathrm{V}(M,\ M_1,\ M_2) \cap \big\{(u_*, \alpha) :\ (u_*,\ \alpha) = (u_{*3},\ \alpha_0(u_{*3}))\big\} \cap \Omega_{24} \\
&\subseteq \mathrm{V}(w_{11},\ M_1) \cap \big\{(u_*, \alpha) :\ (u_*,\ \alpha) = (u_{*3},\ \alpha_0(u_{*3}))\big\} \cap \Omega_{24} \\
&= \mathrm{V}(M_1) \cap \big\{(u_*, \alpha, \eta) :\ (u_*,\ \alpha,\ \eta) = \big(u_{*3},\ \alpha_0(u_{*3}),\ \eta_{01}(u_{*3},\ \alpha_0(u_{*3}))\big)\big\} \\
&\quad \cap \Omega_{24} \\
&= \emptyset.
\end{aligned}$$

Combining the above analysis and from (A3) and (A11), we can get that

$$\mathrm{V}(M,\ M_1,\ M_2) \cap \Omega_{24} = \emptyset, \ \ \mathrm{V}(M,\ M_1) \cap \Omega_{24} \neq \emptyset,$$

which shows that E_* is a cusp of codimension exactly 4.

It is easy to know that $\Omega_{25} \subseteq \mathrm{V}(M,\ M_1) \cap \Omega_{24}$, where Ω_{25} is shown in (1.5), and when $(u_*,\ \alpha,\ \eta) \in \Omega_{25}$ we have $M = M_1 = 0$ and $M_2 \neq 0$, that is E_* is a cusp of codimension exactly 4.

Appendix B: The proof of $\overline{d}_{20} < 0$ and $\overline{d}_{41} < 0$ in (3.26) of Theorem 2

From (3.26), when $\mu = 0$ we have

$$\begin{aligned}
\overline{d}_{20} &= \frac{(\eta+2u_*^3-u_*^2)\overline{d_{200}}}{u_*^2(\eta+u_*^2)(-\eta+2\alpha u_*+u_*^2)^2}, \\
\overline{d}_{41} &= \frac{\overline{d_{410}}}{144u_*^5(\alpha+u_*)^4(\eta+u_*^2)^4(\eta+2u_*^3-u_*^2)(\eta-2\alpha u_*-u_*^2)\overline{d_{200}}^3},
\end{aligned} \tag{B1}$$

where

$$\overline{d_{200}} = \alpha\eta^2 - 3\alpha^2\eta u_* + \alpha^2 u_*^3 - \alpha\eta u_*^3 - 3\alpha\eta u_*^2 + 3\alpha u_*^5 - 3\eta u_*^4 + u_*^6,$$

and we omit the expression of $\overline{d_{410}}$. Notice that, $(u_*, \alpha, \eta) \in \Omega_{25}$, Ω_{25} is shown in (1.5), that is $(u_*, \alpha) = (u_{*3}, \alpha_0(u_{*3})) \in I_3 \times [\frac{48549816647411}{10^{15}}, \frac{48549816647417}{10^{15}}] \triangleq [u_{10}, u_{11}] \times [\alpha_{10}, \alpha_{11}]$, and $\eta = \eta_{00}(u_*, \alpha)$, where $\eta_{00}(u_*, \alpha)$ is shown in (A15).

We first calculate sign$\overline{d_{200}}$. Substitute the parameter $\eta = \eta_{00}(u_*, \alpha)$ into $\overline{d_{200}}$, we can get that $\overline{d_{200}} \mid_{\eta=\eta_{00}(u_*, \alpha)}= \frac{4u_*^3\alpha(u_*+\alpha)^2\overline{d_{201}}}{\eta_{002}^2}$, where $\overline{d_{201}}$ is a polynomial of (u_*, α), we omit the detailed expression. Therefore, we have

$$\text{sign}\left(\overline{d_{200}} \mid_{\eta=\eta_{00}(u_*, \alpha)}\right) = \text{sign}\overline{d_{201}}.$$

By using the same method in Steps 3.1–3.2 of Appendix 1, we can get that $\overline{d_{201}} \in [4.7852072357861 \times 10^{-16}, 4.7852072357899 \times 10^{-16}]$ in $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$. Moreover, combining condition (3.5), we know that $\overline{d_{20}} < 0$ for small μ.

