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ANALYSIS OF A SPATIAL MEMORY MODEL WITH NONLOCAL MATURATION DELAY AND HOSTILE BOUNDARY CONDITION

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Abstract. In this paper, we propose and investigate a memory-based reaction-diffusion equation with nonlocal maturation delay and homogeneous Dirichlet boundary condition. We first study the existence of the spatially inhomogeneous steady state. By analyzing the associated characteristic equation, we obtain sufficient conditions for local stability and Hopf bifurcation of this inhomogeneous steady state, respectively. For the Hopf bifurcation analysis, a geometric method and prior estimation techniques are combined to find all bifurcation values because the characteristic equation includes a non-self-adjoint operator and two time delays. In addition, we provide an explicit formula to determine the crossing direction of the purely imaginary eigenvalues. The bifurcation analysis reveals that the diffusion with memory effect could induce spatiotemporal patterns which were never possessed by an equation without memory-based diffusion. Furthermore, these patterns are different from the ones of a spatial memory equation with Neumann boundary condition.

1. Introduction.

The spatial diffusion of microscopic particles or individuals can be described by reaction diffusion equations [9, 17, 18]. For instance, if the movement flux is assumed to be proportional to the negative gradient of the concentration, then one can derive a standard reaction-diffusion equation; if the movement is

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in an advective environment, then the flux should not only depend on the gradient of the concentration but also on the fluid velocity, which leads to the reaction-diffusion-advection equation [16]; if the movement is affected by chemical signals, then we will arrive at chemotaxis systems [14, 15, 23]. However, these equations cannot reasonably describe highly development animals with memory and cognition. In [20], the following model was proposed to describe the movement of animals with episodic-like spatial memory:

$$\frac{\partial}{\partial t}u(x,t) = D_1\Delta u(x,t) + D_2\mathrm{div}(u(x,t)\nabla u(x,t-\tau)) + g(u(x,t)),\ x\in\Omega,\ t>0,\quad (1)$$

where $u(x,t)$ denotes the population density at position x and time t, $D_1 > 0$ is the Fickian diffusion coefficient, $D_2 \in \mathbb{R}$ is the memory-based diffusion coefficient, $\tau > 0$ is the averaged memory period, g describes the biological birth and death, Ω is a connected and bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$ and Neumann boundary condition is imposed on (1). It has been shown in [20] that the stability of a constant steady state of (1) is completely determined by the relationship between the two diffusion rates, but is independent of time delay. When a maturation delay σ is incorporated into the reaction term g, i.e.,

$$\frac{\partial}{\partial t}u(x,t) = D_1\Delta u(x,t) + D_2\mathrm{div}(u(x,t)\nabla u(x,t-\tau)) + g(u(x,t), u(x,t-\sigma)),\quad (2)$$

the memory delay τ plays an important role on the dynamics of (2). Specifically, under certain conditions, there exists a unique D_2^* such that (2) possesses spatially inhomogeneous periodic solutions for $D_1 > |D_2^*|$, bifurcating from the constant steady state through Hopf bifurcation as (τ,σ) passes through some critical curves. If $D_1 < |D_2^*|$, then all bifurcated periodic solutions through Hopf bifurcation, if exist, must be spatially homogeneous, see [19]. It is argued in [2, 6, 7, 13, 25] that the effect of diffusion and maturation delay σ are not independent of each other, since the individuals located at x in the previous time may move to another place at present. Therefore, the terms that used to describe the intraspecific competition should depend on the population levels in a neighborhood of the original position, and more specifically, it should be a spatial weighted average according to the distance from the original position. In fact, the models with nonlocal reaction terms are more realistic than those with local ones.

In this paper, we introduce this nonlocal effect into the memory-based diffusion population model, and specially consider the homogeneous Dirichlet boundary condition, which means that the external environment of the habitat is hostile and all individuals die when they reach $\partial\Omega$ [4]. The general form of the model is

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \Delta u(x,t) + \mathrm{div}(du(x,t)\nabla u(x,t-\tau)) \\ \qquad\qquad + \lambda u(x,t)F\left(u(x,t), \int_\Omega K(x,y)u(y,t-\sigma)dy\right), & x\in\Omega,\ t>0, \\ u(x,t) = 0, & x\in\partial\Omega,\ t>0. \end{cases}\quad (3)$$

Here d is the ratio of the memory-based diffusion coefficient to the standard diffusion coefficient, $\lambda > 0$ is a scaled constant, $F(\cdot,\cdot)$ represents the survival rate of per $\frac{1}{\lambda}$-individual, $K(x,y)$ accounts for the nonlocal intraspecific competition of the species for resource or space. For instance, when considering both the advantages of local aggregation and the disadvantages of resource depletion caused by high global population, we can choose F in the following form

$$1 + au(x,t) - (1+a)\int_{\Omega} K(x,y)u(y,t-\sigma)dy, \tag{4}$$

where $a > 0$. Furthermore, when competition for space itself rather than resources becomes important, F may take

$$1 + au(x,t) - bu(x,t)^2 - (1+a-b)\int_{\Omega} K(x,y)u(y,t-\sigma)dy, \tag{5}$$

where $b > 0$ and $1 + a - b > 0$, see [2, 7] for more details.

The memory-based diffusion population model with Neumann boundary condition has produced many elegant results [19, 20, 21]. However, there is no relevant research on Dirichlet boundary condition, although it also plays an important role in population ecology. The reason may be that it is difficult to study the existence and stability of the non-zero steady states, because it is usually spatial inhomogeneous under the Dirichlet boundary condition. In this paper, we first use the Lyapunov-Schmidt reduction to show the existence of the inhomogeneous steady state u_λ for λ near a critical number λ_*. Then, we study the local stability of u_λ by investigating the characteristic equation of the linearized system at u_λ, which is the eigenvalue problem of an elliptic operator involving two delays. Sufficient conditions for the local stability of u_λ for any $\tau, \sigma > 0$ are derived. When these sufficient conditions are violated, there exist some critical values for (τ, σ) such that the characteristic equation has purely imaginary roots. The curve in (τ, σ) plane, formed by all these critical values, is referred as the crossing curve in this context, as in [12]. The moving direction of these purely imaginary roots, as (τ, σ) passes through the crossing curves, is also determined.

The main idea in the proof combines the implicit function theorem and some prior estimates, which are initially used in [3] for studying Hopf bifurcation of a delayed diffusive Hutchinson equation, and later be applied to other partial functional differential equations [5, 6, 7, 22, 25]. However, the eigenvalue problem for this model involves two time delays. As a result, the frequency of periodic oscillation (if happens) generated through Hopf bifurcation is not fixed, and therefore the techniques in [3] fail to apply to (3) directly. To overcome this difficulty, the geometric method proposed in [12] for studying transcendental equations with two delays is employed. With the aid of this method, we find all possible values of potential oscillation frequencies when $\lambda = \lambda_*$. Then, the existence of the frequencies of periodic oscillations around u_λ can be proved by implicit function theorem when λ deviates from λ_*. In addition, due to the memorized diffusion term, a lot more prior estimates, such as the estimate on the gradient of eigenfunctions, are required in the proof. We also remark that the method developed here is also applicable to other two-delayed problem with Dirichlet boundary condition without memorized diffusion.

The rest of the paper is organized as follows. In section 2, we show the existence of non-constant steady state for (3). The eigenvalue problem for the linearized equation at this steady state is investigated in Section 3, and much attention is paid on finding the purely imaginary roots. In section 4, we focus on the crossing direction of these purely imaginary roots, as the parameters (τ, σ) vary. Finally, an example is provided in Section 5, and the main results are discussed in Section 6.

2. Existence of positive steady states.

Throughout this paper, we assume

(H1): $F(u,v) \in C^k(\mathbb{R}^2, \mathbb{R}),\ k \geq 3$ and satisfies $F(0,0) = 1$,

(H2): $K(\cdot, y) \in C^{\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, for each $y \in \Omega$; $K(x, \cdot) \in L^{\infty}(\Omega)$ for any $x \in \Omega$.

Denote by $\lambda_* > 0$ the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

and let $\phi > 0$ be the eigenfunction with respect to λ_*. Let $X = H^2(\Omega) \cap H_0^1(\Omega)$, $Y = L^2(\Omega)$ and $C := C([-\max\{\tau, \sigma\}, 0], Y)$. For any space Z, the complexification of Z is defined by $Z_{\mathbb{C}} := Z \oplus \mathrm{i}Z = \{x_1 + \mathrm{i}x_2 | x_1, x_2 \in Z\}$. For a linear operator $L : Z_1 \to Z_2$, we will use $\mathrm{Dom}(L)$, $\mathrm{Ker}(L)$ and $\mathrm{Range}(L)$ to denote its domain, kernel and range space, respectively. For the Hilbert space $Y_{\mathbb{C}}$, define the inner product by $\langle u, v\rangle = \int_{\Omega} \bar{u}(x) v(x) dx$. Let $\mathbb{R}_+$, $\mathbb{N}_0$ and $\mathbb{C}$ denote the sets of nonnegative real numbers, nonnegative integer numbers and complex numbers, respectively. Moreover, for simplicity of the notations, we denote

$$\begin{aligned} r_1 &= \frac{\partial F(0,0)}{\partial u}, \qquad r_2 = \frac{\partial F(0,0)}{\partial v}, \ \rho_0 = -d \int_{\Omega} \phi |\nabla \phi|^2 dx \\ \rho_1 &= r_1 \int_{\Omega} \phi^3 dx, \quad \rho_2 = r_2 \int_{\Omega} \int_{\Omega} K(x,y) \phi^2(x) \phi(y) dx dy. \end{aligned} \tag{7}$$

The steady states of (3) are determined by:

$$\begin{cases} \Delta u + \nabla \cdot (du \nabla u) + \lambda u F\left(u, \int_{\Omega} K(\cdot, y) u(y) dy\right) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{8}$$

Define the nonlinear operator $T : X \times \mathbb{R} \to Y$ by

$$T(u, \lambda) = \Delta u + \nabla \cdot (du \nabla u) + \lambda u F\left(u, \int_{\Omega} K(\cdot, y) u(y) dy\right). \tag{9}$$

Theorem 2.1. *Assume that*

$$\rho_0 + \lambda_*(\rho_1 + \rho_2) \neq 0 \tag{10}$$

Then there exist $\bar{\lambda}^* > \lambda_* > \underline{\lambda}^*$ *and a continuously differentiable mapping* $[\underline{\lambda}^*, \bar{\lambda}^*] \ni \lambda \longmapsto (\xi_\lambda, \alpha_\lambda) \in X_1 \times \mathbb{R}$, *such that, for any* $\lambda \in [\underline{\lambda}^*, \bar{\lambda}^*]$, (3) *has a steady state solution in the form of*

$$u_\lambda = \alpha_\lambda (\lambda - \lambda_*)[\phi + (\lambda - \lambda_*)\xi_\lambda], \tag{11}$$

where

$$\alpha_{\lambda_*} = -\frac{\int_{\Omega} \phi^2 dy}{\rho_0 + \lambda_*(\rho_1 + \rho_2)} \tag{12}$$

and $\xi_{\lambda_*} \in X_1$ *is the unique solution of the equation*

$$(\Delta + \lambda_*)\xi + \alpha_{\lambda_*} d \nabla \cdot (\phi \nabla \phi) + \phi \left[1 + \alpha_{\lambda_*} \lambda_* \left(r_1 \phi + r_2 \int_{\Omega} K(x,y) \phi(y) dy\right)\right] = 0. \tag{13}$$

Moreover, if

(A1): $\rho_0 + \lambda_*(\rho_1 + \rho_2) < 0$,

then $u_\lambda > 0$ *for any* $\lambda \in (\lambda_*, \bar{\lambda}^*]$. *Conversely, if the inequality in* **(A1)** *is reversed, then* $u_\lambda > 0$ *for any* $\lambda \in [\underline{\lambda}^*, \lambda_*)$.

