Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 285 (2021) 17–39

Journal of Differential Equations

www.elsevier.com/locate/jde

Bistable traveling waves in impulsive reaction-advection-diffusion models ☆

Zhenkun Wang [a,b], Hao Wang [c,*]

[a] *College of Automation, Harbin Engineering University, Harbin, Heilongjiang, 150001, PR China*
[b] *College of Mathematical Sciences, Harbin Engineering University, Harbin, Heilongjiang, 150001, PR China*
[c] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada*

Received 22 August 2019; revised 26 August 2020; accepted 2 March 2021

Abstract

In this paper, we study an impulsive reaction-advection-diffusion model given by a reaction-advection-diffusion equation with the Allee effect in high-dimensional space composed with a discrete-time map. We establish the existence, uniqueness up to translation, and global stability of bistable traveling waves for impulsive reaction-advection-diffusion models. The methods involve the monotone semiflow approach, the upper and lower solutions, and spreading speeds of monostable systems. Numerical simulations verify the correctness of our conclusions.

Keywords: Impulsive reaction-advection-diffusion models; Traveling waves; Bistable; Global stability

1. Introduction

In order to study the propagation dynamics of reaction-diffusion systems, Aronson and Weinberger [1,2] pioneered the concept of asymptotic speed of spread, proved that the asymptotic speed of spread can be determined linearly under appropriate conditions, and verified that this speed coincides with the minimal wave speed for traveling wave solutions. For the cooperative

☆ The corresponding author Hao Wang is partially supported by an NSERC Discovery Grant RGPIN-2020-03911 and an NSERC Accelerator Grant RGPAS-2020-00090.
* Corresponding author.
E-mail addresses: wangzhenkun@hrbeu.edu.cn (Z. Wang), hao8@ualberta.ca (H. Wang).

https://doi.org/10.1016/j.jde.2021.03.005
0022-0396/

system, Weinberger proposed to use the idea of abstract dynamic system to deal with it uniformly in [3]. Later, after the development and improvement of Lui, Liang, Yi, Zhao, Li, Lewis et al. [4–9], the current abstract method has formed a theoretical framework to verify the existence of monostable traveling wave solutions. In addition, Fang and Zhao studied the bistable traveling wave solutions of monotone evolution systems by using the monotone semiflow method. In an abstract framework, they establish the existence of bistable traveling wave solutions for discrete-time or continuous-time monotone semiflows in homogeneous and periodic environments. Their theoretical results are also applicable to monotone semiflows with weak compactness.

Clustering is conducive to the growth and survival of the population, but excessive sparseness and overcrowding are not conducive to the survival of the organism. For successful mating, the density needs to be greater than some threshold value, called the Allee effect. It is increasingly recognized that considering the Allee effect has theoretical and practical significance in the study of population dynamics, see [11–16]. A reaction diffusion model with Allee effect can be used to describe the negative growth under low population density. A typical example is the cubic nonlinear $f(u) = u(1-u)(u-\theta)$ with $0 < \theta < 1$, which leads to the so-called "bistable" equation (see [17]), that is, both the extinction steady state 0 and carrying-capacity steady state 1 are locally asymptotically stable. The study of the existence and stability of traveling wave solutions has attracted more and more attention. Many excellent results have been found in [18–33]. Unlike these results, we will study a hybrid PDE/discrete-time system (i.e. impulsive PDE system), which consists of a reaction-advection-diffusion equation and a discrete time map.

Impulsive reaction-diffusion models were only recently studied in the ecological context [34–38]. In their simplest linear form, these equations can be equivalent to linear IDEs (integral difference equations), but their nonlinear extensions typically are not. The study of impulsive reaction-diffusion equations in ecology is still in its infancy, but it is obvious that many problems can be solved by reference to the research results of IDEs. We try to find more commonalities between the two types of equations and extend the mature technology in IDEs to the impulsive reaction-diffusion equation.

Lewis and Li in [34] established an impulsive PDE model to study the persistence and spread of species with birth pulse. This impulsive PDE model combines the characteristics of the reaction-diffusion equation and the difference equation. Considering that the spatial domain is entire space $\mathbb{R}$, they obtained an explicit formula for the spreading speed under appropriate conditions. In addition, their results showed that the spreading speed, can be linearly determined, equal to the minimal wave speed of a class of traveling waves. When considering the bounded domain with Dirichlet boundaries, they obtained the minimum domain size, which is also a threshold condition to distinguish whether a species is persistent or extinct. Using the method in [34], Lin and Wang [35] generalized the conclusions about traveling wave solutions to the model with general response terms and provided the results about the spreading speed and the existence of the traveling wave solution. Vasilyeva et al. in [36] used an impulsive reaction-diffusion equation with non-local impulse to study the population dynamics for a class of species of stream insects with different life stages. Combining the definitions of weak persistence and local persistence, they considered the influence of advection diffusion on population persistence. The space environment is assumed to be one-dimensional in [34–36]. Considering the high dimensional space, Fazly et al. [37,38] established a general impulsive reaction-diffusion-advection system describing population dynamics with two life stages (i.e., impulse birth stage and diffusion stage). This type of models can be used to study the setting of marine (or terrestrial) protected areas, the prevention and control of pests and diseases, and the migration of the birth pulse populations driven by climate change. When considering bounded domain with unfavorable boundary conditions, they studied the effects of the size and geometry of the domain on the persistence of species, obtained that there is an extreme volume size such that if the size, the species is expelled always eliminated regardless of the geometry of the domain. In the entire space $\mathbb{R}^n$, they also proved the existence of a monostable traveling wave solution. Recently, Wu and Zhao [39] extended the relevant conclusions to an impulsive integrodifferential equation model.