Similarly, we can get that $\overline{d_{410}} \mid_{\eta=\eta_{00}(u_*, \alpha)}= \frac{-16384{u_*}^{21}\alpha(\alpha+u_*)^{11}\overline{d_{411}}}{\eta_{002}^{12}}$, where $\overline{d_{411}}$ is a polynomial of (u_*, α), and we omit the detailed expression. Therefore, we have

$$\text{sign}\left(\overline{d_{410}} \mid_{\eta=\eta_{00}(u_*, \alpha)}\right) = -\text{sign}\overline{d_{411}},$$

moreover, we can get that $\overline{d_{411}} \in [2.113486566436 \times 10^{-96}, 2.113486566447 \times 10^{-96}]$ in $[u_{00}, u_{01}] \times [\alpha_{00}, \alpha_{01}]$, which shows that $\overline{d_{41}} < 0$ for small μ.

Appendix C: The proof of nondegeneracy condition (3.29) of Theorem 2

In order to show that the nondegeneracy condition (3.29) holds when $(u_*, \alpha, \eta) \in \Omega_{25}$, where Ω_{25} is shown in (1.5), we just need to show that $\overline{f}_1\overline{f}_{21}\overline{f}_{22} \neq 0$. Notice that, $\Omega_{25} \subseteq \mathrm{V}(M, M_1, r_{11}, r_{12}, r_{22}) \cap \Omega_{24}$, so that we first prove $\mathrm{V}(M, M_1, r_{11}, r_{12}, r_{22}, \overline{f}_1) \cap \Omega_{24} = \emptyset$.

By eliminating variables in the order $\eta \prec \alpha \prec u_*$, we have

$$\begin{aligned}
\text{res}(M, \overline{f}_1, \eta) &= 18239578490850508800\alpha^{10}u_*^{96}(3u_* - 1)^6(\alpha + u_*)^{40} \\
&\quad (\alpha + u_*^2)^{21} r_{31}, \\
\text{res}(r_{11}, r_{31}, \alpha) &= C_0(u_* - 3)^{18}(u_* - 1)^{17}u_*^{407}(u_* + 5)(3u_* - 1)^{18} r_{32}, \\
\text{res}(r_{22}, r_{32}, u_*) &\neq 0,
\end{aligned}$$

where C_0 is a positive constant, r_{31} and r_{32} are polynomials of (u_*, α) and u_*, respectively, and we omit their complicated expressions.

From Appendix 1 **Step 1**, we know that lcoeff$(M, \eta) > 0$ and lcoeff$(r_{11}, \alpha) < 0$ in Ω_{25}. Similarly, from Theorem 1 in Chen and Zhang (2009) we can get that

$$\begin{aligned}
&\mathrm{V}(M, M_1, r_{11}, r_{12}, r_{22}, \overline{f}_1) \cap \Omega_{24} \\
&\quad = \mathrm{V}(M, M_1, r_{11}, r_{12}, r_{22}, \overline{f}_1, r_{31}, r_{32}) \cap \Omega_{24} = \emptyset,
\end{aligned} \tag{C1}$$

which shows that $\overline{f}_1 \neq 0$ in Ω_{25}.

Second, we prove $\mathrm{V}(M, M_1, r_{11}, r_{12}, r_{22}, \overline{f}_{21}\overline{f}_{22}) \cap \Omega_{24} = \emptyset$. Similarly, by eliminating variables in the order $\eta \prec \alpha \prec u_*$, we have

$$\begin{aligned}
\mathrm{res}(M, \overline{f}_{21}\overline{f}_{22}, \eta) &= 21743271936\alpha^7 u_*^{48}(3u_* - 1)^3(\alpha + u_*)^{21}(\alpha + u_*^2)^{11} r_{41},\\
\mathrm{res}(r_{11}, r_{41}, \alpha) &= 36762585439786041826344960000(u_* - 3)^8(u_* - 1)^7 u_*^{175}\\
&\quad (3u_* - 1)^9 r_{42} r_{22} = 0,\\
\mathrm{res}(r_{22}, r_{42}, u_*) &\neq 0,
\end{aligned}$$

where r_{41} and r_{42} are polynomials of (u_*, α) and u_*, respectively, and we omit their complicated expressions. From $\mathrm{V}(r_{11}, r_{41}) \neq \emptyset$, we have

$$\mathrm{V}(M, \overline{f}_{21}\overline{f}_{22}) \cap \Omega_{24} = \big(\mathrm{V}(M, \overline{f}_{21}) \cup \mathrm{V}(M, \overline{f}_{22})\big) \cap \Omega_{24} \neq \emptyset. \tag{C2}$$