Proof. Since $D_uT(0,\lambda_*) = \Delta + \lambda_*$ is a symmetric Fredholm operator from X to Y, we have the following decompositions:

$$X = \mathrm{Ker}(\Delta + \lambda_*) \oplus X_1, \qquad Y = \mathrm{Ker}(\Delta + \lambda_*) \oplus Y_1, \tag{14}$$

where

$$\mathrm{Ker}(\Delta + \lambda_*) = \mathrm{span}\{\phi\}, \quad X_1 = \left\{ y \in X : \int_\Omega \phi(x)y(x)dx = 0 \right\},$$
$$Y_1 = \mathrm{Range}(\Delta + \lambda_*) = \left\{ y \in Y : \int_\Omega \phi(x)y(x)dx = 0 \right\}.$$

Therefore, for any $u \in X$, there exists a unique decomposition:

$$u = u_1 + u_2, \quad u_1 \in \mathrm{Ker}(\Delta + \lambda_*), u_2 \in X_1.$$

Denote by P the projection operator from Y to Y_1. It is clear that $T(u,\lambda) = 0$ if and only if

$$PT(u_1 + u_2, \lambda) = 0, \quad (I - P)T(u_1 + u_2, \lambda) = 0. \tag{15}$$

Note that $PT(0,\lambda) = 0$ and $PT_u(0,\lambda_*) = \Delta + \lambda_*$ is bijective from X_1 to Y_1. Thus, from the implicit function theorem, there exist a neighborhood U of $(0,\lambda_*)$ in $\mathrm{Ker}(\Delta + \lambda_*) \times \mathbb{R}$ and a unique Fréchet differentiable function $f : U \to X_1$ such that $u_2 = f(u_1, \lambda)$ and

$$PT(u_1 + f(u_1,\lambda), \lambda) = 0, \qquad \forall\ (u_1, \lambda) \in U. \tag{16}$$

Taking the Fréchet derivative of the both side of (16) with respect to u_1 at $(0, \lambda_*)$ gives that

$$f_{u_1}(0,\lambda_*)\phi = 0, \qquad \forall\ \phi \in \mathrm{Ker}(\Delta + \lambda_*). \tag{17}$$

Moreover, since $T(0,\lambda) = 0$, it follows from the uniqueness of the implicit function that $f(0,\lambda) = 0$ for λ close to λ_* and thus we have

$$f_\lambda(0,\lambda_*) = 0. \tag{18}$$

Submitting $u_2 = f(u_1,\lambda)$ into the second equation of (15), it remains to solve $(u_1,\lambda) \in U$ from the following equation

$$(I - P)T(u_1 + f(u_1,\lambda), \lambda) = 0 \tag{19}$$

Clearly, $u_1 \in \mathrm{Ker}(\Delta + \lambda_*)$ if and only if $u_1 = s\phi$ for some $s \in \mathbb{R}$. Similar as the proof of the Crandall-Rabinowitz bifurcation theorem [8], we define a new function $h(s,\lambda) \in C^1(\mathbb{R}^2; \mathbb{R})$ by

$$h(s,\lambda) = \begin{cases} \dfrac{1}{s} g(s,\lambda), & \textit{if } s \neq 0, \\ g_s(0,\lambda), & \textit{if } s = 0, \end{cases} \tag{20}$$

where $g(s,\lambda) = \langle \phi, T(s\phi + f(s\phi,\lambda),\lambda)\rangle$. Direct calculation derives that

$$h(0,\lambda_*) = g_s(0,\lambda_*) = \int_\Omega \phi(\Delta + \lambda_*)\phi dx = 0,$$
$$h_s(0,\lambda_*) = \frac{1}{2} g_{ss}(0,\lambda_*) = \rho_0 + \lambda_*(\rho_1 + \rho_2).$$

From (10), we know $h_s(0,\lambda_*) \neq 0$. Based on the implicit function theorem, there exist a $\delta > 0$ and a unique continuously differentiable mapping $\lambda \longmapsto s_\lambda$ from $[\lambda_* - \delta, \lambda_* + \delta]$ to $\mathbb{R}$, such that $h(s_\lambda, \lambda) = 0$. Accordingly, we obtain

$$g(s_\lambda, \lambda) = 0, \qquad \forall\ \lambda \in [\lambda_* - \delta, \lambda_* + \delta],$$

and therefore

$$T(s_\lambda\phi + f(s_\lambda\phi,\lambda),\lambda) = 0, \qquad \forall\, \lambda \in [\lambda_* - \delta, \lambda_* + \delta],$$

which means $u_\lambda := s_\lambda\phi + f(s_\lambda\phi,\lambda)$ is a solution of (8). Next, we shall express u_λ in a more precise form.

Note that $s_{\lambda_*} = 0$ and together with (17), (18), it is reasonable to suppose that, when λ close to λ_*, (8) has a solution

$$u = \alpha(\lambda-\lambda_*)[\phi + (\lambda-\lambda_*)\xi], \qquad \alpha \in \mathbb{R},\ \xi \in X_1. \tag{21}$$

Submitting (21) into (8), it is easy to check that u is a steady state solution if and only if (ξ,α,λ) is a zero of the function $m : X_1 \times \mathbb{R}^2 \to Y$, which is defined by

$$m(\xi,\alpha,\lambda) = (\Delta+\lambda_*)\xi + \alpha d\,\nabla\cdot\big(m_1(\xi,\lambda)\nabla m_1(\xi,\lambda)\big) + m_1(\xi,\lambda) + \lambda m_1(\xi,\lambda) m_2(\xi,\alpha,\lambda),$$

where $m_1(\xi,\lambda) = \phi + (\lambda-\lambda_*)\xi$ and

$$m_2(\xi,\alpha,\lambda) = \begin{cases} \dfrac{F\left(u, \int_\Omega K(\cdot,y)u(y)dy\right) - 1}{\lambda-\lambda_*}, & \text{if } \lambda \neq \lambda_*, \\ \alpha\left(r_1\phi + r_2\displaystyle\int_\Omega K(\cdot,y)\phi(y)dy\right), & \text{if } \lambda = \lambda_*. \end{cases}$$

Recall that $\Delta + \lambda_*$ is bijective from X_1 to Y_1. Then, α_{λ_*} and ξ_{λ_*} are well defined. Moreover, it is easy to verify $m(\xi_{\lambda_*},\alpha_{\lambda_*},\lambda_*) = 0$. Taking the Fréchet derivative of m with respect to (ξ,α) at $(\xi_{\lambda_*},\alpha_{\lambda_*},\lambda_*)$ gives that

$$D_{(\xi,\alpha)}m(\xi_{\lambda_*},\alpha_{\lambda_*},\lambda_*)[\eta,\varepsilon] = (\Delta+\lambda_*)\eta + \varepsilon d\nabla\cdot(\phi\nabla\phi) + \varepsilon\lambda_*\phi\left(r_1\phi + r_2\int_\Omega K(x,y)\phi(y)dy\right)$$

Due to $d\nabla\cdot(\phi\nabla\phi) + \lambda_*\phi(r_1\phi + r_2\int_\Omega K(x,y)\phi(y)dy) \notin Y_1$, it follows that $D_{(\xi,\alpha)}m(\xi_{\lambda_*}, \alpha_{\lambda_*},\lambda)$ is bijective from $X_1 \times \mathbb{R} \to Y$. Then from the implicit function theorem, there exist $\bar\lambda^* > \lambda_* > \underline{\lambda}^*$ and a continuously differential mapping $\lambda \longmapsto (\xi_\lambda,\alpha_\lambda)$ from $[\underline{\lambda}^*,\bar\lambda^*]$ to $X_1\times\mathbb{R}$ such that

$$m(\xi_\lambda,\alpha_\lambda,\lambda) = 0,$$

and hence $u_\lambda := \alpha_\lambda(\lambda-\lambda_*)[\phi + (\lambda-\lambda_*)\xi_\lambda]$ is a steady state solution of (3) for any $\lambda \in [\underline{\lambda}^*,\lambda^*]$. Furthermore, when **(A1)** holds, we have $\alpha_{\lambda_*} > 0$. Thus, by choosing $\bar\lambda_*$ sufficiently close to λ_*, $u_\lambda > 0$ for $\lambda \in (\lambda_*,\bar\lambda^*]$ follows directly from the continuity of $\lambda \longmapsto \alpha_\lambda$. A similar discussion can be used when the inequality of **(A1)** is reversed, resulting in $u_\lambda > 0$ for $\lambda \in [\underline{\lambda}^*,\lambda_*)$. □

3. Eigenvalues and stability analysis.

In this section, we will consider the stability of the bifurcated steady state u_λ, when Ω is a bounded open set in $\mathbb{R}$. Without loss of generality, we assume **(A1)** holds. The linearized equation of (3) at u_λ is given by

$$\begin{aligned}\frac{\partial}{\partial t}v(x,t) =& \Delta v(x,t) + \nabla\cdot(du_\lambda(x)\nabla v(x,t-\tau)) + \nabla\cdot(dv(x,t)\nabla u_\lambda(x)) \\ &+ \lambda F\left(u_\lambda(x), \int_\Omega K(x,y)u_\lambda(y)dy\right)v(x,t) + \lambda\vartheta^1_\lambda(x)u_\lambda(x)v(x,t) \\ &+ \lambda\vartheta^2_\lambda(x)u_\lambda(x)\int_\Omega K(x,y)v(y,t-\sigma)dy\end{aligned} \tag{22}$$

where $\vartheta^i_\lambda(x) = \frac{\partial}{\partial x_i}F\left(u_\lambda(x), \int_\Omega K(x,y)u_\lambda(y)dy\right),\ i = 1,2$. For any $\lambda > \lambda_*$ and $(\tau,\sigma) \in \mathbb{R}^2_+$, we are looking for $\mu \in \mathbb{C}$ and $\psi \in X_{\mathbb{C}}\backslash\{0\}$ such that

$$\Pi(\mu,\lambda,\tau,\sigma)\psi = 0. \tag{23}$$

where

$$\begin{aligned}\Pi(\mu,\lambda,\tau,\sigma)\psi =&\Delta\psi+\nabla\cdot(du_\lambda\nabla\psi)e^{-\mu\tau}+\nabla\cdot(d\psi\nabla u_\lambda)\\&+\lambda F\left(u_\lambda,\int_\Omega K(\cdot,y)u_\lambda(y)dy\right)\psi+\lambda\vartheta^1_\lambda u_\lambda\psi\\&+\lambda\vartheta^2_\lambda u_\lambda\int_\Omega K(\cdot,y)\psi(y)dye^{-\mu\sigma}-\mu\psi.\end{aligned}\tag{24}$$

The complex number μ is referred as an eigenvalue associated with (22). Firstly, we have the following two estimates for the solution $(\mu_\lambda,\psi_\lambda)$ of (23).

Lemma 3.1. *Assume that* ***(A1)*** *holds. If*

$$\textbf{(A2):}\quad |d|<d_*:=\frac{1}{\max_\lambda\max_x u_\lambda(x)},$$

then there exists a constant C*, such that for any* $(\mu_\lambda,\lambda,\tau_\lambda,\sigma_\lambda,\psi_\lambda)\in\mathbb{C}\times(\lambda_*,\bar\lambda_*]\times\mathbb{R}^2_+\times X_\mathbb{C}\backslash\{0\}$ *with* $\mathrm{Re}\mu_\lambda\geq 0$ *satisfying* (23),

$$\|\nabla\psi_\lambda\|_{Y_\mathbb{C}}\leq C\|\psi_\lambda\|_{Y_\mathbb{C}}.\tag{25}$$

Proof. According to the continuity of $[\lambda^*,\bar\lambda^*]\ni\lambda\longmapsto(\xi_\lambda,\alpha_\lambda)\in X\times\mathbb{R}_+$ and the embedding theorem [1], it follows that there is a constant $C_0=C_0(\gamma,\bar\lambda_*,\Omega)>0$, such that for any $\lambda\in(\lambda_*,\bar\lambda_*]$,

$$|\alpha_\lambda|\leq C_0,\quad |\xi_\lambda|_{1+\gamma}\leq C_0,\quad |u_\lambda|_{1+\gamma}\leq C_0,\tag{26}$$

where $0<\gamma<\frac{1}{2}$. Note that u_λ solves (8), i.e.,

$$(1+du_\lambda)\Delta u_\lambda+d\nabla u_\lambda\cdot\nabla u_\lambda+\lambda u_\lambda F\left(u_\lambda,\int_\Omega K(\cdot,y)u_\lambda(y)dy\right)=0,$$

and **(H1)**, **(H2)** and **(A2)** hold. By the regularity theory for elliptic equations [11], one can obtain $u_\lambda\in C^{2+\beta}(\bar\Omega)$, where $0<\beta<\min\{\alpha,1/2\}$. Moreover, there is a constant $C_1=C_1(\beta,d,\Omega,C_0)$, such that for any $\lambda\in(\lambda_*,\bar\lambda_*]$,

$$|u_\lambda|_{2+\beta}\leq C_1.\tag{27}$$

On the other hand, there also exists a constant $C_2=C_2(\bar\lambda_*,\Omega,C_1)>0$, such that for any $\lambda\in(\lambda_*,\bar\lambda_*]$,

$$\left\|\lambda F\left(u_\lambda,\int_\Omega K(\cdot,y)u_\lambda(y)dy\right)\right\|_\infty\leq C_2,\ \|\vartheta^i_\lambda\|_\infty\leq C_2,\ \|\lambda\vartheta^i_\lambda u_\lambda\|_\infty\leq C_2,\ i=1,2.\tag{28}$$

Since

$$\begin{aligned}\int_\Omega\bar\psi\nabla\cdot(\psi\nabla u_\lambda)dx&=\int_\Omega\bar\psi\nabla\psi\cdot\nabla u_\lambda dx+\int_\Omega|\psi|^2\Delta u_\lambda dx\\&=-\int_\Omega\psi\nabla\cdot(\bar\psi\nabla u_\lambda)dx+\int_\Omega|\psi|^2\Delta u_\lambda dx,\end{aligned}$$

we have

$$\mathrm{Re}\left\{\int_\Omega\bar\psi\nabla\cdot(d\psi\nabla u_\lambda)dx\right\}=\frac{d}{2}\int_\Omega|\psi|^2\Delta u_\lambda dx.\tag{29}$$

Taking the inner product of ψ_λ with both sides of $\Pi(\mu_\lambda,\lambda,\tau_\lambda,\sigma_\lambda)\psi_\lambda=0$, and from (29), we get

$$\begin{aligned}\|\nabla\psi_\lambda\|^2_{Y_{\mathbb{C}}}=&-d\int_\Omega u_\lambda|\nabla\psi_\lambda|^2dx\mathrm{Re}\{e^{-\mu_\lambda\tau_\lambda}\}+\frac{d}{2}\int_\Omega\Delta u_\lambda|\psi_\lambda|^2dx\\&+\int_\Omega\left[\lambda F\big(u_\lambda,\int_\Omega K(\cdot,y)u_\lambda(y)dy\big)+\lambda\vartheta^1_\lambda u_\lambda-\mathrm{Re}\mu_\lambda\right]\left|\psi_\lambda\right|^2dx\\&+\lambda\mathrm{Re}\left\{\int_\Omega\int_\Omega\vartheta^2_\lambda(x)u_\lambda(x)\bar{\psi_\lambda}(x)K(x,y)\psi_\lambda(y)dxdye^{-\mu_\lambda\sigma_\lambda}\right\}.\end{aligned}\tag{30}$$

It then follows from $\mathrm{Re}\mu_\lambda>0$ and (30) that

$$\|\nabla\psi_\lambda\|^2_{Y_{\mathbb{C}}}\le|d|\max_{\bar\Omega}\{u_\lambda(x)\}\|\nabla\psi_\lambda\|^2_{Y_{\mathbb{C}}}+\left(\frac{|d|}{2}C_1+2C_2+C_2|\Omega|\|K\|_{\infty\times\infty}\right)\|\psi_\lambda\|^2_{Y_{\mathbb{C}}},$$

from which we obtain

$$\|\nabla\psi_\lambda\|^2_{Y_{\mathbb{C}}}\le\frac{|d|C_1/2+C_2(2+|\Omega|\|K\|_{\infty\times\infty})}{1-|d|\max_{\bar\Omega}\{u_\lambda(x)\}}\|\psi_\lambda\|^2_{Y_{\mathbb{C}}}.\tag{31}$$

This completes the proof. □

Lemma 3.2. *Assume that **(A1)** and **(A2)** hold. If* $(\mu_\lambda,\lambda,\tau_\lambda,\sigma_\lambda,\psi_\lambda)\in\mathbb{C}\times(\lambda_*,\bar\lambda_*]\times\mathbb{R}^2_+\times X_{\mathbb{C}}\backslash\{0\}$ *satisfies* (23) *with* $\mathrm{Re}\mu_\lambda\ge0$, *then* $\left|\frac{\mu_\lambda}{\lambda-\lambda_*}\right|$ *is bounded for* $\lambda\in(\lambda_*,\bar\lambda_*]$.