This paper is a follow-up to [37,38]. We focus on the traveling wave solution of an impulsive reaction-advection-diffusion system with a bistable structure in high-dimensional habitats. We re-introduce all notations in [37]. The population of a species at the beginning of year m is denoted by $N_m(x)$. We consider the dynamics of a population at the beginning of a reproductive stage within a period. We use g to describe the population density at the end of a reproductive stage as a function of the population density at the beginning of the stage. At the end of this year, the density $u^{(m)}(x, 1)$ provides the population density for the start of year $m + 1$, denoted by $N_{m+1}(x)$. We study the following impulsive reaction-advection-diffusion system for any $m \in \mathbb{N}$,

$$\begin{cases} u_t^{(m)} = div(A\nabla u^{(m)} - au^{(m)}) + f(u^{(m)}), \quad (x, t) \in \Omega \times (0, 1], \\ u^{(m)}(x, 0) = g(N_m(x)), \quad x \in \Omega, \\ N_{m+1}(x) = u^{(m)}(x, 1), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, A is a constant symmetric positive definite matrix and a is a constant vector. For convenience, we rewrite the above mathematical model as

$$\begin{cases} u_t = div(A\nabla u - au) + f(u), \quad (x, t) \in \Omega \times (0, 1], \\ u(x, 0) = g(N_m(x)), \quad x \in \Omega, \\ N_{m+1}(x) = u(x, 1), \quad x \in \Omega. \end{cases} \tag{1.2}$$

The above equation (1.2) defines a recurrence relation for $N_m(x)$ as

$$N_{m+1}(x) = Q[N_m(x)] \text{ for } x \in \Omega \subset \mathbb{R}^n, \tag{1.3}$$

where $m \geq 0$ and Q is an operator that depends on A, a, f, g.

Unlike the previous papers [37,38], we choose $f(\cdot)$ as Nagumo (or Huxley) type nonlinearity instead of Fisher-KPP type, such as $f(u) = u(1 - u)(u - \theta)$ where $0 < \theta < 1$. Clearly, the PDE in (1.2) has three constant solutions: $u = 0, \theta, 1$. According to the linearized stability analysis, 0 and 1 are stable, and θ is unstable.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and assumptions that will be used later. In Section 3, we establish the existence of bistable traveling waves for impulsive PDE systems. In Section 4, we study the global stability and uniqueness up to translation for bistable traveling waves. We additionally provide numerical simulations for illustration in Section 5. Finally, we summarize the main findings and mention some future work.

2. Notations and assumptions

We first introduce some notations. Let $C := BC(\mathbb{R}^n, \mathbb{R})$ be the set of all bounded and continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ equipped with the compact open topology, that is, a sequence ϕ_m converges to ϕ in C if and only if $\phi_m(x)$ converges to $\phi(x)$ in $\mathbb{R}$ uniformly for x in any compact subset of $\mathbb{R}^n$. Note $C_+ = \{\phi \in C : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}^n\}$. Clearly, C_+ is a closed cone in the normed vector space C. For any $\phi_1, \phi_2 \in C$, we denote $\phi_1 \geq \phi_2$ if $\phi_1 - \phi_2 \in C_+$, and $\phi_1 > \phi_2$ if $\phi_1 \geq \phi_2$ and $\phi_1 \neq \phi_2$. For given numbers $0 < \alpha < \beta$, we define $C_\beta = \{\phi \in C : 0 \leq \phi \leq \beta\}$ and $C_{[\alpha,\beta]} = \{\phi \in C : \alpha \leq \phi \leq \beta\}$.

The function

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}\sqrt{det(A)}} e^{-\frac{1}{4}\langle A^{-1}(\frac{x-ta}{\sqrt{t}}), \frac{x-ta}{\sqrt{t}}\rangle} & (x \in \mathbb{R}^n, t > 0) \\ 0 & (x \in \mathbb{R}^n, t < 0) \end{cases}$$

is called the fundamental solution of the Kolmogorov's equation $u_t - div(A\nabla u - au) = 0$. Using the solution maps $\{\Phi(x,t)\}_{t\geq 0}$ of the Kolmogorov's equation, we write (1.2) in integral form:

$$u(x,t;g) = \Phi * g + \int_0^t \Phi(t-s) * f(u(s))ds, \tag{2.1}$$

where $\Phi(x) * \phi(x) = \int_{\mathbb{R}^n} \Phi(x-y)\phi(y)dy$. Define $Q_t[g] := u(x,t;g)$ for all $g \in C_+$, where $C_+ \subset C$ is the set of all bounded nonnegative functions, and this integral operator is continuous and compact with respect to its integrand. Let $Q_1[g(N_m(x))]$ be $u(x,1)$ with $u(x,t)$ the solution of the initial value problem

$$u_t = div(A\nabla u - au) + f(u), \quad u(x,0) = g(N_m(x)).$$

Note $k(x,t)$ is the Green function of $u_t = div(A\nabla u - au)$, $k_1 * g(u) = \int_{\mathbb{R}^n} k(x-y,1)g(u(y))dy$, and $k_1 ** f(u) = \int_0^1 \int_{\mathbb{R}^n} k(x-y, 1-s) f(u(s,y))dyds$, where $k_1(x) = k(x,1)$. Then we are able to derive an explicit relation between the initial value $g(N_m(x))$ and time-one solution map $u(x,1;g(N_m(x)))$. It reads

$$Q[\cdot] = Q_1[g(\cdot)] = (k_1 * g + k_1 ** f)[\cdot].$$

Define $Q[N_m(x)] := Q_1[g(N_m(x))]$ and $N_{m+1} = Q[N_m(x)] = Q^{m+1}[N_0]$, where Q^m is the $m-th$ iteration of Q. For any given $y \in \mathbb{R}^n$, define the translation operator T_y by $T_y[\phi] = \phi(x-y)$ for all $x \in \mathbb{R}^n$ and $\phi \in C$.