By using pseudo-division, we can get that

$$\mathrm{prem}(M, \overline{f}_{21}, \eta) = -u_* (\alpha + u_*)^2 \left(\alpha + u_*^2\right)\left(u_*^2 - 3\eta\right),$$

and from (3.29) we have $\mathrm{lcoeff}(\overline{f}_{21}, \eta) = \alpha \neq 0$, then we can get that

$$\mathrm{V}(M, \overline{f}_{21}) = \mathrm{V}(\overline{f}_{21}, (u_*^2 - 3\eta)) \subseteq \{\eta : \eta = \tfrac{u_*^2}{3}\}.$$

Moreover, from (A17), when $u_* \in [u_{10}, u_{11}]$ we have $\frac{u_*^2}{3} \doteq 0.00166258 > \eta_{00}(u_{*3}, \alpha_0(u_{*3}))$, which shows that

$$\begin{aligned}
&\mathrm{V}(M, M_1, r_{11}, r_{12}, r_{22}, \overline{f}_{21}) \cap \Omega_{24}\\
&\quad = \mathrm{V}(M, M_1, r_{11}, r_{12}, r_{22}, \overline{f}_{21}, (u_*^2 - 3\eta)) \cap \Omega_{24} = \emptyset.
\end{aligned} \tag{C3}$$

By using pseudo-division again, we can get that

$$\mathrm{prem}(\overline{f}_{22}, M, \eta) = -\frac{9216u_*^{19}(\alpha+u_*)^6\left(\alpha+u_*^2\right)^4}{\alpha^7(\alpha+3u_*)^{11}} w_{12},$$

where w_{12} is 1-order function of η, and the coefficients are functions of (u_*, α). Then we can get that

$$\mathrm{V}(\overline{f}_{22}, M) = \mathrm{V}(M, w_{12}) \subseteq \mathrm{V}(w_{12}).$$

and from $w_{12} = 0$ we can get a unique root $\eta = \eta_{02}(u_*, \alpha)$.

By using the method in Appendix 1 Steps 3.1–3.2, we can also show that $\eta_{02}(u_*, \alpha)$ is well defined and $\eta_{02}(u_{*3}, \alpha_0(u_{*3})) \in [0.01413234, 0.01413235]$ in $[u_{10}, u_{11}] \times [\alpha_{10}, \alpha_{11}]$. Hence, we have $\eta_{02}(u_{*3}, \alpha_0(u_{*3})) > \eta_0$, that is, $\big(u_{*3}, \alpha_0(u_{*3}), \eta_{02}\big(u_{*3}, \alpha_0(u_{*3})\big)\big) \notin \Omega_{24}$.

Therefore, we can get that

$$\begin{aligned} &\mathrm{V}(M,\ M_1,\ r_{11},\ r_{12},\ r_{22},\ \overline{f}_{22}) \cap \Omega_{24} \\ &\subseteq \mathrm{V}(M_1,\ r_{11},\ r_{12},\ r_{22},\ w_{12}) \cap \Omega_{24} \\ &= \mathrm{V}(M_1) \cap \{(u_*, \alpha, \eta) :\ (u_*,\ \alpha,\ \eta) \\ &= (u_{*3},\ \alpha_0(u_{*3}),\ \eta_{02}(u_{*3},\ \alpha_0(u_{*3})))\} \cap \Omega_{24} = \emptyset. \end{aligned} \tag{C4}$$

From (C1), (C3) and (C4), we can obtain that $\overline{f}_1 \overline{f}_{21} \overline{f}_{22} \neq 0$ when $(u_*,\ \alpha,\ \eta) \in \Omega_{25}$, that is the nondegeneracy condition (3.29) holds.