Proof. For each fixed $\lambda\in(\lambda_*,\bar\lambda_*]$, we define the linear self-conjugate operator $H_\lambda:X_{\mathbb{C}}\to Y_{\mathbb{C}}$ by

$$H_\lambda(\psi)=\nabla\cdot\big((1+du_\lambda)\nabla\psi\big)+\lambda\psi F\left(u_\lambda,\int_\Omega K(\cdot,y)u_\lambda(y)dy\right).\tag{32}$$

Note that $u_\lambda>0$ and $H_\lambda(u_\lambda)=0$, we have 0 is the principal eigenvalues of H_λ, and therefore $\langle\psi,H_\lambda(\psi)\rangle\le0$ for any $\psi\in X_{\mathbb{C}}$. Without loss of generality, we may assume $\|\psi_\lambda\|_{Y_{\mathbb{C}}}=1$. Then, from (32) and $\Pi(\mu_\lambda,\lambda,\tau_\lambda,\sigma_\lambda)\psi_\lambda=0$, we can derive that

$$\begin{aligned}0\ge\langle\psi_\lambda,H_\lambda(\psi_\lambda)\rangle=&\mu_\lambda-(e^{-\mu_\lambda\tau_\lambda}-1)\big\langle\psi_\lambda,\nabla\cdot(du_\lambda\nabla\psi_\lambda)\big\rangle-\big\langle\psi_\lambda,\nabla\cdot(d\psi_\lambda\nabla u_\lambda)\big\rangle\\&-\lambda\Big\langle\psi_\lambda,\vartheta^1_\lambda u_\lambda\psi_\lambda+\vartheta^2_\lambda u_\lambda\int_\Omega K(\cdot,y)\psi_\lambda(y)dye^{-\mu_\lambda\sigma_\lambda}\Big\rangle.\end{aligned}$$

Applying the regularity theory for elliptic equations to $m(\xi_\lambda,\alpha_\lambda,\lambda)=0$, we obtain $\xi_\lambda\in C^{2+\beta}(\bar\Omega)$ for any $0<\beta<\min\{\alpha,\frac12\}$. Moreover, there exists a constant $C_3=C_3(\beta,d,\Omega,C_0)$, such that for any $\lambda\in(\lambda_*,\bar\lambda_*]$,

$$|\xi_\lambda|_{2+\beta}\le C_3,\quad|m_1(\xi_\lambda,\lambda)|_{2+\beta}\le C_3.\tag{33}$$

Therefore, based on Lemma 3.1 and $\mathrm{Re}\mu_\lambda>0$, we arrive at the following inequality

$$\begin{aligned}0\le&\mathrm{Re}\left(\frac{\mu_\lambda}{\lambda-\lambda_*}\right)\\\le&\alpha_\lambda\mathrm{Re}\left\{d(1-e^{-\mu_\lambda\tau_\lambda})\big\langle\nabla\psi_\lambda,m_1(\xi_\lambda,\lambda)\nabla\psi_\lambda\big\rangle+\frac{d}{2}\big\langle\psi_\lambda,\psi_\lambda\Delta m_1(\xi_\lambda,\lambda)\big\rangle\right\}\\&+\alpha_\lambda\mathrm{Re}\left\{\lambda\Big\langle\psi_\lambda,\vartheta^1_\lambda m_1(\xi_\lambda,\lambda)\psi_\lambda+\vartheta^2_\lambda m_1(\xi_\lambda,\lambda)\int_\Omega K(\cdot,y)\psi_\lambda(y)dye^{-\mu_\lambda\sigma_\lambda}\Big\rangle\right\}\end{aligned}$$

$$\le \alpha_\lambda |d| \left(2C\|m_1(\xi_\lambda,\lambda)\|_\infty + \frac{1}{2}\|\Delta m_1(\xi_\lambda,\lambda)\|_\infty \right) + \alpha_\lambda \lambda \left(\|\vartheta^1_\lambda\|_\infty + \|\vartheta^2_\lambda\|_\infty \|K\|_{\infty\times\infty}|\Omega| \right) \|m_1(\xi_\lambda,\lambda)\|_\infty.$$

Similarly, we have

$$\begin{aligned} &\left| \mathrm{Im}\left(\frac{\mu_\lambda}{\lambda-\lambda_*} \right) \right| \\ =&\alpha_\lambda \Bigg| \mathrm{Im}\big\{ -de^{-\mu_\lambda\tau_\lambda}\langle \nabla\psi_\lambda, m_1(\xi_\lambda,\lambda)\nabla\psi_\lambda\rangle + d\langle \psi_\lambda, \nabla\psi_\lambda\cdot\nabla m_1(\xi_\lambda,\lambda)\rangle \big\} \\ &+ \lambda \mathrm{Im}\left\{ \left\langle \psi_\lambda, \vartheta^2_\lambda m_1(\xi_\lambda,\lambda)\int_\Omega K(\cdot,y)\psi_\lambda(y)dy e^{-\mu_\lambda\sigma_\lambda} \right\rangle \right\} \Bigg| \\ \le& \alpha_\lambda |d| C \Big(\|m_1(\xi_\lambda,\lambda)\|_\infty + \|\nabla m_1(\xi_\lambda,\lambda)\|_\infty \Big) \\ &+ \alpha_\lambda\lambda \|\vartheta^2_\lambda\|_\infty \|K\|_{\infty\times\infty}|\Omega| \|m_1(\xi_\lambda,\lambda)\|_\infty. \end{aligned}$$

With the aid of the estimates (26), (28) and (33), it is evident that $\left|\mathrm{Re}\left(\frac{\mu_\lambda}{\lambda-\lambda_*}\right)\right|$ and $\left|\mathrm{Im}\left(\frac{\mu_\lambda}{\lambda-\lambda_*}\right)\right|$ are bounded for $\lambda \in (\lambda_*, \bar\lambda_*]$, and so is $\left|\dfrac{\mu_\lambda}{\lambda-\lambda_*}\right|$. □

Theorem 3.3. *Assume that **(A1)** and **(A2)** hold. Then there exists $\tilde\lambda^* \in (\lambda_*, \bar\lambda^*]$, such that zero is not an eigenvalue of (22) for any $\tau,\sigma \in \mathbb{R}_+$ and $\lambda \in (\lambda_*, \tilde\lambda^*]$.*

Proof. If the assertion does not hold, then there exists a sequence $\{(\lambda_n,\tau_n,\sigma_n,\psi_{\lambda_n})\}_{n=1}^\infty \subset (\lambda_*,\bar\lambda^*]\times\mathbb{R}^2_+\times X_\mathbb{C}\backslash\{0\}$ such that $\lim_{n\to\infty}\lambda_n = \lambda_*$, $\|\psi_{\lambda_n}\|^2_{Y_\mathbb{C}} = \|\phi\|^2_{Y_\mathbb{C}}$ and

$$\Pi(0,\lambda_n,\tau_n,\sigma_n)\psi_{\lambda_n} = 0. \tag{34}$$

Note that $X_\mathbb{C} = \big(\mathrm{Ker}(\Delta+\lambda_*)\big)_\mathbb{C} \oplus (X_1)_\mathbb{C}$. We write $\psi_{\lambda_n} = \beta_{\lambda_n}\phi + (\lambda_n-\lambda_*)z_{\lambda_n}$, where $\beta_{\lambda_n} \ge 0$ and $z_{\lambda_n} \in (X_1)_\mathbb{C}$. Then, from Theorem 2.1 and (34), we obtain:

$$\begin{cases} H_1(z_{\lambda_n},\beta_{\lambda_n},\lambda_n) := (\Delta+\lambda_*)z_{\lambda_n} + \alpha_{\lambda_n} d\,\nabla\cdot\Big(m_1(\xi_{\lambda_n},\lambda_n)\nabla(\beta_{\lambda_n}\phi + (\lambda_n-\lambda_*)z_{\lambda_n}) \Big) \\ \qquad + \alpha_{\lambda_n} d\,\nabla\cdot\Big([\beta_{\lambda_n}\phi + (\lambda_n-\lambda_*)z_{\lambda_n}]\nabla m_1(\xi_{\lambda_n},\lambda_n) \Big) \\ \qquad + \big[1+\lambda_n m_2(\xi_{\lambda_n},\alpha_{\lambda_n}\lambda_n)\big]\big[\beta_{\lambda_n}\phi + (\lambda_n-\lambda_*)z_{\lambda_n}\big] \\ \qquad + \lambda_n\alpha_{\lambda_n}\vartheta^1_{\lambda_n} m_1(\xi_{\lambda_n},\lambda_n)\big[\beta_{\lambda_n}\phi + (\lambda_n-\lambda_*)z_{\lambda_n}\big] \\ \qquad + \lambda_n\alpha_{\lambda_n}\vartheta^2_{\lambda_n} m_1(\xi_{\lambda_n},\lambda_n)\int_\Omega K(\cdot,y)[\beta_{\lambda_n}\phi(y) + (\lambda_n-\lambda_*)z_{\lambda_n}(y)]dy \\ \qquad = 0, \\ H_2(z_{\lambda_n},\beta_{\lambda_n},\lambda_n) := (\beta^2_{\lambda_n}-1)\|\phi\|^2_{Y_\mathbb{C}} + (\lambda_n-\lambda_*)^2\|z_{\lambda_n}\|^2_{Y_\mathbb{C}} = 0. \end{cases} \tag{35}$$

By the second equation of (35), it is clear that $|\beta_{\lambda_n}| \le 1$. Multiplying the both sides of the first equation of (35) by $\bar z_{\lambda_n}$ and integrating over Ω, we have

$$\begin{aligned} &\|\nabla z_{\lambda_n}\|^2_{Y_\mathbb{C}} \\ \le& \lambda_*\|z_{\lambda_n}\|^2_{Y_\mathbb{C}} + (\lambda_n-\lambda_*)\alpha_{\lambda_n}|d|\|m_1(\xi_{\lambda_n},\lambda_n)\|_\infty\|\nabla z_{\lambda_n}\|^2_{Y_\mathbb{C}} \\ &+ \alpha_{\lambda_n}|d|\Big(\|\nabla m_1(\xi_{\lambda_n},\lambda_n)\|_\infty\|\nabla\phi\|_{Y_\mathbb{C}} + \|m_1(\xi_{\lambda_n},\lambda_n)\|_\infty\|\Delta\phi\|_{Y_\mathbb{C}} \Big)\|z_{\lambda_n}\|_{Y_\mathbb{C}} \\ &+ \alpha_{\lambda_n}|d|\Big(\|\nabla m_1(\xi_{\lambda_n},\lambda_n)\|_\infty\|\nabla\phi\|_{Y_\mathbb{C}} + \|\Delta m_1(\xi_{\lambda_n},\lambda_n)\|_\infty\|\phi\|_{Y_\mathbb{C}} \Big)\|z_{\lambda_n}\|_{Y_\mathbb{C}} \end{aligned}$$

$$
\begin{aligned}
&+ (\lambda_n - \lambda_*)\frac{|d|}{2}\alpha_{\lambda_n}\|\Delta m_1(\xi_{\lambda_n},\lambda_n)\|_\infty\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}\\
&+ \Big[\|1+\lambda_n m_2(\xi_{\lambda_n},\alpha_{\lambda_n}\lambda_n)+\lambda_n\alpha_{\lambda_n}\vartheta^1_{\lambda_n}m_1(\xi_{\lambda_n},\lambda_n)\|_\infty+\|h_{\lambda_n}\|_\infty\\
&+\|\lambda_n\alpha_{\lambda_n}\vartheta^2_{\lambda_n}m_1(\xi_{\lambda_n},\lambda_n)\|_\infty\|K\|_\infty|\Omega|\Big]\Big[\|\phi\|_{Y_{\mathbb{C}}}\|z_{\lambda_n}\|_{Y_{\mathbb{C}}}+(\lambda_n-\lambda_*)\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}\Big]\\
\le{}& \lambda_*\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}+(\lambda_n-\lambda_*)M_1\|\nabla z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}+M_2\|z_{\lambda_n}\|_{Y_{\mathbb{C}}}+(\lambda_n-\lambda_*)M_3\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}},
\end{aligned}\tag{36}
$$

for some constants $M_1, M_2, M_3 > 0$. Since $\lim_{n\to\infty} 1-(\lambda_n-\lambda_*)M_1 = 1 > 0$, we derive, from (36), that there is a positive integer N_1, such that

$$
\begin{aligned}
\|\nabla z_{\lambda_n}\|^2_{Y_{\mathbb{C}}} &\le \frac{M_2}{1-(\lambda_n-\lambda_*)M_1}\|z_{\lambda_n}\|_{Y_{\mathbb{C}}}+\frac{\lambda_*+(\lambda_n-\lambda_*)M_3}{1-(\lambda_n-\lambda_*)M_1}\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}\\
&\le 2M_2\|z_{\lambda_n}\|_{Y_{\mathbb{C}}}+2\lambda_*\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}
\end{aligned}\tag{37}
$$

for $n > N_1$. Combining (36) and (37), we have

$$
\begin{aligned}
\|\nabla z_{\lambda_n}\|^2_{Y_{\mathbb{C}}} &\le \lambda_*\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}+2M_2\|z_{\lambda_n}\|_{Y_{\mathbb{C}}}+(\lambda_n-\lambda_*)(2\lambda_*M_1+M_3)\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}\\
&:= \lambda_*\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}+M_4\|z_{\lambda_n}\|_{Y_{\mathbb{C}}}+(\lambda_n-\lambda_*)M_5\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}
\end{aligned}\tag{38}
$$

for $n > N_1$. Let $\lambda_{**} > \lambda_*$ be second eigenvalue of the operator $-\Delta$. Then

$$
\langle\psi,-\Delta\psi\rangle \ge \lambda_{**}\langle\psi,\psi\rangle,\quad \forall\ \psi\in(X_1)_{\mathbb{C}}.\tag{39}
$$