Let $\beta \in \mathbb{R}_+$ and Q be a map from C_β to C_β with $Q(0) = 0$ and $Q(\beta) = \beta$. Let E be the set of all fixed points of Q restricted on $[0,\beta]$. In order to study traveling wave solutions, we always assume that the domain $\Omega = \mathbb{R}^n$ in this paper. Plane wave is only one type of traveling wave solution. According to tradition, the terminology traveling wave solution or traveling waves are still used in this paper to refer to plane waves. We follow the definition of traveling wave solution in [38].

Definition 1. We say that $N_m(x)$ is a traveling wave solution of (1.3) in $e = (e_1, \cdots, e_n) \in \mathbb{R}^n$ direction if there exist a function W and a constant c such that $N_m(x) = W(x \cdot e - cm)$ and $Q[W(x \cdot e - cm)] = W(x \cdot e - (m+1)c)$ for all integers m.

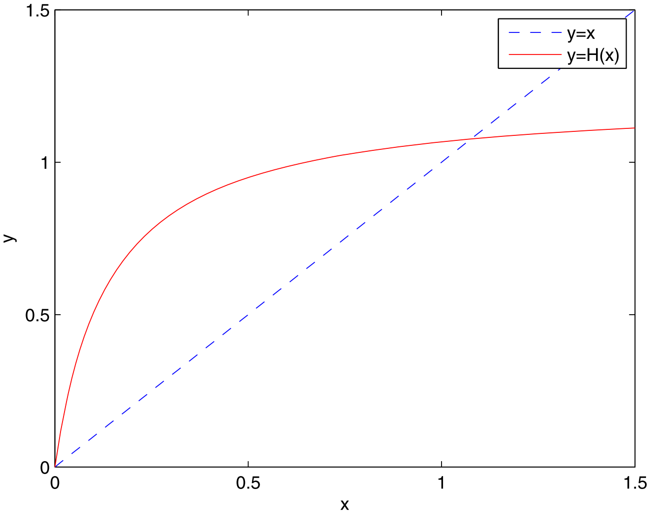


Fig. 1. The map H for the system (2.3), where $f(x)$ satisfies $(f_1 - f_3)$ and $g(x)$ satisfies (g_1).

We first consider the following impulsive system without spatial dispersion:

$$\begin{cases} \dfrac{du}{dt} = f(u), \quad t \in (0, 1], \\ u(0) = g(N_m(x)), \\ N_{m+1}(x) = u(1). \end{cases} \tag{2.2}$$

Let F be the time-1 mapping of the ordinary differential equation in (2.2). Similar to [36], system (2.2) can be transformed into a discrete-time system

$$N_{m+1} = F(g(N_m)) := H(N_m), \tag{2.3}$$

where $g(N)$ is a differentiable function $g(0) = 0$, $g(N) > 0$ for $N \geq 0$.

The nature of the operator H depends on the properties of the functions f and g. With regard to the properties of f and g, we propose the following assumptions:

(f1) $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $f(1) = 0$ and $f'(0) > 0$, $f'(1) < 0$.
(f2) $f(u) \leq f'(0)u, \forall u \in [0, 1]$, where $f'(0) = \lim_{u \to 0+} f'(u)$.
(f3) $\forall \epsilon > 0, \exists \delta > 0$, such that $f(u) \geq (f'(0) - \epsilon)u$ for all $u \in [0, \delta]$.
(g1) $g(u)$ is a positive function for $u \geq 0$, $g(0) = 0 < g'(0)$, and $g'(u) > 0$, $g''(u) \leq 0$ for $u > 0$.

Assume that f satisfies (f1-f3), then F satisfies $F(0) = 0$, $F(N) > N$ for $N > 0$, and F is strictly monotone increasing. In addition, the derivative function F' is monotone decreasing. By the properties of F and g, we can get $H(0) = 0$, $H'(0) = F'(0)g'(0) > 0$, $H''(N) = F''[g(N)][g'(N)]^2 + F'[g(N)]g''(N) < 0$.

Based on the above assumptions, the typical $H(N)$ curve is shown in Fig. 1.

Model (2.2) always has the trivial equilibrium 0, and a positive constant equilibrium of (2.2) is a root of the equilibrium equation $H(N) = N$. The local stability of the equilibrium point x^* is determined by the linear system of the above system at that point, and is determined only by whether $|H'(x^*)|$ is less than one. According to $H''(N) < 0$, and $H(0) = 0$ if $N = 0$, the necessary condition for the existence of a positive equilibrium point is $H'(0) > 1$. Otherwise, 0 is the only fixed point and is stable. In addition, it can be seen that the positive fixed point is unique if it exists. Binding properties of monotone concave operator, we have the following observation about the system (2.2).

Lemma 2.1.

1. *If* $H'(0) > 1$ *then there exists a unique* $\beta > 0$ *with* $H(\beta) = \beta$.
2. *If* $H'(0) > 1$ *and* $N_0 > 0$, *then* $N_{m+1} > N_m$ *and* $\lim_{m\to\infty} N_m = \beta$.