Appendix D: The proof of $0 < \hat{b}_{11}\ |_{\lambda=0} < 2\sqrt{2}$ in (3.55)

Notice that, in (3.55) we have

$$\begin{aligned} \widehat{d_1} &= \alpha^2\eta^3 - 4\alpha^3\eta^2u^* - 7\alpha^2\eta^2(u^*)^2 + 2(u^*)^7(\alpha^2 + 2\eta) + 4\eta(u^*)^5(2\alpha^2 + \alpha \\ &\quad -\eta) + 2\alpha\eta(u^*)^3(2\alpha^2 - \alpha\eta - 2\eta) - \alpha(u^*)^6(\alpha - 8\eta) + \alpha\eta(u^*)^4(7\alpha - 8\eta), \\ \widehat{d_2} &= 3\alpha^4\eta^4 + 2(u^*)^{11}(39\alpha^3 - 50\alpha\eta) + 6\alpha^2\eta^3(u^*)^2(\eta - 13\alpha^2) + (u^*)^{12}(88\alpha^2 \\ &\quad -2\alpha - 24\eta) + \alpha^3\eta^3u^*(11\eta - 12\alpha^2) + \alpha(u^*)^9(28\alpha^3 + \alpha^2(68\eta + 1) \\ &\quad -172\alpha\eta + 42\eta^2) + \alpha(u^*)^8(\alpha^3(48\eta + 5) - 254\alpha^2\eta - 6\alpha\eta(8\eta + 1) \\ &\quad +106\eta^2) + 2\alpha^2(u^*)^7(3\alpha^3 - 40\alpha^2\eta - \alpha\eta(29\eta + 24) + 144\eta^2) - 4(12\alpha^3 \\ &\quad +3\alpha^2\eta - 50\alpha\eta + 21\eta^2)\alpha^2\eta(u^*)^5 - 2\alpha^2\eta^2(u^*)^4(12\alpha^3 - 112\alpha^2 + 17\alpha\eta \\ &\quad +21\eta) + 6\alpha^3\eta^2(u^*)^3(13\alpha^2 - 18\eta) + 2(u^*)^{10}(6\alpha^4 + 11\alpha^3 - 36\alpha^2\eta \\ &\quad -15\alpha\eta + 18\eta^2) + 2\alpha\eta(u^*)^6(12\alpha^4 - 3\alpha^3(2\eta + 15) + 101\alpha^2\eta + 29\alpha\eta \\ &\quad -21\eta^2) + 34\alpha(u^*)^{13} + 4(u^*)^{14}. \end{aligned} \tag{D1}$$

First, we prove $\hat{b}_{11}\ |_{\lambda=0} > 0$. Substituting $\eta = \overline{\eta}$ in (1.7) into $\widehat{d_1}$, we have

$$\begin{aligned} \widehat{d_1} = &\frac{\left(\hat{d}_{11} + \hat{d}_{12}\sqrt{(u^*)^2((3\alpha^2+\alpha(u^*+3)u^*+3(u^*)^3)^2 - 4\alpha u^*(\alpha^2+3\alpha(u^*)^2+(u^*)^3))}\right)}{-2\alpha^2} \\ &\times (u^*)^2(\alpha^2 + \alpha(u^*)^2 + \alpha u^* + (u^*)^3), \end{aligned}$$

where

$$\begin{aligned} \hat{d}_{11} &= 9\alpha^5u^* + 6\alpha^4(u^*)^3 + 34\alpha^4(u^*)^2 + 36\alpha^3(u^*)^4 + 45\alpha^3(u^*)^3 + 10\alpha^2(u^*)^6 \\ &\quad +66\alpha^2(u^*)^5 + 24\alpha^2(u^*)^4 + 33\alpha(u^*)^7 + 52\alpha(u^*)^6 + 36(u^*)^8 + \alpha^3(u^*)^5, \\ \hat{d}_{12} &= -3\alpha^3 - \alpha^2(u^*)^2 - 9\alpha^2u^* - 7\alpha(u^*)^3 - 8\alpha(u^*)^2 - 12(u^*)^4. \end{aligned}$$

Through simple calculation, we have

$$\begin{aligned} &\hat{d}_{11}^2 - \hat{d}_{12}^2(u^*)^2((3\alpha^2 + \alpha(u^* + 3)u^* + 3(u^*)^3)^2 - 4\alpha u^*(\alpha^2 + 3\alpha(u^*)^2 + (u^*)^3)) \\ &\quad = -32\alpha^2(u^*)^4(\alpha + u^*)^3(\alpha^3 + 6\alpha^2(u^*)^2 + \alpha(u^*)^4 + 4\alpha(u^*)^3 + 4(u^*)^5) < 0, \end{aligned}$$

and $\hat{d}_{12} < 0$, which show that $\widehat{d_1} > 0$.