It now follows from (38) and (39) that

$$
\lambda_{**}\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}} \le \langle z_{\lambda_n},-\Delta z_{\lambda_n}\rangle = \|\nabla z_{\lambda_n}\|^2_{Y_{\mathbb{C}}} \le \lambda_*\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}+M_4\|z_{\lambda_n}\|_{Y_{\mathbb{C}}}+(\lambda_n-\lambda_*)M_5\|z_{\lambda_n}\|^2_{Y_{\mathbb{C}}}
$$

or equivalently,

$$
\|z_{\lambda_n}\|_{Y_{\mathbb{C}}} \le \frac{M_4}{\lambda_{**}-\lambda_*}+\frac{\lambda_n-\lambda_*}{\lambda_{**}-\lambda_*}M_5\|z_{\lambda_n}\|_{Y_{\mathbb{C}}},\qquad n > N_1,
$$

which means that $\{z_{\lambda_n}\}$ is bounded in $Y_{\mathbb{C}}$. Applying the standard regularity theory and embedding theorem to the first equation of (35), it can be also seen that $\{z_{\lambda_n}\}$ is bounded in $C^{2+\beta}(\bar{\Omega})$, where $0 < \beta < \min\{\alpha, 1/2\}$. Therefore, there exists a subsequence, still denoted by $\{(z_{\lambda_n},\beta_{\lambda_n},\lambda_n)\}_{n=1}^\infty$, such that

$$
(z_{\lambda_n},\beta_{\lambda_n},\lambda_n)\to(z_*,1,\lambda_*)\ \text{in}\ C^2(\bar{\Omega})\times\mathbb{R}^2.
$$

Taking the limit of the equation $H_1(z_{\lambda_n},\beta_{\lambda_n},\lambda_n)=0$ in $C(\bar{\Omega})\times\mathbb{R}^2$ gives that

$$
(\Delta+\lambda_*)z_*+2\alpha_{\lambda_*}d\nabla\cdot(\phi\nabla\phi)+2\lambda_*\alpha_{\lambda_*}\phi(r_1\phi+\int_\Omega K(\cdot,y)\phi((y)dy)+\phi=0.
$$

This, together with (12), implies

$$
\alpha_{\lambda_*}(\rho_0+\lambda_*\rho_1+\lambda_*\rho_2)=0,
$$

which contradicts **(A1)**. □

Remark 1. Using a similar argument as the proof of Theorem 3.3 and Lemma 3.2, we claim that under the assumption of **(A1)**, there is a constant $\bar{\lambda}^*_0\in(\lambda_*,\bar{\lambda}^*]$, such that all the eigenvalues of (22) have negative real parts for $\tau=\sigma=0$ and $\lambda\in(\lambda_*,\bar{\lambda}^*_0]$.

Next, we are about to find the pure imaginary eigenvalues of (22) for λ close to λ_*. Suppose that $\mu = \mathrm{i}\omega$, $\omega > 0$ is an eigenvalue of (22) with eigenfunction ψ. In the light of Lemma 3.2, we set

$$\begin{cases} \omega = h(\lambda - \lambda_*), \quad h > 0, \\ \psi = \beta\phi + (\lambda - \lambda_*)z, \quad \beta \geq 0, \ z \in (X_1)_{\mathbb{C}}, \\ \|\psi\|^2_{Y_{\mathbb{C}}} = \beta^2\|\phi\|^2_{Y_{\mathbb{C}}} + (\lambda - \lambda_*)^2\|z\|^2_{Y_{\mathbb{C}}} = \|\phi\|^2_{Y_{\mathbb{C}}}. \end{cases} \tag{40}$$

Substituting (40) into (23) gives

$$\begin{cases} g_1(z, \beta, h, \theta_1, \theta_2, \lambda) := (\Delta + \lambda_*)z + \alpha_\lambda d\nabla \cdot \Big(m_1(\xi_\lambda, \lambda)\nabla\big(\beta\phi + (\lambda - \lambda_*)z\big)\Big)e^{-\mathrm{i}\theta_1} \\ \qquad + \alpha_\lambda d\nabla \cdot \Big(\big(\beta\phi + (\lambda - \lambda_*)z\big)\nabla m_1(\xi_\lambda, \lambda)\Big) \\ \qquad + [\beta\phi + (\lambda - \lambda_*)z]\big\{1 + \lambda m_2(\xi_\lambda, \alpha_\lambda\lambda) + \lambda\alpha_\lambda\vartheta^1_\lambda m_1(\xi_\lambda, \lambda) - \mathrm{i}h\big\} \\ \qquad + \lambda\alpha_\lambda\vartheta^2_\lambda m_1(\xi_\lambda, \lambda)\displaystyle\int_\Omega K(\cdot, y)\big[\beta\phi(y) + (\lambda - \lambda_*)z(y)\big]dy e^{-\mathrm{i}\theta_2} \\ \qquad = 0, \\ g_2(z, \beta, h, \theta_1, \theta_2, \lambda) := (\beta^2 - 1)\|\phi\|^2_{Y_{\mathbb{C}}} + (\lambda - \lambda_*)^2\|z\|^2_{Y_{\mathbb{C}}} = 0. \end{cases} \tag{41}$$

where $\theta_1 = \omega\tau$ and $\theta_2 = \omega\sigma$. If there exists $(z, \beta, h, \theta_1, \theta_2, \lambda) \in (X_1)_{\mathbb{C}} \times \mathbb{R}^2_+ \times [0, 2\pi) \times [0, 2\pi) \times \mathbb{R}_+$ solving (41), then $\mu = \mathrm{i}\omega = \mathrm{i}h(\lambda - \lambda_*)$ is an eigenvalue of (22) when $(\lambda, \tau, \sigma) = (\lambda, \tau_n, \sigma_m)$ and $\psi = \beta\phi + (\lambda - \lambda_*)z$, where

$$\tau_n = \frac{\theta_1 + 2n\pi}{\omega}, \quad \sigma_m = \frac{\theta_2 + 2m\pi}{\omega}, \quad n, m \in \mathbb{N}_0. \tag{42}$$

Define $G : (X_1)_{\mathbb{C}} \times \mathbb{R}^2_+ \times [0, 2\pi) \times [0, 2\pi) \times \mathbb{R}_+ \to Y_{\mathbb{C}} \times R$ by $G = (g_1, g_2)$.

For the purpose of seeking zeros of G, we consider the following auxiliary equation:

$$D(h, \theta_1, \theta_2) := P_0(h) + P_1(h)e^{-\mathrm{i}\theta_1} + P_2(h)e^{-\mathrm{i}\theta_2} = 0, \tag{43}$$

where

$$P_0(h) = \alpha_{\lambda_*}\lambda_*\rho_1 - \mathrm{i}h\int_\Omega \phi^2 dx, \ P_1(h) = \alpha_{\lambda_*}\rho_0, \ P_2(h) = \alpha_{\lambda_*}\lambda_*\rho_2.$$

Lemma 3.4. *If*

$$\textbf{(A3):} \ |\rho_0| + \lambda_*(|\rho_2| - |\rho_1|) > 0.$$

is satisfied, then $(h, \theta_1^{h\pm}, \theta_2^{h\pm})$ *is the root of* (43), *where*

$$\begin{aligned} h \in H &= \left\{ h \in \mathbb{R}_+\backslash\{0\} : \frac{\alpha^2_{\lambda_*}h_1}{(\int_\Omega \phi^2 dx)^2} \geq h^2 \geq \max\left\{0, \frac{\alpha^2_{\lambda_*}h_2}{(\int_\Omega \phi^2 dx)^2}\right\}\right\} \\ \theta_1^{h\pm} &= \arg\big(\arg(P_1(h)) - \arg(P_0(h)) \pm \varphi_1(h) - \pi\big) \\ \theta_2^{h\pm} &= \arg\big(\arg(P_2(h)) - \arg(P_0(h)) \mp \varphi_2(h) - \pi\big) \end{aligned} \tag{44}$$

with

$$\begin{aligned} h_1 &= (|\rho_0| + \lambda_*|\rho_2|)^2 - (\lambda_*\rho_1)^2, \quad h_2 = (|\rho_0| - \lambda_*|\rho_2|)^2 - (\lambda_*\rho_1)^2 \\ \varphi_1(h) &= arccos\left[\frac{|P_1(h)|^2 + |P_0(h)|^2 - |P_2(h)|^2}{2|P_0(h)P_1(h)|}\right] \\ \varphi_2(h) &= arccos\left[\frac{|P_2(h)|^2 + |P_0(h)|^2 - |P_1(h)|^2}{2|P_0(h)P_2(h)|}\right]. \end{aligned} \tag{45}$$

Proof. The equation (43) can be regarded as characteristic equation of functional differential equations involving two discrete delays, so the geometric method proposed in [12] can be used to find the root (h, θ_1, θ_2) for (43). We consider the three terms, $P_0(h)$, $P_1(h)e^{-i\theta_1}$ and $P_2(h)e^{-i\theta_2}$ in (43), as three vectors in complex plane, with the magnitudes $|P_0(h)|$, $|P_1(h)|$ and $|P_2(h)|$, respectively. Then, any solution of (43) must put these vectors connect to each other and form a triangle as shown in Figure 1. Hence, (h, θ_1, θ_2) is a zero of $D(h, \theta_1, \theta_2)$ if and only if h satisfies

$$|P_0(h)| + |P_1(h)| \geq |P_2(h)|,$$
$$|P_0(h)| + |P_2(h)| \geq |P_1(h)|,$$
$$|P_1(h)| + |P_2(h)| \geq |P_0(h)|,$$

from which we derive $h \in H$. Clearly, if **(A3)** holds, then $H \neq \emptyset$.

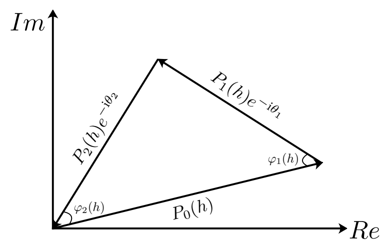


FIGURE 1. Triangle formed by $P_0(h)$, $P_1(h)e^{-i\theta_1}$ and $P_2(h)e^{-i\theta_2}$

For each given $h \in H$, let $\varphi_1(h)$, $\varphi_2(h)$ be the angles formed by $P_0(h)$, $P_1(h)e^{-i\theta_1}$ and $P_0(h)$, $P_2(h)e^{-i\theta_2}$, respectively. By the law of cosine, we know $\varphi_1(h)$ and $\varphi_2(h)$ can be represented by $|P_0(h)|$, $|P_1(h)|$ and $|P_2(h)|$, as in (45). It is easy to see that $(\theta_1, \theta_2) \in [0, 2\pi) \times [0, 2\pi)$ solving (43) for a given $h \in H$, must satisfy

$$\begin{aligned} \arg\left(P_1(h)e^{-i\theta_1}\right) - \arg\left(P_0(h)\right) \pm \varphi_1(h) = \pi, \\ \arg\left(P_2(h)e^{-i\theta_2}\right) - \arg\left(P_0(h)\right) \mp \varphi_2(h) = \pi, \end{aligned} \tag{46}$$

where $\arg : \mathbb{C} \to [0, 2\pi)$ denotes the principle value of the argument of complex number. From (46), one can solve (θ_1, θ_2), which is given by (44). ☐

Remark 2. It should be noted that **(A1)** and **(A3)** can be satisfied simultaneously. In fact, when $d > 0$, we have $\rho_0 < 0$, the region in (ρ_1, ρ_2) plane that meet both **(A1)** and **(A3)** is enclosed by

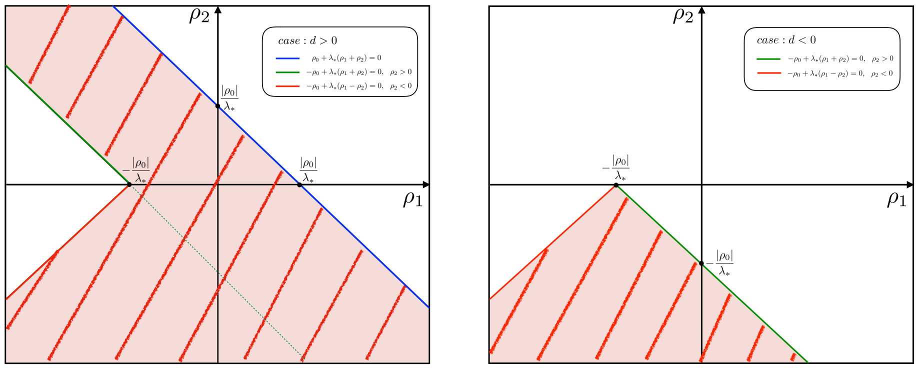
$$\begin{cases} \rho_0 + \lambda_*(\rho_1 + \rho_2) = 0 \\ -\rho_0 + \lambda_*(\rho_1 + \rho_2) = 0, \ \rho_2 > 0 \\ -\rho_0 + \lambda_*(\rho_1 - \rho_2) = 0, \ \rho_2 < 0, \end{cases}$$

which has been shown in Figure 2(a). Similarly, when $d < 0$, the region surrounded by

$$\begin{cases} -\rho_0 + \lambda_*(\rho_1 + \rho_2) = 0, \ \rho_2 < 0 \\ -\rho_0 + \lambda_*(\rho_1 - \rho_2) = 0, \ \rho_2 < 0, \end{cases}$$

satisfies **(A1)** and **(A3)**, see Figure 2(b).