There is growing evidence that the Allee effect is widespread in many populations. We add an Allee effect to the impulsive system, such that growth is negative when the population density is too low. Then we propose the following assumptions:

(F1) $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $f(\theta) = 0$, $f(1) = 0$ and $f'(0) < 0$, $f'(\theta) > 0$, $f'(1) < 0$.
(F2) $f(u) \le f'(\theta)(u - \theta)$ for all $u \in [\theta, 1]$ and $f(u) \ge f'(\theta)(u - \theta)$ for all $u \in [0, \theta]$.
(F3) $\forall\epsilon > 0, \exists\delta > 0$, such that $f(u) \ge (f'(\theta) - \epsilon)(u - \theta)$ for all $u \in [\theta, \theta + \delta]$ and $f(u) \le (f'(\theta) - \epsilon)(u - \theta)$ for all $u \in [\theta - \delta, \theta]$.

In subsequent sections, we assume that f satisfies (F1-F3) and g satisfies (g1) and take a new map H. In order to study the existence of bistable traveling wave solutions, we make the following biologically reasonable assumptions about the properties of $H(N)$:

(H1) $H(0) = 0$, $H(\alpha) = \alpha$, $H(\beta) = \beta$, and $0 < H'(0) < 1$, $H'(\alpha) > 1, 0 < H'(\beta) < 1$.
(H2) $H'(N) > 0$ for $N \ge 0$, $H''(N) > 0$ for $N \in (0, \alpha)$, and $H''(N) < 0$ for $N \in (\alpha, \beta)$.

Based on the above assumptions, the curve $H(N)$ can be represented by the following Fig. 2.

The purpose of this paper is to study the traveling wave solutions of (1.3) when Q has a bistable structure. Restricting Q on $\mathbb{R}$ will produce a mapping $\overline{Q} : \mathbb{R} \to \mathbb{R}$. We assume that $\overline{Q}$ has the following properties:

(H) (Bistability) $\overline{Q}$ admits exactly three fixed points $\beta > \alpha > 0$, and $\overline{Q}[N]$ is nondecreasing in $N \in [0, \beta]$. Moreover, fixed points 0 and β are stable and α is unstable in the sense that

$$\overline{Q}'[0] < 1, \overline{Q}'[\beta] < 1, \text{ and } \overline{Q}'[\alpha] > 1. \tag{2.4}$$

Let $\phi(x) \in C_\beta$ be a continuous initial function such that

$$a_- = \limsup_{x\cdot e\to-\infty} \phi(x) < \alpha < \liminf_{x\cdot e\to\infty} \phi(x) = a_+. \tag{2.5}$$

We assume that a_- is not too far from 0 and a_+ is not too far from β. It means that the two limiting values fall into the attracting basins of 0 and β respectively. In this paper, we only consider that the system has three constant equilibrium points. Therefore, this assumption is always valid.

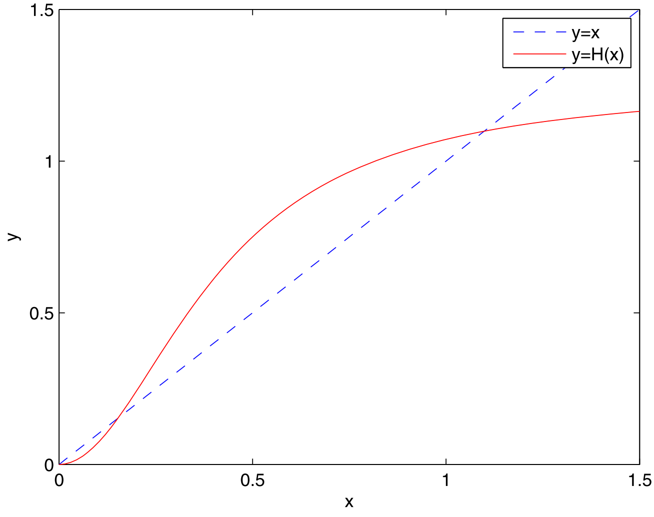


Fig. 2. The map H for the system (2.3), where $f(x)$ satisfies $(F_1 - F_3)$ and $g(x)$ satisfies (g_1).

One of the most important concepts is the asymptotic wave speed, $c^*(e)$, defined for each unit vector $e \in \mathbb{R}^N$. Weinberger [3] defined the asymptotic speed of spread $c^*(e)$ for each unit vector $e \in \mathbb{R}^n$. He proved that if $N_0(x)$ has bounded support and is positive on a set with positive measure, then $c^*(e)$ is the spreading speed of $N_m(x)$ defined by the recursion (1.3). There is a classic and well-understood description. $c^*(e)$ is the spreading speed of $N_m(x)$ means that if one runs at a speed greater than $c^*(e)$ in the direction e, then for large m, $N_m(x)$ is near 0 and if one runs at a speed less than $c^*(e)$ in the direction e, then $N_m(x)$ is near β. In addition, the monotonicity of the function $\varphi \in C(\mathbb{R}^n, \mathbb{R})$ usually expresses monotonicity in the direction e, for example, for any $x, y \in \mathbb{R}^n$, $e \cdot (y - x)(\varphi(y) - \varphi(x)) > 0$ is established, and we call the function φ a monotone increasing function (in the direction e).