Similarly, we can prove that $\widehat{d_2} < 0$. Combining the condition $\eta = \overline{\eta}$ and $0 < u^* < \frac{1}{3}$, we have $\hat{b}_{11} \mid_{\lambda=0} > 0$.

Second, we prove that $\hat{b}_{11} \mid_{\lambda=0} < 2\sqrt{2}$, from (3.55), which is equivalent to proving that

$$
\begin{aligned}
&72\hat{d}_1^3 u^*(\alpha + u^*)(\eta + 2(u^*)^3 - (u^*)^2) + \hat{d}_2^2(\eta + (u^*)^2)^2 \\
&\quad = -\frac{\left(\hat{d}_{21} + \hat{d}_{22}\sqrt{(u^*)^2((3\alpha^2 + \alpha(u^*+3)u^* + 3(u^*)^3)^2 - 4\alpha u^*(\alpha^2 + 3\alpha(u^*)^2 + (u^*)^3))}\right)}{2\alpha^7} \\
&\quad\times 9(u^*)^9(\alpha + u^*)^5(\alpha + (u^*)^2)^3 < 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{d}_{21} =& -6561\alpha^{17}u^* - 729\alpha^{16}((u^*)^2(27u^* + 67)) - 81\alpha^{15}((u^*)^3(297(u^*)^2 \\
&+ 2121u^* + 1385)) - 9\alpha^{14}((u^*)^4(1755(u^*)^3 + 27819(u^*)^2 + 54939u^* \\
&- 13213)) + \alpha^{13}(-6075(u^*)^4 - 196857(u^*)^3 - 902097(u^*)^2 + 173511u^* \\
&+ 1295173)(u^*)^5 - \alpha^{12}(1377(u^*)^5 + 90441(u^*)^4 + 883203(u^*)^3 \\
&+ 311171(u^*)^2 - 5142519u^* - 3530471)(u^*)^6 + \alpha^{11}(-171(u^*)^6 \\
&- 24339(u^*)^5 - 501571(u^*)^4 - 934249(u^*)^3 + 8332381(u^*)^2 \\
&+ 16569133u^* + 5489045)(u^*)^7 + \alpha^{10}(-9(u^*)^7 - 3561(u^*)^6 \\
&- 165393(u^*)^5 - 913593(u^*)^4 + 6980391(u^*)^3 + 32872783(u^*)^2 \\
&+ 29149703u^* + 5440095)(u^*)^8 + \alpha^9(-219(u^*)^7 - 29363(u^*)^6 \\
&- 441427(u^*)^5 + 3095111(u^*)^4 + 35618269(u^*)^3 + 66318797(u^*)^2 \\
&+ 32229157u^* + 3444736)(u^*)^9 + \alpha^8(-2169(u^*)^7 - 106569(u^*)^6 \\
&+ 611581(u^*)^5 + 22621213(u^*)^4 + 83767639(u^*)^3 + 82507335(u^*)^2 \\
&+ 22611288u^* + 1290240)(u^*)^{10} + \alpha^7(-10251(u^*)^7 - 489(u^*)^6 \\
&+ 8326479(u^*)^5 + 63373303(u^*)^4 + 118457877(u^*)^3 + 64564384(u^*)^2 \\
&+ 9351168u^* + 221184)(u^*)^{11} + \alpha^6(-12315(u^*)^6 + 1609645(u^*)^5 \\
&+ 28651757(u^*)^4 + 103141317(u^*)^3 + 104191032(u^*)^2 + 29649136u^* \\
&+ 1769472)(u^*)^{13} + \alpha^5(120231(u^*)^5 + 7137447(u^*)^4 + 54545143(u^*)^3 \\
&+ 102913984(u^*)^2 + 53464944u^* + 6225920)(u^*)^{15} + \alpha^4(749493(u^*)^4 \\
&+ 16252821(u^*)^3 + 62466696(u^*)^2 + 59444000u^* + 12527104)(u^*)^{17} \\
&+ 3\alpha^3(705861(u^*)^3 + 7236576(u^*)^2 + 13658976u^* + 5208064)(u^*)^{19} \\
&+ 216\alpha^2(15687(u^*)^2 + 75738u^* + 56192)(u^*)^{21} + 3888\alpha(765u^* \\
&+ 1408)(u^*)^{23} + 1119744(u^*)^{25},
\end{aligned}
$$