As a direct consequence of Lemma 3.4, one can show the existence of the zeros of G when $\lambda = \lambda_*$.



(a) (b)

FIGURE 2. Regions in the (ρ_1, ρ_2) plane that satisfy both **(A1)** and **(A3)**: (a) $d > 0$ and (b) $d < 0$.

Lemma 3.5. *Assume that **(A1)**, **(A2)** and **(A3)** hold. Then* $G(z, \beta, h, \theta_1, \theta_2, \lambda_*) = 0$ *if and only if* $h \in H$ *and*

$$(z, \beta, h, \theta_1, \theta_2) = (z^{h\pm}, 1, h, \theta_1^{h\pm}, \theta_2^{h\pm}),$$

where $\theta_i^{h\pm}$, $i = 1, 2$ *are defined by* (44), $z^{h\pm}$ *is the unique solution of the equation*

$$\begin{aligned} 0 =&(\Delta + \lambda_*)z + \alpha_{\lambda_*} d\,\nabla\cdot(\phi\nabla\phi) + \phi\Big[1 + \lambda_*\alpha_{\lambda_*}\big(r_1\phi + r_2\int_\Omega K(\cdot, y)\phi(y)dy\big)\Big] \\ &+ \alpha_{\lambda_*} d\,\nabla\cdot(\phi\nabla\phi)e^{-\mathrm{i}\theta_1^{h\pm}} + \lambda_*\alpha_{\lambda_*}\phi\Big(r_1\phi + r_2\int_\Omega K(\cdot, y)\phi(y)dy e^{-\mathrm{i}\theta_2^{h\pm}}\Big) - \mathrm{i}h\phi \end{aligned} \tag{47}$$

Proof. If $\lambda = \lambda_*$, from the second equation of (41), we have $\beta = 1$. Submitting $\lambda = \lambda_*$ and $\beta = 1$ into the first equation of (41), it follows that

$$\begin{aligned} g_1(z, 1, h, \theta_1, \theta_2, \lambda_*) :=&(\Delta + \lambda_*)z + \alpha_{\lambda_*} d\,\nabla\cdot(\phi\nabla\phi)e^{-\mathrm{i}\theta_1} + \alpha_{\lambda_*} d\,\nabla\cdot(\phi\nabla\phi) \\ &+ \phi\left[1 + \lambda_*\alpha_{\lambda_*}(r_1\phi + r_2\int_\Omega K(\cdot, y)\phi(y)dy)\right] \\ &+ \lambda_*\alpha_{\lambda_*}\phi\left[r_1\phi + r_2\int_\Omega K(\cdot, y)\phi(y)dy e^{-\mathrm{i}\theta_2}\right] - \mathrm{i}h\phi = 0. \end{aligned} \tag{48}$$

Multiplying the both side of (48) by ϕ and then integrating over Ω, we know (48) is solvable if and only if there exists $(h, \theta_1, \theta_2) \in \mathbb{R}_+ \times [0, 2\pi) \times [0, 2\pi)$ satisfying (43). which is the case if **(A3)** is satisfied, by Lemma 3.4. Once $(\theta_1^{h\pm}, \theta_2^{h\pm})$ is determined, one can solve z to get $z^{h\pm}$ by (48), that is, $z^{h\pm}$ satisfies (47). □

Remark 3. We also remark that **(A1)** and **(A3)** are coincident with (3.5) in Theorem 3.3 in [7], when $d = 0$.

Before proving the existence of solutions of $G(z, \beta, h, \theta_1, \theta_2, \lambda) = 0$ for $\lambda > \lambda_*$, we need the following result.

Lemma 3.6. *Assume that **(A1)**, **(A2)** and **(A3)** hold. Then* $\sin(\theta_1^{h\pm} - \theta_2^{h\pm}) = 0$ *for* $h \in H$ *if and only if*

$$h^2 = \frac{\alpha_{\lambda_*}^2 h_1}{(\int_\Omega \phi^2 dx)^2} \quad \text{or} \quad h^2 = \frac{\alpha_{\lambda_*}^2 h_2}{(\int_\Omega \phi^2 dx)^2}, \tag{49}$$

where h_1 *and* h_2 *are given by* (45).

Proof. Since $(h, \theta_1^{h\pm}, \theta_2^{h\pm})$ is a solution of (43) for $h \in H$, we know

$$\left(\alpha_{\lambda_*}\lambda_*\rho_1 - \mathrm{i}h\int_\Omega \phi^2 dx\right) e^{\mathrm{i}\theta_1^{h\pm}} + \alpha_{\lambda_*}\rho_0 + \alpha_{\lambda_*}\lambda_*\rho_2 e^{-\mathrm{i}(\theta_2^{h\pm} - \theta_1^{h\pm})} = 0 \tag{50}$$

Then, separating the real and imaginary parts of (50) leads to

$$\begin{aligned} \alpha_{\lambda_*}\lambda_*\rho_1 \cos\theta_1^{h\pm} + h\int_\Omega \phi^2 dx \sin\theta_1^{h\pm} &= -\alpha_{\lambda_*}\rho_0 - \alpha_{\lambda_*}\lambda_*\rho_2\cos(\theta_2^{h\pm} - \theta_1^{h\pm}) \\ \alpha_{\lambda_*}\lambda_*\rho_1 \sin\theta_1^{h\pm} - h\int_\Omega \phi^2 dx \cos\theta_1^{h\pm} &= \alpha_{\lambda_*}\lambda_*\rho_2\sin(\theta_2^{h\pm} - \theta_1^{h\pm}). \end{aligned} \tag{51}$$

From (51), we have

$$(\alpha_{\lambda_*}\lambda_*\rho_1)^2 + h^2\left(\int_\Omega \phi^2 dx\right)^2 = (\alpha_{\lambda_*}\rho_0)^2 + (\alpha_{\lambda_*}\lambda_*\rho_2)^2 + 2\alpha_{\lambda_*}\lambda_*\rho_0\rho_2\cos(\theta_2^{h\pm} - \theta_1^{h\pm})$$

This implies $\sin(\theta_1^{h\pm} - \theta_2^{h\pm}) = 0$ if and only if

$$(\alpha_{\lambda_*}\lambda_*\rho_1)^2 + h^2\left(\int_\Omega \phi^2 dx\right)^2 = \alpha_{\lambda_*}^2(\rho_0 \pm \lambda_*\rho_2)^2. \tag{52}$$

that is,

$$h^2 = \frac{\alpha_{\lambda_*}^2 h_1}{(\int_\Omega \phi^2 dx)^2} \quad \text{or} \quad h^2 = \frac{\alpha_{\lambda_*}^2 h_2}{(\int_\Omega \phi^2 dx)^2}.$$

□

Now, we are ready to prove the main results in this section.

Theorem 3.7. *Assume that **(A1)**, **(A2)** and **(A3)** hold, then there exist a connected region* $I := \{(\lambda, h) \mid \lambda \in [\lambda_*, \bar{\bar{\lambda}}_*],\ h \in H_\lambda\}$ *and continuously differentiable mappings* $I \ni (\lambda, h) \longmapsto (z^{\lambda h\pm}, \beta^{\lambda h\pm}, \theta_1^{\lambda h\pm}, \theta_2^{\lambda h\pm}) \in (X_1)_{\mathbb{C}} \times \mathbb{R}_+ \times [0, 2\pi) \times [0, 2\pi)$ *such that*

$$G(z^{\lambda h\pm}, \beta^{\lambda h\pm}, h, \theta_1^{\lambda h\pm}, \theta_2^{\lambda h\pm}, \lambda) = 0, \tag{53}$$

where $\bar{\bar{\lambda}}_* > \lambda_*$ *is a constant, and* H_λ *is an interval for each* $\lambda \in [\lambda_*, \bar{\bar{\lambda}}_*]$*. Moreover,* $G(z, \beta, h, \theta_1, \theta_2, \lambda) = 0$ *for* $\lambda \in [\lambda_*, \bar{\bar{\lambda}}_*]$ *if and only if*

$$(z, \beta, h, \theta_1, \theta_2) = (z^{\lambda h\pm}, \beta^{\lambda h\pm}, h, \theta_1^{\lambda h\pm}, \theta_2^{\lambda h\pm}), \quad h \in H_\lambda.$$

Proof. For each given $h \in H$, denote by $T^{h\pm} = (T_1^{h\pm}, T_2^{h\pm}) : (X_1)_{\mathbb{C}} \times \mathbb{R}_+ \times [0, 2\pi) \times [0, 2\pi) \to Y_{\mathbb{C}} \times \mathbb{R}$ the Fréchet derivative of $G(z, \beta, h, \theta_1, \theta_2, \lambda_*)$ with respect to

$(z,\beta,\theta_1,\theta_2)$ at $(z^{h\pm},1,\theta_1^{h\pm},\theta_2^{h\pm})$. Then,

$$\begin{cases} T_1^{h\pm}(\eta,\kappa,\Theta_1,\Theta_2) = & (\Delta+\lambda_*)\eta+\kappa\Big\{\alpha_{\lambda_*}d\nabla\cdot(\phi\nabla\phi)e^{-\mathrm{i}\theta_1^{h\pm}}+\alpha_{\lambda_*}d\nabla\cdot(\phi\nabla\phi) \\ & +\phi\left[1+\lambda_*\alpha_{\lambda_*}\left(r_1\phi+r_2\int_\Omega K(\cdot,y)\phi(y)dy\right)\right] \\ & +\lambda_*\alpha_{\lambda_*}\phi\left[r_1+r_2\int_\Omega K(\cdot,y)\phi(y)dye^{-\mathrm{i}\theta_2^{h\pm}}\right]-\mathrm{i}h\phi\Big\} \\ & -\mathrm{i}\Theta_1\alpha_{\lambda_*}d\nabla\cdot(\phi\nabla\phi)e^{-\mathrm{i}\theta_1^{h\pm}}-\mathrm{i}\Theta_2\lambda_*\alpha_{\lambda_*}r_2\phi\int_\Omega K(\cdot,y)\phi(y)dye^{-\mathrm{i}\theta_2^{h\pm}} \\ T_2^{h\pm}(\eta,\kappa,\Theta_1,\Theta_2) = & 2\kappa\|\phi\|_{Y_{\mathbb{C}}}^2. \end{cases} \tag{54}$$

We first show that $T^{h\pm}$ is a bijection from $(X_1)_{\mathbb{C}}\times\mathbb{R}_+\times[0,2\pi)\times[0,2\pi)$ to $Y_{\mathbb{C}}\times\mathbb{R}$, for any $h\in\mathrm{int}H$. Clearly, $T^{h\pm}$ is a surjective operator. It remains to show that it is an injection. Let $T^{h\pm}(\eta,\kappa,\Theta_1,\Theta_2)=0$, from the second equation of (35), we have $\kappa=0$. Correspondingly, one can obtain

$$(\Delta+\lambda_*)\eta-\mathrm{i}\Theta_1\alpha_{\lambda_*}d\nabla\cdot(\phi\nabla\phi)e^{-\mathrm{i}\theta_1^{h\pm}}-\mathrm{i}\Theta_2\lambda_*\alpha_{\lambda_*}r_2\phi\int_\Omega K(\cdot,y)\phi(y)dye^{-\mathrm{i}\theta_2^{h\pm}}=0,$$

and hence

$$\begin{cases}\Theta_1\rho_0+\Theta_2\lambda_*\rho_2\cos(\theta_2^{h\pm}-\theta_1^{h\pm})=0\\ \Theta_2\lambda_*\rho_2\sin(\theta_2^{h\pm}-\theta_1^{h\pm})=0.\end{cases}$$

According to Lemma 3.6, it follows that $\sin(\theta_2^{h\pm}-\theta_1^{h\pm})\neq 0$ for $h\in\mathrm{int}H$. Then, we have $\Theta_2=0$ and thus $\Theta_1=0$ and $\eta=0$. Therefore, $T^{h\pm}$ is bijective for $h\in\mathrm{int}H$.