3. Existence of bistable traveling wave solutions

In this section, we prove the existence of the bistable traveling wave solutions for the system (1.3). Suppose Q is an operator from C_β to C_β. In order to use the theoretical framework developed in [10], we need to make the following assumptions about Q:

(A1) (Translation invariance) $T_y[Q[\phi]] = Q[T_y[\phi]], \forall \phi \in C_\beta, y \in \mathbb{R}^n$, where T_y is defined by $T_y[\phi] = \phi(x - y)$.
(A2) (Continuity) $Q : C_\beta \to C_\beta$ is continuous with respect to the compact open topology.
(A3) (Monotonicity) Q is order preserving in the sense that $Q[\phi] \geq Q[\psi]$ whenever $\phi \geq \psi$ in C_β.
(A4) (Compactness) $Q : C_\beta \to C_\beta$ is compact with respect to the compact open topology.
(A5) (Bistability) Two fixed points 0 and β are strongly stable from above and below, respectively, for the map $\overline{Q} : [0, \beta] \to [0, \beta]$, that is, there exists a number $\delta > 0$ such that

$$\overline{Q}[\eta] < \eta, \overline{Q}[\beta - \eta] > \beta - \eta, \forall \eta \in (0, \delta],$$

and the set $E \setminus \{0, \beta\}$ is totally unordered.

(A6) (Counter-propagation) For each $\alpha \in E \setminus \{0, \beta\}$, $c_-^*(\alpha, \beta) + c_+^*(0, \alpha) > 0$, where $c_-^*(\alpha, \beta)$ and $c_+^*(0, \alpha)$ represent the upstream and downstream spreading speeds of monostable subsystem $\{Q^m\}_{m\geq 0}$ restricted on $C_{[\alpha,\beta]}$ and C_α, respectively.

The following lemma for monotone semiflows reveals the existence of bistable traveling wave solutions.

Lemma 3.1. *(Theorem 3.1 in [10]) Assume that Q satisfies (A1-A6). Then there exists $c \in \mathbb{R}$ such that the discrete semiflow $\{Q^m\}_{m\geq 0}$ admits a nondecreasing traveling wave with speed c and connecting 0 to β.*

From the assumption (A5), we can get two monostable subsystems: $\{Q^m\}_{m\geq 0}$ restricted to C_α and $C_{[\alpha,\beta]}$, respectively. Refer to the definition of asymptotic speed of spread in monostable systems (see [38]), we can define asymptotic speed of spread for two monostable subsystems, respectively. Let $\phi_\alpha^+ \in C_\beta$ be a nondecreasing initial function that satisfies the following property:

$$(P1)\ \phi_\alpha^+(x) = \alpha, \forall x \in S_1 = \{x \mid x \cdot e \geq 1\} \text{ and}$$
$$\phi_\alpha^+(x) = \inf\{\alpha, \delta\}, \forall x \in S_2 = \{x \mid x \cdot e \leq 0\}.$$

We define

$$c_+^*(0, \alpha) := \sup\left\{c \in \mathbb{R} : \lim_{m\to\infty, x\cdot e\leq cm} Q^m\left[\phi_\alpha^+\right](x) = 0\right\}, \tag{3.1}$$

which is the downstream asymptotic speed of ϕ_α^+. Similarly, choose a nondecreasing initial function $\phi_\alpha^- \in C_\beta$ with the following property

$$(P2)\ \phi_\alpha^-(x) = \alpha, \forall x \in S_3 = \{x \mid x \cdot e \leq -1\} \text{ and}$$
$$\phi_\alpha^-(x) = \sup\{\alpha, \beta - \delta\}, \forall x \in S_4 = \{x \mid x \cdot e \geq 0\}.$$

We define

$$c_-^*(\alpha, \beta) := \sup\left\{c \in \mathbb{R} : \lim_{m\to\infty, x\cdot e\geq -cm} Q^m\left[\phi_\alpha^-\right](x) = \beta\right\}, \tag{3.2}$$

which is the upstream asymptotic speed of ϕ_α^-.

The following lemma illustrates the relationship between the wave speeds of monostable traveling wave and the asymptotic speeds defined in (3.1) and (3.2).

Lemma 3.2. *Suppose that Q satisfies (A3). Then the following proposition is true:*

1. *If $W(x \cdot e - cm)$ is a monotone traveling wave connecting 0 to α of the discrete semiflow $\{Q^m\}_{m\geq 0}$, then $c \geq c_+^*(0, \alpha)$.*
2. *If $W(x \cdot e - cm)$ is a monotone traveling wave connecting α to β of the discrete semiflow $\{Q^m\}_{m\geq 0}$, then $c \leq -c_-^*(\alpha, \beta)$.*

Proof. (1) Since $W(-\infty) = 0$ and $W(+\infty) = \alpha$, one can find a translation of W, still denoted by W, such that $\phi_\alpha^+ \geq W$, where ϕ_α^+ satisfies $(P1)$, and it is easy to verify that such ϕ_α^+ can always be found. According to the contradiction method, we first assume that $c < c_+^*(0, \alpha)$. Choose $\frac{q}{p} \in \left(c, c_+^*(0, \alpha)\right)$ with $p, q \in \mathbb{Z}$. From (3.1), we can obtain that

$$
\begin{aligned}
0 &= \lim_{m\to\infty} Q^{pm}\left[\phi_\alpha^+\right]\left(\frac{q}{p}\cdot pme\right) \geq \lim_{m\to\infty} Q^{pm}[W](qme) \\
&= \lim_{m\to\infty} W(qme\cdot e - pcm) = \lim_{m\to\infty} W(m(q-pc)) = W(+\infty) = \alpha.
\end{aligned}
$$