and $\hat{d}_{22}$ is also a polynomial of $(\alpha,\ u^*)$, we omit its expression, and $\hat{d}_1$ and $\hat{d}_2$ are showing in (D1).

By simple calculation, we have

$$\begin{aligned}&\hat{d}_{21}^2-\hat{d}_{22}^2(u^*)^2((3\alpha^2+\alpha(u^*+3)u^*+3(u^*)^3)^2-4\alpha u^*(\alpha^2+3\alpha(u^*)^2+(u^*)^3))\\&\quad=16384\alpha^7\hat{d}_{23}(u^*)^{10}(1-3u^*)(\alpha+u^*)^7(\alpha+(u^*)^2)(\alpha^3+6\alpha^2(u^*)^2+\alpha(u^*)^4\\&\qquad+4\alpha(u^*)^3+4(u^*)^5)^2,\end{aligned}$$

where

$$\begin{aligned}\hat{d}_{23}=&\ \alpha^5(-3u^*-5)+\alpha^4(-3(u^*)^2-30u^*+13)u^*+\alpha^3(-(u^*)^2+81u^*\\&+10)(u^*)^2+4\alpha^2(9u^*+35)(u^*)^4+2\alpha(37u^*+32)(u^*)^5+64(u^*)^7.\end{aligned}$$

Notice that, $\hat{d}_{21}$ is a 17-order polynomial of α with coefficients are functions of u^*, where the coefficients of (0–13)-order terms are positive, (15-17)-order terms are negative and 14-order term is uncertain when $0<u^*<\frac{1}{3}$. By using the Descartes' rule of signs, we know that $\hat{d}_{21}$ has a unique positive real root $\widehat{\alpha}_1$, and

$$\hat{d}_{21}>0\ (=0,\ \text{or}\ <0)\quad \text{if}\quad 0<\alpha<\widehat{\alpha}_1\ (\alpha=\widehat{\alpha}_1,\ \text{or}\ \alpha>\widehat{\alpha}_1).$$

Similarly, $\hat{d}_{22}$ (or $\hat{d}_{23}$) also has a unique positive real root $\widehat{\alpha}_2$ (or $\widehat{\alpha}_3$), and

$$\begin{aligned}&d_{22}<0\ (=0,\ \text{or}\ >0)\quad \text{if}\quad 0<\alpha<\widehat{\alpha}_2\ (\alpha=\widehat{\alpha}_2,\ \text{or}\ \alpha>\widehat{\alpha}_2);\\&d_{23}>0\ (=0,\ \text{or}\ <0)\quad \text{if}\quad 0<\alpha<\widehat{\alpha}_3\ (\alpha=\widehat{\alpha}_3,\ \text{or}\ \alpha>\widehat{\alpha}_3).\end{aligned}\tag{D2}$$

Hence, by using the same method as in the analysis of sign(J) in proof (II) of Theorem 3, we can get that $\widehat{\alpha}_2<\widehat{\alpha}_3<\widehat{\alpha}_1$. Therefore, we can get that $\hat{d}_{21}+\hat{d}_{22}\sqrt{(u^*)^2((3\alpha^2+\alpha(u^*+3)u^*+3(u^*)^3)^2-4\alpha u^*(\alpha^2+3\alpha(u^*)^2+(u^*)^3))}>0$, then we have $72\hat{d}_1^3u^*(\alpha+u^*)(\eta+2(u^*)^3-(u^*)^2)+\hat{d}_2^2(\eta+(u^*)^2)^2<0$, that is $\hat{b}_{11}\mid_{\lambda=0}<2\sqrt{2}$. We finish the proof.

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