Applying the implicit function theorem to $G(z,\beta,h,\theta_1,\theta_2,\lambda)$, it can be seen that for each fixed $h_*\in\mathrm{int}H$, there exist an open set $U^{h\pm}$ of $\mathbb{R}_+^2$ containing (λ_*,h_*) and a unique continuously differentiable function $(z^{\lambda h\pm},\beta^{\lambda h\pm},\theta_1^{\lambda h\pm},\theta_2^{\lambda h\pm})$ from $U^{h\pm}$ to $(X_1)_{\mathbb{C}}\times\mathbb{R}_+\times[0,2\pi)\times[0,2\pi)$ such that

$$G(z^{\lambda h\pm},\beta^{\lambda h\pm},h,\theta_1^{\lambda h\pm},\theta_2^{\lambda h\pm},\lambda)=0 \quad \text{for } (\lambda,h)\in U^{h\pm}. \tag{55}$$

Regarding $\{\lambda_*\}\times H$ as a bounded set in (λ,h)-plane. For sufficiently small $\varepsilon>0$, there are finitely many open set $U^{h_n\pm}$, $n=1,2,\cdots,N$ such that

$$\{\lambda_*\}\times[a+\varepsilon,b-\varepsilon]\subset\bigcup_{n=1}^N U^{h_n\pm},$$

where

$$a=\max\left\{0,\frac{\alpha_{\lambda_*}\sqrt{h_2}}{\int_\Omega\phi^2dx}\right\},\qquad b=\frac{\alpha_{\lambda_*}\sqrt{h_1}}{\int_\Omega\phi^2dx}$$

are the two boundary points of H. Then one can always find a $\bar{\bar{\lambda}}^*>\lambda_*$, such that for any $\lambda\in[\lambda_*,\bar{\bar{\lambda}}^*]$, there exists a connected interval $\tilde{H}_\lambda\in\mathbb{R}_+\backslash\{0\}$ of h, such that

$$G(z^{\lambda h\pm},\beta^{\lambda h\pm},h,\theta_1^{\lambda h\pm},\theta_2^{\lambda h\pm},\lambda)=0 \quad \text{for } \lambda\in[\lambda_*,\bar{\bar{\lambda}}^*] \text{ and } h\in\tilde{H}_\lambda. \tag{56}$$

see Figure 3 for the illustration. Based on the implicit function theorem, the interval $\tilde{H}_\lambda$ for each $\lambda\in[\lambda_*,\bar{\bar{\lambda}}^*]$ can be continuously extend to the maximal connected interval H_λ, the endpoint of which is 0 or some constant $h_e^\lambda>0$ that satisfies the Fréchet derivative of $G(z,\beta,h_e^\lambda,\theta_1,\theta_2,\lambda)$ with respect to $(z,\beta,\theta_1,\theta_2)$ at $(z^{\lambda h_e^\lambda\pm},\beta^{\lambda h_e^\lambda\pm},\theta_1^{\lambda h_e^\lambda\pm},\theta_2^{\lambda h_e^\lambda\pm})$ is irreversible.

Now, we suppose that $\{(z^\lambda,\beta^\lambda,h^\lambda,\theta_1^\lambda,\theta_2^\lambda,\lambda)\}$ are the solutions of $G(z,\beta,h,\theta_1,\theta_2,\lambda)$ for $\lambda\in[\lambda_*,\bar{\bar{\lambda}}_*]$. It is clear that $\{(\beta^\lambda,h^\lambda,\theta_1^\lambda,\theta_2^\lambda,\lambda)\}$ are bounded in $\mathbb{R}^5$. Moreover,

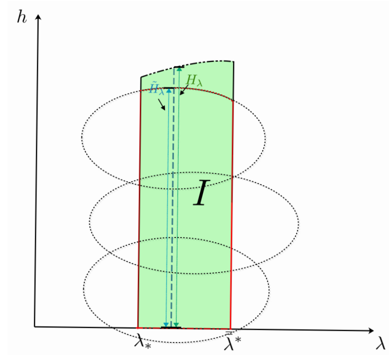


FIGURE 3. The area painted green is the connected region I of (λ, h).

similar as the proof in Theorem 3.3, we can obtain that $\{z^\lambda\}$ and $\{|\nabla z^\lambda|\}$ are bounded in $(Y_1)_{\mathbb{C}}$. Note that $\{(z^\lambda, \beta^\lambda, h^\lambda, \theta_1^\lambda, \theta_2^\lambda, \lambda)\}$ satisfy the following equation:

$$(\Delta + \lambda_*)z^\lambda = -\frac{g(z^\lambda, \beta^\lambda, h^\lambda, \theta_1^\lambda, \theta_2^\lambda, \lambda)}{1 + (\lambda - \lambda_*)\alpha_\lambda d m_1(\xi_\lambda, \lambda)e^{-\mathrm{i}\theta_1^\lambda}}, \tag{57}$$

where $g : (X_1)_{\mathbb{C}} \times \mathbb{R}_+^2 \times [0, 2\pi) \times [0, 2\pi) \times \mathbb{R}_+ \to (Y_1)_{\mathbb{C}}$ is defined by

$$\begin{aligned} g(z, \beta, h, \theta_1, \theta_2, \lambda) =& \alpha_\lambda d\,\nabla \cdot (m_1(\xi_\lambda, \lambda)\nabla\beta\phi)e^{-\mathrm{i}\theta_1} - \lambda_*(\lambda - \lambda_*)\alpha_\lambda d m_1(\xi_\lambda, \lambda)e^{-\mathrm{i}\theta_1} z \\ &+ \alpha_\lambda d(\lambda - \lambda_*)\,\nabla m_1(\xi_\lambda, \lambda) \cdot \nabla z e^{-\mathrm{i}\theta_1} \\ &+ \alpha_\lambda d\,\nabla \cdot ((\beta\phi + (\lambda - \lambda_*)z)\nabla m_1(\xi_\lambda, \lambda)) \\ &+ [\beta\phi + (\lambda - \lambda_*)z]\left\{1 + \lambda m_2(\xi_\lambda, \alpha_\lambda\,\lambda) + \lambda\alpha_\lambda\vartheta_\lambda^1 m_1(\xi_\lambda, \lambda) - \mathrm{i}h\right\} \\ &+ \lambda\alpha_\lambda\vartheta_\lambda^2 m_1(\xi_\lambda, \lambda)\int_\Omega K(\cdot, y)[\beta\phi(y) + (\lambda - \lambda_*)z(y)]dy e^{-\mathrm{i}\theta_2}. \end{aligned}$$

Then, according to the continuity of $\lambda \longmapsto (\xi_\lambda, \alpha_\lambda)$ in $X_1 \times \mathbb{R}_+$ and the boundedness of $\{(z^\lambda, \beta^\lambda, h^\lambda, \theta_1^\lambda, \theta_2^\lambda, \lambda)\}$ in $(H^1)_{\mathbb{C}} \times \mathbb{R}_+^2 \times [0, 2\pi) \times [0, 2\pi) \times \mathbb{R}_+$, it can be proved that there exists a subsequence $\{(z^{\lambda_n}, \beta^{\lambda_n}, h^{\lambda_n}, \theta_1^{\lambda_n}, \theta_2^{\lambda_n}, \lambda_n)\}_{n=1}^\infty$ of $\{(z^\lambda, \beta^\lambda, h^\lambda, \theta_1^\lambda, \theta_2^\lambda, \lambda)\}$, such that $\lim_{n\to\infty} \beta^{\lambda_n} = \beta^* = 1$ and

$$\begin{aligned} &\frac{g(z^{\lambda_n}, \beta^{\lambda_n}, h^{\lambda_n}, \theta_1^{\lambda_n}, \theta_2^{\lambda_n}, \lambda_n)}{1 + (\lambda_n - \lambda_*)\alpha_{\lambda_n} d m_1(\xi_{\lambda_n}, \lambda_n)e^{-\mathrm{i}\theta_1^{\lambda_n}}} \overset{n\to\infty}{\longrightarrow} \\ &\alpha_{\lambda_*} d\,\nabla \cdot (\phi\nabla\phi)e^{-\mathrm{i}\theta_1^*} + \alpha_{\lambda_*} d\,\nabla \cdot (\phi\nabla\phi) + \\ &\phi\left[1 + \lambda_*\alpha_{\lambda_*}(r_1\phi + r_2\int_\Omega K(\cdot, y)\phi(y)dy) - \mathrm{i}h^*\right] \\ &+ \lambda_*\alpha_{\lambda_*}\phi\left[r_1\phi + r_2\int_\Omega K(\cdot, y)\phi(y)dy e^{-\mathrm{i}\theta_2^*}\right] \end{aligned}$$

in $(Y_1)_{\mathbb{C}}$, for some $(h^*, \theta_1^*, \theta_2^*) \in \mathbb{R}_+ \times [0, 2\pi) \times [0, 2\pi)$. Due to $(\Delta + \lambda_*)^{-1}$ subject to homogeneous Dirichlet boundary condition is a linear bounded operator from $(Y_1)_{\mathbb{C}}$ to $(X_1)_{\mathbb{C}}$, we have

$$z^{\lambda_n} \longrightarrow z^* \quad \text{in } (X_1)_{\mathbb{C}},$$

where z^* satisfies

$$\begin{aligned}0 =&(\Delta+\lambda_*)z^* + \alpha_{\lambda_*} d\,\nabla\cdot(\phi\nabla\phi)e^{-\mathrm{i}\theta_1^*} + \alpha_{\lambda_*} d\,\nabla\cdot(\phi\nabla\phi)\\ &+\phi\left[1+\lambda_*\alpha_{\lambda_*}(r_1\phi + r_2\int_\Omega K(\cdot,y)\phi(y)dy) - \mathrm{i}h^*\right]\\ &+\lambda_*\alpha_{\lambda_*}\phi\left[r_1\phi + r_2\int_\Omega K(\cdot,y)\phi(y)dye^{-\mathrm{i}\theta_2^*}\right].\end{aligned} \tag{58}$$

Then, based on the Lemma 3.5, it can be seen there exists a $h \in H$ such that

$$(z^*, \beta^*, h^*, \theta_1^*, \theta_2^*) = (z^{h\pm}, 1, h, \theta_1^{h\pm}, \theta_2^{h\pm}).$$

Thus we arrive at the conclusion that $(z^{\lambda h\pm}, \beta^{\lambda h\pm}, h, \theta_1^{\lambda h\pm}, \theta_2^{\lambda h\pm}),\ h \in H_\lambda$ are the all solutions of (41) for $\lambda \in [\lambda_*, \bar{\bar{\lambda}}_*]$. □

Corollary 1. *Assume that **(A1)**, **(A2)** and **(A3)** hold, then for each fixed* $\lambda \in (\lambda_*, \bar{\bar{\lambda}}_*]$, *there exists an interval* H_λ, *such that* $\mu = \mathrm{i}\omega,\ \omega > 0$ *is an eigenvalue of equation* (22) *if and only if*

$$\omega = \omega^{\lambda h} := h(\lambda-\lambda_*),\ \ h \in H_\lambda,$$

and

$$\psi = c\psi^{\lambda h\pm},\ \ \tau = \tau_n^{\lambda h\pm} := \frac{\theta_1^{\lambda h\pm}+2n\pi}{h(\lambda-\lambda_*)},\ \ \sigma = \sigma_m^{\lambda h\pm} := \frac{\theta_2^{\lambda h\pm}+2m\pi}{h(\lambda-\lambda_*)},\ \ n, m \in \mathbb{N}_0,$$

where c *is a nonzero constants,* $\psi^{\lambda h\pm} = \beta^{\lambda h\pm}\phi + (\lambda-\lambda_*)z^{\lambda h\pm}$, *and* $z^{\lambda h\pm}, \beta^{\lambda h\pm}, h, \theta_1^{\lambda h\pm}, \theta_2^{\lambda h\pm}$ *are given by Theorem 3.7.*

It is observed in the proof of Lemma 3.5 that $G(z, 1, h, \theta_1, \theta_2, \lambda_*) = 0$ has no solutions in $(X_1)_{\mathbb{C}} \times \mathbb{R}_+\backslash\{0\} \times [0, 2\pi) \times [0, 2\pi)$, as long as

$$\textbf{(A4):}\ |\rho_0| + \lambda_*(|\rho_2| - |\rho_1|) \le 0,$$

holds. This, together with Theorem 3.3, suggest u_λ is stable for sufficiently small$\lambda > \lambda_*$ and all $\tau, \sigma \ge 0$, if **(A1)** and **(A4)** are satisfied. We shall prove this is the case in next theorem.

Theorem 3.8. *Assume that **(A1)**, **(A2)** and **(A4)** hold. Then, there exists* $\hat{\lambda}^* \in (\lambda_*, \bar{\lambda}^*]$, *such that all the eigenvalues of* (22) *have negative real parts for any* $(\lambda, \tau, \sigma) \in (\lambda_*, \hat{\lambda}^*] \times \mathbb{R}_+^2$.