Another proposition can be proved similarly, we omit it. □

Lemma 3.3. *Assume that (H1-H2) hold. If $g'(\alpha)e^{f'(\theta)} > e^{\frac{\langle a,e\rangle^2}{4\langle Ae,e\rangle}}$, then the following two statements are valid:*

1. *If $W(x\cdot e - cm)$ is a nondecreasing traveling wave of (1.3) with $W(-\infty) = 0$ and $W(+\infty) = \alpha$, then $c > 0$.*
2. *If $W(x\cdot e - cm)$ is a nondecreasing traveling wave of (1.3) with $W(-\infty) = \alpha$ and $W(+\infty) = \beta$, then $c < 0$.*

Proof. (1) System (1.2) has monostable dynamics on C_α. To better understand the dynamics of this subsystem, we make the transform $\tilde{N}_m = \alpha - N_m$ and define $\tilde{g}(\tilde{N}_m) = g(\alpha) - g(\alpha - \tilde{N}_m)$ with $\tilde{g}(0) = g(0) = 0$, $\tilde{g}(\alpha) = g(\alpha)$. Then the dynamics is equivalent to that of the following system on C_α:

$$
\begin{cases}
u_t = div(A\nabla u - au) + f(u), & (x,t)\in\Omega\times(0,1],\\
u(x,0) = \tilde{g}(\tilde{N}_m(x)), & x\in\Omega,\\
\tilde{N}_{m+1}(x) = u(x,1), & x\in\Omega.
\end{cases}
\tag{3.3}
$$

Because $f(x)$ satisfies hypothesis (F3), we know that for any small $\epsilon > 0$, there exists $\delta \in (0,\beta)$ and a linear operator $L_\epsilon = f'(\theta) - \epsilon$ such that $L_\epsilon\phi \to f'(\theta)\phi$ for all $\phi \in C_\beta$ as $\epsilon \to 0$ and

$$
f(\theta+\phi) \geq L_\epsilon(\phi) \quad \text{and} \quad f(\theta-\phi) \leq -L_\epsilon(\phi), \quad \forall\phi\in C_\delta.
$$

From assumption (H1), we see that 0 is a linearly unstable steady state of the system

$$
\begin{cases}
\dfrac{du}{dt} = f(u), & t\in(0,1],\\
u(0) = \tilde{g}(\tilde{N}_m(x)),\\
\tilde{N}_{m+1}(x) = u(1).
\end{cases}
\tag{3.4}
$$

Consider the linearized system of (3.3) at the equilibrium 0:

$$\begin{cases} u_t = div(A\nabla u - au) + f'(\theta)u, & (x,t) \in \Omega \times (0,1], \\ u(x,0) = \tilde{g}'(0)\tilde{N}_m(x), & x \in \Omega, \\ \tilde{N}_{m+1}(x) = u(x,1), & x \in \Omega. \end{cases} \tag{3.5}$$

Let $L[\cdot]$ be the solution map of the linear system (3.5). According to Theorem 3.10 in [7], Theorem 3.1 and Theorem 4.1 in [38], for each unit direction vector e in the habitat, (3.5) has the upstream spreading speed $c^*(-e)$ and downstream spreading speed $c^*(e)$. For every $c \geq c^*(e)$ there exists a traveling wave solution $W(x \cdot e - cm)$ with $W(-\infty) = \alpha$ and $W(+\infty) = 0$. In addition, for every $c \geq c^*(-e)$ there exists a traveling wave solution $\bar{W}(x \cdot e + cm)$ with $\bar{W}(-\infty) = 0$ and $\bar{W}(+\infty) = \alpha$.

As a result, system (3.3) has a downstream spreading speed $\tilde{c}^*_{\alpha,0}(e) = 2\sqrt{\langle Ae, e\rangle} \times \sqrt{f'(\theta) + \ln(g'(\alpha))} + e \cdot a > 0$ and $c \geq \tilde{c}^*_{\alpha,0}(e) > 0$.

(2) Similarly, system (1.2) has monostable dynamics on $C_{[\alpha,\beta]}$. We make the transform $\bar{N}_m = N_m - \alpha$ and $\bar{g}(\bar{N}_m) = g(\bar{N}_m + \alpha)$. Then the dynamics are equivalent to that of the following system on $C_{\beta-\alpha}$:

$$\begin{cases} u_t = div(A\nabla u - au) + f(u), & (x,t) \in \Omega \times (0,1], \\ u(x,0) = \bar{g}(\bar{N}_m(x)), & x \in \Omega, \\ \bar{N}_{m+1}(x) = u(x,1), & x \in \Omega. \end{cases} \tag{3.6}$$

System (3.6) has two steady states $S_1 = 0$ and $S_2 = \beta - \alpha$. Consider the linearized system of (3.6) at the equilibrium S_1:

$$\begin{cases} u_t = div(A\nabla u - au) + f'(\theta)u, & (x,t) \in \Omega \times (0,1], \\ u(x,0) = \bar{g}'(0)\bar{N}_m(x), & x \in \Omega, \\ \bar{N}_{m+1}(x) = u(x,1), & x \in \Omega. \end{cases} \tag{3.7}$$

By Theorem 3.1 and Theorem 4.1 in [38], for system (1.2) restricted on $[\alpha, \beta]$, that is, there is a number $c^*_{\alpha,\beta}(-e)$ such that $-c \geq c^*_{\alpha,\beta}(-e)$, that is,

$$c \leq -c^*_{\alpha,\beta}(-e) = -2\sqrt{\langle Ae, e\rangle}\sqrt{f'(\theta) + \ln(g'(\alpha))} + e \cdot a < 0. \quad \square$$

Remark 1. According to the conclusions and proofs of Lemmas 3.2 and 3.3, and combined with the existence conditions of the traveling wave solution of the monostable subsystem, we obtain the estimation of the spreading speed of the bistable traveling wave solution, that is, $-c^*_{\alpha,\beta}(-e) < c^*_{0,\beta}(e) < c^*_{0,\alpha}(e)$.