Proof. Suppose there exists a sequence $\{(\mu_{\lambda_n}, \lambda_n, \tau_{\lambda_n}, \sigma_{\lambda_n}, \psi_{\lambda_n})\}_{n=1}^\infty$ solves equation (23) with

$$\lim_{n\to\infty}\lambda_n = \lambda_*, \quad \mathrm{Re}\mu_{\lambda_n} \ge 0, \quad \|\psi_{\lambda_n}\|_{Y_{\mathbb{C}}}^2 = \|\phi\|_{Y_{\mathbb{C}}}^2.$$

Similar as the proof in Theorem 3.3, let

$$\begin{cases}\mu_{\lambda_n} = h_{\lambda_n}(\lambda_n-\lambda_*),\\ \psi_{\lambda_n} = \beta_{\lambda_n}\phi + (\lambda_n-\lambda_*)z_{\lambda_n}, \quad \beta \ge 0,\ z \in (X_1)_{\mathbb{C}},\\ \|\psi_{\lambda_n}\|_{Y_{\mathbb{C}}}^2 = \beta_{\lambda_n}^2\|\phi\|_{Y_{\mathbb{C}}}^2 + (\lambda_n-\lambda_*)^2\|z_{\lambda_n}\|_{Y_{\mathbb{C}}}^2 = \|\phi\|_{Y_{\mathbb{C}}}^2.\end{cases} \tag{59}$$

Substituting (59) into (23), we can see that the sequence $\{h_{\lambda_n}, \lambda_n, \tau_{\lambda_n}, \sigma_{\lambda_n}, \beta_{\lambda_n}, z_{\lambda_n}\}_{n=1}^{\infty}$ satisfies the following equations:

$$\begin{cases} P_1(h,\lambda,\tau,\sigma,\beta,z) :=(\Delta+\lambda_*)z + \alpha_\lambda d\,\nabla\cdot(m_1(\xi_\lambda,\lambda)\nabla(\beta\phi+(\lambda-\lambda_*)z))e^{-h(\lambda-\lambda_*)\tau} \\ \qquad + \alpha_\lambda d\,\nabla\cdot((\beta\phi+(\lambda-\lambda_*)z)\nabla m_1(\xi_\lambda,\lambda)) \\ \qquad + [\beta\phi+(\lambda-\lambda_*)z]\left\{1+\lambda m_2(\xi_\lambda,\alpha_\lambda\lambda)+\lambda\alpha_\lambda\vartheta_\lambda^1 m_1(\xi_\lambda,\lambda)-h\right\} \\ \qquad + \lambda\alpha_\lambda\vartheta_\lambda^2 m_1(\xi_\lambda,\lambda)\int_\Omega K(\cdot,y)[\beta\phi(y)+(\lambda-\lambda_*)z(y)]dy e^{-h(\lambda-\lambda_*)\sigma} \\ \qquad =0, \\ P_2(h,\lambda,\tau,\sigma,\beta,z) :=(\beta^2-1)\|\phi\|_{Y_{\mathbb{C}}}^2 + (\lambda-\lambda_*)^2\|z\|_{Y_{\mathbb{C}}}^2 = 0. \end{cases} \tag{60}$$

Note that $\text{Re}\mu_{\lambda_n} \geq 0$. It then follows that $\{h_{\lambda_n}, e^{-h_{\lambda_n}(\lambda_n-\lambda_*)\tau_{\lambda_n}}, e^{-h_{\lambda_n}(\lambda_n-\lambda_*)\sigma_{\lambda_n}}, \beta_{\lambda_n}\}$ are bounded in $\mathbb{C}_+^3\times\mathbb{R}$. Along the same lines as in the proof of Theorem 3.7, we can obtain $\{z_{\lambda_n}\}$ and $\{|\nabla z_{\lambda_n}|\}$ are bounded in $(Y_1)_{\mathbb{C}}$. Thus, there is a subsequence, still denoted by $\{h_{\lambda_n}, \lambda_n, e^{-h_{\lambda_n}(\lambda_n-\lambda_*)\tau_{\lambda_n}}, e^{-h_{\lambda_n}(\lambda_n-\lambda_*)\sigma_{\lambda_n}}, \beta_{\lambda_n}, z_{\lambda_n}\}$, that converges to

$$(h^*, \lambda^*, t_1^* e^{-i\theta_1^*}, t_2^* e^{-i\theta_2^*}, \beta^*, z_*) \in \mathbb{C}\times\mathbb{R}_+\times\mathbb{C}^2\times\mathbb{R}_+\times(X_1)_{\mathbb{C}}$$

as $n\to\infty$, where $\beta^*=1$, $t_i^*\in[0,1]$ and $\theta_i^*\in[0,2\pi)$. Taking the limit of the equation

$$P_1(h_{\lambda_n}, \lambda_n, \tau_{\lambda_n}, \sigma_{\lambda_n}, \beta_{\lambda_n}, z_{\lambda_n}) = 0$$

in $\mathbb{C}\times\mathbb{R}_+^4\times(Y_1)_{\mathbb{C}}$, we have

$$\begin{aligned} -(\Delta+\lambda_*)z_* &= \alpha_{\lambda_*} d\nabla\cdot(\phi\nabla\phi) + \lambda_*\alpha_{\lambda_*}\phi(r_1\phi + \int_\Omega K(\cdot,y)\phi((y)dy) + \phi \\ &\quad + \alpha_{\lambda_*} d\nabla\cdot(\phi\nabla\phi)t_1^* e^{-i\theta_1^*} + \lambda_*\alpha_{\lambda_*}\phi(r_1\phi + \int_\Omega K(\cdot,y)\phi((y)dy)t_2^* e^{-i\theta_2^*}) - h_*\phi. \end{aligned}$$

Thus,

$$\alpha_{\lambda_*}\lambda_*\rho_1 - h_*\int_\Omega \phi^2 dx + \alpha_{\lambda_*}\rho_0 t_1^* e^{-i\theta_1^*} + \alpha_{\lambda_*}\lambda_*\rho_2 t_2^* e^{-i\theta_2^*} = 0. \tag{61}$$

We claim that $h_*\neq 0$. Otherwise, considering $\lambda_*\rho_1$, $\rho_0 t_1^* e^{-i\theta_1^*}$ and $\lambda_*\rho_2 t_2^* e^{-i\theta_2^*}$ as three vectors on complex plane, it then follows from **(A4)** that (61) holds if and only if

$$t_1^* = t_2^* = 1,\ |\rho_0| + \lambda_*(|\rho_2| - |\rho_1|) = 0.$$

Consequently, we derive $\theta_1^* = \theta_2^* = 0$, or equivalently,

$$\lambda_*(\rho_1+\rho_2) + \rho_0 = 0,$$

which is contrary to **(A1)**. Now we regard the three terms $\alpha_{\lambda_*}\lambda_*\rho_1 - h_*\int_\Omega\phi^2 dx$, $\alpha_{\lambda_*}\rho_0 t_1^* e^{-i\theta_1^*}$ and $\alpha_{\lambda_*}\lambda_*\rho_2 t_2^* e^{-i\theta_2^*}$ in (61) as vectors on complex plane. Similarly, we can get

$$0 < |h_*|^2(\int_\Omega \phi^2)^2 \leq \alpha_{\lambda_*}^2(|\rho_0| + \lambda_*|\rho_1|)^2 - \alpha_{\lambda_*}^2\lambda_*|\rho_0|^2 \leq 0.$$

which is also a contradiction. □

In the remainder of this paper, we refer the set

$$\mathcal{T}_\lambda =: \{(\tau_n^{\lambda h\pm}, \sigma_m^{\lambda h\pm})|\ n,m\in\mathbb{N}_0, h\in H_\lambda\},\quad \lambda\in(\lambda_*, \bar{\bar{\lambda}}_*]$$

as the crossing curves. We summarize Theorem 3.3, Corollary 1 and Theorem 3.8, arriving at the following conclusions on the stability of u_λ.

Theorem 3.9. *Assume that* ***(A1)*** *and* ***(A2)*** *hold.*

(1) *If* ***(A3)*** *is satisfied, then for* $\lambda \in (\lambda_*, \bar{\bar{\lambda}}^*]$, *the positive steady state* $u_\lambda(x)$ *is locally asymptotically stable for* (τ, σ) *close to* $(0,0)$*; The stability of* u_λ *is reversed with the occurrence of purely imaginary roots, only if* (τ, σ) *crosses* $\mathcal{T}_\lambda$.

(2) *If* ***(A4)*** *is satisfied, then for any* $(\lambda, \tau, \sigma) \in (\lambda_*, \hat{\lambda}^*] \times \mathbb{R}^2_+$, *the positive steady state* $u_\lambda(x)$ *is locally asymptotically stable.*

4. **Crossing direction.** In this section, we shall show the direction of the pure imaginary eigenvalue of (22) passes through the imaginary axis in complex plane, as (τ, σ) deviates from $\mathcal{T}_\lambda$. From (24), it can be seen that the eigenvalues μ of (22) with $(\lambda, \tau, \sigma, \psi) \in \mathbb{R}^3_+ \times X_\mathbb{C}$ must satisfy

$$\mathcal{D}(\mu, \lambda, \tau, \sigma) := \mathcal{P}_0(\mu, \lambda) + \mathcal{P}_1(\mu, \lambda) e^{-\mu\tau} + \mathcal{P}_2(\mu, \lambda) e^{-\mu\sigma} = 0, \tag{62}$$

where

$$\begin{aligned}
\mathcal{P}_0(\mu, \lambda) =& \int_\Omega \phi(x) \Big[\nabla \cdot (d\psi(x) \nabla u_\lambda(x)) - (\lambda_* + \mu)\psi(x) \Big] dx \\
&+ \int_\Omega \lambda \phi(x) \left[F\left(u_\lambda(x), \int_\Omega K(x,y) u_\lambda(y) dy \right) \psi(x) + \vartheta^1_\lambda(x) u_\lambda(x) \psi(x) \right] dx, \\
\mathcal{P}_1(\mu, \lambda) =& \int_\Omega \phi(x) \nabla \cdot (d u_\lambda(x) \nabla \psi(x)) dx, \\
\mathcal{P}_2(\mu, \lambda) =& \lambda \int_\Omega \int_\Omega \phi(x) \vartheta^2_\lambda(x) u_\lambda(x) K(x,y) \psi(y) dx dy.
\end{aligned}$$

Therefore, if $\frac{\partial \mathcal{D}}{\partial \mu}(\mathrm{i}\omega^{\lambda h}, \lambda, \tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) \neq 0$ for $(\tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) \in \mathcal{T}_\lambda$, then the equation (22) has a pure imaginary eigenvalue

$$\mu_\lambda(\tau, \sigma) = \alpha_\lambda(\tau, \sigma) + \mathrm{i}\omega_\lambda(\tau, \sigma)$$

in the neighbourhood of $(\tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm})$ with the corresponding eigenfunction $\psi_\lambda(\tau, \sigma)$, such that

$$\alpha_\lambda(\tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) = 0, \quad \omega_\lambda(\tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) = \omega^{\lambda h}, \quad \psi_\lambda(\tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) = \psi^{\lambda h \pm}.$$

As in [12], we call the direction of the crossing curve $\mathcal{T}_\lambda$ that corresponds to increasing ω the *positive direction,* and the region on the left-hand (right-hand) side when we move along the positive direction of the curve *the region on the left (right).* Based on the implicit function theorem, it follows that if

$$\mathrm{Im} \left\{ \overline{\frac{\partial \mathcal{D}}{\partial \tau}} (\mathrm{i}\omega^{\lambda h}, \lambda, \tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) \frac{\partial \mathcal{D}}{\partial \sigma} (\mathrm{i}\omega^{\lambda h}, \lambda, \tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) \right\} \neq 0,$$

then (τ, σ) could regarded as the functions of $(\alpha_\lambda, \omega_\lambda)$ in the neighborhood of $(0, \omega^{\lambda h})$. Note that $\left(\frac{\partial \sigma}{\partial \omega_\lambda}(0, \omega^{\lambda h}), -\frac{\partial \tau}{\partial \omega_\lambda}(0, \omega^{\lambda h}) \right)$ is the normal vector of $\mathcal{T}_\lambda$ at $(\tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm})$ pointing to the region on the right. Then, a directly calculation gives that

$$\begin{aligned}
&\mathrm{Sign} \left\{ \left(\frac{\partial \alpha_\lambda}{\partial \tau}(\tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}), \frac{\partial \alpha_\lambda}{\partial \sigma}(\tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) \right) \cdot \left(\frac{\partial \sigma}{\partial \omega_\lambda}(0, \omega^{\lambda h}), -\frac{\partial \tau}{\partial \omega_\lambda}(0, \omega^{\lambda h}) \right) \right\} \\
&\quad = \mathrm{Sign} \left\{ \mathrm{Im} \left\{ \overline{\frac{\partial \mathcal{D}}{\partial \tau}} (\mathrm{i}\omega^{\lambda h}, \lambda, \tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) \frac{\partial \mathcal{D}}{\partial \sigma} (\mathrm{i}\omega^{\lambda h}, \lambda, \tau_n^{\lambda h \pm}, \sigma_m^{\lambda h \pm}) \right\} \right\} \\
&\quad = \mathrm{Sign} \left\{ \mathrm{Im} \left\{ \overline{\mathrm{P}_1(\mathrm{i}\omega^{\lambda h}, \lambda)} \mathrm{P}_2(\mathrm{i}\omega^{\lambda h}, \lambda) e^{\mathrm{i}(\theta_1^{\lambda h \pm} - \theta_2^{\lambda h \pm})} \right\} \right\}
\end{aligned} \tag{63}$$

Similar as Lemma 3.6, we can carry out that $\sin(\theta_1^{\lambda h\pm} - \theta_2^{\lambda h\pm}) \neq 0$ for $(\lambda, h) \in \mathrm{int}I$. Since

$$\lim_{\lambda\to\lambda_*} \frac{\overline{\mathrm{P}_1(\mathrm{i}\omega^{\lambda h}, \lambda)}\mathrm{P}_2(\mathrm{i}\omega^{\lambda h}, \lambda)}{\lambda(\lambda-\lambda_*)^2} = \rho_0\rho_2, \tag{64}$$

we may consider $\bar{\bar{\lambda}}_* > \lambda_*$ sufficiently small and then arrive at the following results.