Theorem 1. *Assume that (H1-H2) hold, then there exists $c \in \mathbb{R}$ such that system (1.1) admits a traveling wave connecting 0 to β with speed $c_{0,\beta}$.*

Proof. Let Q_t denote the solution operator of the reaction-advection-diffusion equation in (1.2). Easy to verify that the discrete semiflow $\{Q^m\}_{m\geq 0}$ on C_β satisfies (A1)-(A4) with $Q[N_m] = Q_1[g(N_m)]$. In fact, $N_{m+1}(x) = Q[N_m(x)] \in [0, \beta]$ as $N_m(x) \in [0, \beta]$. According to the theory of partial differential equation [[41], Section 8.2, Lemma 2.1], one can get that Q_t is continuous and compact in the topology of uniform convergence on every bounded domain. Since g is continuous, we have that Q is continuous and compact in the topology of uniform convergence on every bounded domain.

It is assumed that u_0, v_0 are nonnegative continuous functions on $\mathbb{R}^n$ and $u_0(x) \geq v_0(x)$ for all $x \in \mathbb{R}^n$. We use $Q_t[u_0]$ and $Q_t[v_0]$ to represent the solutions of $u_t = \Delta u + ce \cdot \nabla u + f(u)$ with initial conditions u_0 and v_0. By the monotonicity assumption on g and the comparison principle for reaction-diffusion equations, one can get $g(u_0(x)) \geq g(v_0(x))$ for all $x \in \mathbb{R}^n$, and $Q_t[u_0] \geq Q_t[v_0]$ for all $x \in \mathbb{R}^n, 0 < t \leq 1$. This result implies that the operator $Q = Q_1[g(\cdot)]$ has the same monotonicity property, $Q[u] \geq Q[v]$, that is, Q is order preserving.

According to the translation invariance of the convolution and (2.1), one can get

$$T_y[\Phi(x) * \phi(x)] = T_y[\int_{\mathbb{R}^n} \Phi(x-z)\phi(z)dz] = \int_{\mathbb{R}^n} \Phi(x-y-z)\phi(z)dz,$$

$$\Phi(x) * T_y[\phi(x)] = \int_{\mathbb{R}^n} \Phi(x-z)\phi(z-y)dz = \int_{\mathbb{R}^n} \Phi(x-y-w)\phi(w)dw.$$

Hence, we have $T_y[Q[\phi]] = Q[T_y[\phi]]$.

According to the stability of equilibrium points, it is obvious that $\overline{Q}'(0)$ and $\overline{Q}'(\beta)$ are nonnegative numbers between 0 and 1. Then there exist $0 < \theta_1 < 1$ and $0 < \delta \ll 1$ such that $\overline{Q}'(u) < \theta_1, u \in [-\delta, \delta]$ and $\overline{Q}'(u) < \theta_1, u \in [\beta - \delta, \beta + \delta]$. By the continuous differentiability of $\overline{Q}$, it then follows that there exists $\delta > 0$ such that

$$\overline{Q}(\eta) = \overline{Q}(0) + \int_0^1 \overline{Q}'(s\eta)\eta ds < \eta \int_0^1 \overline{Q}'(s\eta)ds < \theta_1\eta < \eta$$

and

$$\overline{Q}(\beta - \eta) = \overline{Q}(\beta) + \int_0^1 \overline{Q}'(\beta - s\eta)(-\eta)ds > \beta - \theta_1\eta > \beta - \eta$$

for all $\eta \in (0, \delta]$, and hence, 0 is strongly stable from above and β is strongly stable from below for the map $\overline{Q}$. For this system, the set $E \setminus \{0, \beta\} = \alpha$, and it is totally unordered.

Next, we show that (A6) holds. For this system, we have $\alpha = E \setminus \{0, \beta\}$. $\{Q^m\}_{m\geq 0} : [\alpha, \beta] \to [\alpha, \beta]$ has monostable dynamics, where α is unstable and β is stable. By the theory developed in [38], $\{Q^m\}_{m\geq 0}$ admits upstream and downstream spreading speeds $c_-^*(\alpha, \beta)$ and $c_+^*(\alpha, \beta)$. Note that $\{Q^m\}_{m\geq 0} : [0, \alpha] \to [0, \alpha]$ also has monostable dynamics, where 0 is stable and α is unstable. Similarly, this monostable subsystem also admits a upstream and downstream spreading speeds $c_-^*(0, \alpha)$ and $c_+^*(0, \alpha)$. Due to the definitions of spreading speeds for monostable systems (see [7,9]), one can find that $c_{0,\alpha}^*(e)$, the downstream spreading speed of the monostable subsystem restricted to C_α, is a lower bound of $c_+^*(0, \alpha)$. A similar observation holds for $c_-^*(\alpha, \beta)$. According to Lemma 3.3, we obtain

$$c_-^*(\alpha, \beta) + c_+^*(0, \alpha) \geq c_{\alpha,\beta}^*(-e) + c_{0,\alpha}^*(e) > 0.$$

Consequently, Theorem 1 completes the proof. □

Remark 2. When (A6) is verified, the condition

$$g'(\alpha)e^{f'(\theta)} > e^{\frac{\langle a,e\rangle^2}{4\langle Ae,e\rangle}}$$

in Lemma 3.3 is not necessary, that is, one of the $c_-^*(\alpha,\beta)$ and $c_+^*(0,\alpha)$ may have a negative value.