Theorem 4.1. *Assume that **(A1)**, **(A2)** and **(A3)** hold, if $\frac{\partial \mathcal{D}}{\partial \mu}(\mathrm{i}\omega^{\lambda h}, \lambda, \tau_n^{\lambda h\pm}, \sigma_m^{\lambda h\pm}) \neq 0$ for some $(\tau_n^{\lambda h\pm}, \sigma_m^{\lambda h\pm}) \in \mathcal{T}_\lambda$ and $h \in \mathrm{int}H_\lambda$, then the equation (22) has a eigenvalue $\mu_\lambda(\tau,\sigma) = \alpha_\lambda(\tau,\sigma) + \mathrm{i}\omega_\lambda(\tau,\sigma)$ in a neighbourhood of $(\tau_n^{\lambda h\pm}, \sigma_m^{\lambda h\pm})$ that satisfies $\alpha_\lambda(\tau_n^{\lambda h\pm}, \sigma_m^{\lambda h\pm}) = 0$, $\omega_\lambda(\tau_n^{\lambda h\pm}, \sigma_m^{\lambda h\pm}) = \omega^{\lambda h}$. Moreover, $\mu_\lambda(\tau,\sigma)$ cross the imaginary axis from left to right, as (τ,σ) passes through the crossing curve to the region on the right (left) whenever*

$$\rho_0\rho_2 \sin\left(\theta_1^{h\pm} - \theta_2^{h\pm}\right) > 0 \ (< 0),$$

where $h \in \mathrm{int}H$ and $\theta_i^{h\pm}$, $i = 1, 2$ are given by Lemma 3.5.

5. **An example.** In this section, we apply the results obtained in the previous sections to the memory-based reaction-diffusion model with the growth function (5) and the kernel function $K(x,y) = 1/\pi$ used in [10]:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + d(u(x,t)\nabla u_x(x,t-\tau))_x + \lambda u(x,t)\Big(1 + au(x,t)\Big) \\ \quad - \lambda u(x,t)\Big(bu^2(x,t) + (1+a-b)\displaystyle\int_0^\pi \frac{u(y,t-\sigma)}{\pi}dy\Big), \ x \in (0,\pi), \ t > 0, \\ u(0,t) = u(\pi,t) = 0, \ t > 0. \end{cases} \tag{65}$$

It is clear that $\lambda_* = 1$ and $\phi(x) = \sin x$. After some simple calculations, we have

$$r_1 = \frac{\partial F(0,0)}{\partial x_1} = a, \qquad r_2 = \frac{\partial F(0,0)}{\partial x_2} = -(1+a-b)$$
$$\rho_0 = -\frac{2d}{3}, \qquad \rho_1 = \frac{4a}{3}, \qquad \rho_2 = -(1+a-b).$$

As a consequence of Theorem 2.1, we have the following results on the existence of positive steady state.

Theorem 5.1. *If $a + 3b - 3 - 2d \neq 0$, then there exist $\bar{\lambda}^* > \lambda_* > \underline{\lambda}^*$ such that the model has one steady state solution $u_\lambda(x)$ for any $\lambda \in [\underline{\lambda}^*, \bar{\lambda}^*]$. Moreover, if $a + 3b - 3 - 2d < 0$, then **(A1)** is satisfied, and therefore $u_\lambda(x)$ is positive for $\lambda \in (\lambda_*, \bar{\lambda}^*]$; otherwise, $u_\lambda(x)$ is positive for $\lambda \in [\underline{\lambda}^*, \lambda_*)$.*

Under the condition $a + 3b - 3 - 2d < 0$, we also have

$$|\rho_0| + \lambda_*(|\rho_2| - |\rho_1|) = \frac{1}{3}\left[2|d| - (a + 3b - 3)\right] > 2|d| - 2d \geq 0,$$

which means **(A3)** is true. Then, according to Theorem 3.9, we obtain the following results.

Theorem 5.2. *Assume that $a+3b-3-2d < 0$ and **(A2)** hold, then for $\lambda \in (\lambda_*, \bar{\bar{\lambda}}^*]$, the positive steady state $u_\lambda(x)$ is locally asymptotically stable for (τ,σ) close to $(0,0)$; In addition, (22) has purely imaginary eigenvalues if and only if (τ,σ) lies on the crossing curves $\mathcal{T}_\lambda$.*

For the purpose of numerically verifying Theorem 5.1 and Theorem 5.2, we choose $a = 0.3$, $b = 0.5$, $d = 2$. It is easy to check $a+3b-3-2d = -5.2 < 0$, then the positive steady state $u_\lambda(x)$ should exist when $\lambda > \lambda_*$ and its stability depends on the value of (τ, σ). Thus, it is required to identify the crossing curve $\mathcal{T}_\lambda$. Unfortunately, this is difficult to achieve, because all functions $(\omega^{\lambda h}, \tau_n^{\lambda h\pm}, \sigma_n^{\lambda h\pm})$ do not have explicit expressions. However, it is known that $\mathcal{T}_\lambda$ is the perturbation of the crossing curve of (3.22), which is actually the special case $\lambda = \lambda_*$, as shown in Figure 4. Thus, $\mathcal{T}_\lambda$, for λ sufficiently close to λ_*, can be interpreted by the crossing curves of (43), since $(\omega^{\lambda h}, \tau_n^{\lambda h\pm}, \sigma_n^{\lambda h\pm})$ are continuous with respect to λ. It can be also verified $\rho_0\rho_2 \sin\left(\theta_1^{h\pm} - \theta_2^{h\pm}\right) > 0$, which means that, as (τ, σ) passes through the crossing curve to the region on the right, the eigenvalues will cross the imaginary axis from left to right.

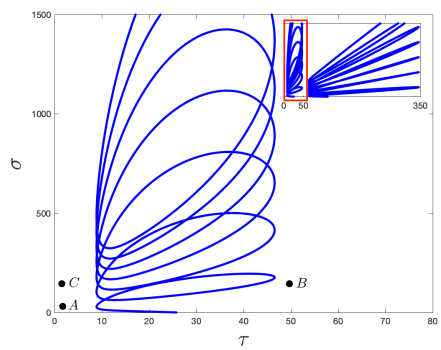
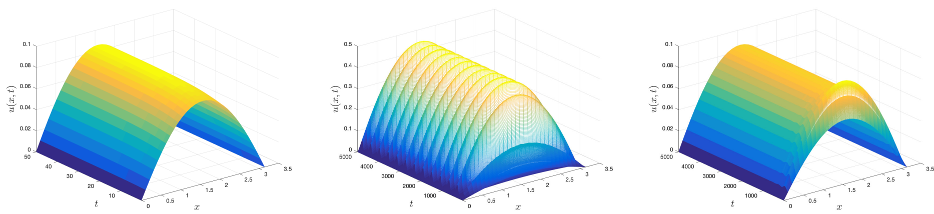


FIGURE 4. Approximation of the crossing curve $\mathcal{T}_\lambda$. Here $\lambda = 1.1$, $a = 0.3$, $b = 0.5$, $d = 2$. Two crossing curves of (43) are plotted in the top right corner, and one of it (in the red box) is enlarged in the figure.

We selected three values of (τ, σ), corresponding to the three points $A(1, 1)$, $B(50, 100)$ and $C(1, 100)$ in Figure 4, to simulate the solutions of (43). According to the shape of the crossing curve, the positive steady state is stable when $(\tau, \sigma) = (1, 1)$ and $(1, 100)$, which are located at the left side of crossing curve. As (τ, σ) passes through the crossing curve, the steady state loses its stability due to the occurrence of Hopf bifurcation, resulting in the generation of a stable spatially inhomogeneous periodic solution. See the solutions of (43) for different choice of (τ, σ) in Figure 5.

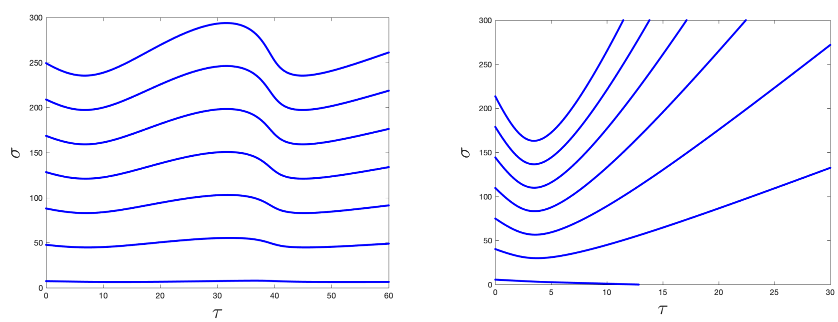
It is remarked that the crossing curves $\mathcal{T}_\lambda$ can also tell if Hopf bifurcation will happen when $\tau = 0$ or $\sigma = 0$. Specifically, Hopf bifurcation exists when $\tau = 0$ ($\sigma = 0$ resp.), as long as the crossing curve intersects the σ-axis (τ-axis resp.), and these intersections are exactly the critical values of Hopf bifurcation. For instance, in Figure 4, when $\sigma = 0$, there exists $\tau^* \in (20, 30)$ such that (65) will undergo Hopf bifurcation at $\tau = \tau^*$. If $\tau = 0$, there is no intersection of crossing curve and σ-axis, indicating no Hopf bifurcation in this case. There are also many other shapes of the crossing curves, if the values of parameters are altered. This is shown in Figure 6. In particular, Figure 6-(a) exhibits the crossing curves for small $d = 0.1$, and there are many intersections of these curves with σ-axis. This suggest the occurrence of Hopf bifurcation for $\tau = 0$ and small d, which is expected from the results in [5].

We also choose another set of parameters $a = -0.3$, $b = 0.5$, $d = 0.2$ to check the theoretical results, even though they have no biological significance for $a < 0$.



(a) $A(1,1)$ (b) $B(50,100)$ (c) $C(1,100)$

Figure 5. Let $\lambda = 1.1$, $a = 0.3$, $b = 0.5$, $d = 2$. (a,c) When (τ, σ) are located at the left side of crossing curve, the positive steady state is stable. (b) A stable spatially inhomogeneous periodic solution is generated, when (τ, σ) passes through the crossing curve.



(a) $a = 0.3, b = 0.5, d = 0.1$ (b) $a = 2, b = 0.5, d = 2$

Figure 6. The crossing curves of (43) for other choices of parameters. Here, $\lambda = 1.1$.

In this case, the conditions **(A1)** and **(A4)** are satisfied. Again, let $\lambda = 1.1$, that is close to λ_*. The simulation in Figure 7 shows that the positive steady state is still stable for large $\tau = \sigma = 150$. In this case, the time delays cannot change the stability of the steady state.

6. **Discussion.** In this paper, we propose and study a spatial memory model with nonlocal maturation delay and hostile boundary condition. We prove the existence of a positive steady state u_λ, and investigate its local stability. Compared with the previous works [3, 7] on the bifurcation analysis for the inhomogeneous steady states of general diffusion equations, the associated eigenvalue problem here is rather complicated, since it includes two time delays and the linear operator $\Pi(\mu, \lambda, \tau, \sigma)\psi$ is not self-conjugate. Therefore, the method proposed in [3] cannot be simply extended to analyze the purely imaginary eigenvalues. To overcome these difficulties, we employ the geometric method in [12], prior estimation techniques and the idea in [3], to obtain the sufficient conditions for both local stability of u_λ and occurrence of purely imaginary eigenvalues at u_λ. The critical values of (τ, σ) such that (23) has purely imaginary roots, form crossing curves $\mathcal{T}_\lambda$ in (τ, σ)-plane. The shape and location of $\mathcal{T}_\lambda$ for λ sufficiently close to λ_* can be inferred from the crossing curves of

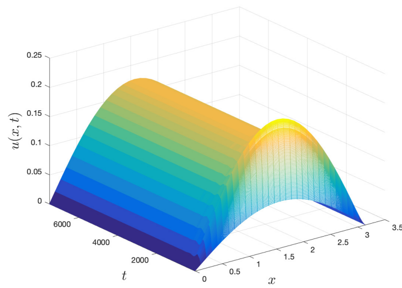


FIGURE 7. Let $\lambda = 1.1$, $a = -0.3$, $b = 0.5$, $d = 0.2$, the positive steady state u_λ is still stable for sufficient large $\tau = \sigma = 100$.

a transcendental equation with two delays, that is, (43). Spatially inhomogeneous periodic solutions are generated through Hopf bifurcation, as (τ, σ) crosses these curves. The method developed here is also applicable for studying other reaction-diffusion equations with two delays and Dirichlet boundary condition in the absence or presence of memorized diffusion.

From Theorem 3.9, we know that the steady state u_λ of (3) with $d = 0$ is asymptotically stable, when $|\rho_2| < |\rho_1|$. If the memorized diffusion is considered, then (**A4**) can be violated by choosing properly large d. The first statement in Theorem 3.9 implies that u_λ may become unstable, leading to periodic oscillations, for some (τ, σ). This means that complicated spatial-temporal patterns can be induced by memorized diffusion.

By comparing the dynamics of the memory-based reaction-diffusion equation under Neumann or Dirichlet boundary conditions, it is found that the inhomogeneous periodic solutions respectively bifurcated from constant steady states (through Hopf or Turing-Hopf bifurcation) and non-constant steady states (through Hopf bifurcation) have different spatiotemporal structures. Specifically, when the spatial region is one-dimensional, the inhomogeneous periodic solutions under Neumann boundary condition can be approximated as $u_0 + a\cos(nx/\pi)\cos(\omega t)$, where u_0 is the constant steady state, and forms a checkerboard-liked pattern, see [19]. However, under the Dirichlet boundary condition, the inhomogeneous periodic solutions should be approximately as $a \sin x + b \sin x \cos \omega t$, which generates a striped pattern, see Figure 5 (b).

It has to be admitted that the direction and stability of bifurcated periodic solutions are not studied here, as in [3, 5]. Many theories need to be developed for such equations, like center manifold theorem and formal adjoint theory, since these theories cannot be directly obtained or extended from those for standard partial functional differential equations in [24].

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