Remark 3. Similarly, if we consider the nonincreasing traveling waves, we can define the numbers $c_+^*(\alpha,\beta)$ and $c_-^*(0,\alpha)$. Then (A6) should be replaced with $c_+^*(\alpha,\beta)+c_-^*(0,\alpha)>0$.

Remark 4.

1. If the continuous initial function $\phi \in C_\beta$ satisfies

$$\limsup_{x\cdot e\to-\infty}\phi(x)<\alpha<\liminf_{x\cdot e\to\infty}\phi(x),$$

then system (1.3) has a nondecreasing traveling wave $W(x\cdot e-c_{0,\beta}m)$ with $W(-\infty)=0$ and $W(+\infty)=\beta$.
2. If the continuous initial function $\phi \in C_\beta$ satisfies

$$\liminf_{x\cdot e\to-\infty}\phi(x)>\alpha>\limsup_{x\cdot e\to\infty}\phi(x),$$

then system (1.3) has a nonincreasing traveling wave $\overline{W}(x\cdot e-c_{\beta,0}m)$ with $\overline{W}(-\infty)=\beta$ and $\overline{W}(+\infty)=0$.
3. If the continuous initial function $\phi \in C_\beta$ satisfies

$$\limsup_{|x\cdot e|\to+\infty}\phi(x)<\alpha \text{ and } \phi(x)>\alpha \text{ for } |x\cdot e|<l,$$

where l are some positive number. Then system (1.3) has a nondecreasing traveling wave $W(x\cdot e-c_{0,\beta}m)$ with $W(-\infty)=0$ and $W(+\infty)=\beta$ and nonincreasing traveling wave $\overline{W}(x\cdot e-c_{\beta,0}m)$ with $\overline{W}(-\infty)=\beta$ and $\overline{W}(+\infty)=0$.

4. Global stability and uniqueness of bistable traveling wave solutions

In this section, we study the global stability and uniqueness of bistable traveling waves for system (1.3). A nondecreasing function $W(x\cdot e-cm)$ on $\mathbb{R}$ is a traveling wave solution of (1.3) with $W(-\infty)=0$ and $W(+\infty)=\beta$ in the direction of the unit vector $e\in\mathbb{R}^n$ if there exists a constant c such that $N_m(x)=W(x\cdot e-cm)$ satisfies the recursion $Q[W(x\cdot e-cm)]=W(x\cdot e-c(m+1))$ for every m. Letting $z=x\cdot e-c(m+1)$, we transform (1.3) into the following system

$$W_{m+1}(z)=T_{-c}Q[W_m](z), m\geq 0. \tag{4.1}$$

In what follows, we denote $U_m(x,\psi) = U_m(x, 1; g(N_m))$, where $U_m(x, 1; g(N_m)) = N_{m+1}(x)$ is the solution of (1.2) with initial data $\psi(x) = N_0(x) = W_0(x)$.

Lemma 4.1. *If $\psi \in C_\beta$ is nondecreasing and satisfies*

$$\limsup_{x\cdot e\to -\infty} \psi(x) < \alpha < \liminf_{x\cdot e\to\infty} \psi(x), \tag{4.2}$$

then $\forall\varepsilon > 0$, there exists $\tilde{z} = \tilde{z}(\varepsilon,\psi) > 0$ such that $W(z - \tilde{z}) - \varepsilon \le W_0(z,\psi) \le W(z + \tilde{z}) + \varepsilon$.

Proof. It is easy to see that

$$\limsup_{x\cdot e\to\infty} \psi(x) < \beta + \varepsilon = \lim_{z\to\infty} W(z) + \varepsilon,$$

$$\liminf_{x\cdot e\to -\infty} \psi(x) > 0 - \varepsilon = \lim_{z\to -\infty} W(z) - \varepsilon.$$

Then there exists $Z_0 > 0$ such that $W(z) - \varepsilon \le \psi(z) := \psi(z \cdot e + c(m + 1) \cdot e) \le W(z) + \varepsilon$ holds for all $|z| \ge Z_0$. By the monotonicity of ψ and W, there exists $\tilde{z} > 0$ such that $W(z - \tilde{z}) - \varepsilon \le W_0(z,\psi) \le W(z + \tilde{z}) + \varepsilon$. □

We re-introduced the definition of upper and lower solutions in [30], which will be used later.

Definition 2. A function sequence $W_m^+(z)$ $(W_m^-(z)) \in C(\mathbb{R},\mathbb{R})$, $m \ge 0$, is an upper (lower) solution of (4.1) if $W_m^+(z)$ $(W_m^-(z))$ satisfies

$$W_{m+1}^+ \ge Q\left[W_m^+\right](z + c)\ \ (W_{m+1}^- \le Q\left[W_m^-\right](z + c)), \quad m \ge 0.$$

Lemma 4.2. *There exist positive numbers σ, $\varepsilon_0 \in (0, 1)$, and δ_0 such that $\forall\varepsilon \in (0, \varepsilon_0)$ and $\forall\hat{z} \in \mathbb{R}$, the functions*

$$W_m^\pm(z,\varepsilon,\delta_0) = W\left(z \pm \hat{z} \pm \varepsilon\left(1 - e^{-\sigma m}\right)\right) \pm \delta_0 e^{-\sigma m}, \forall z \in \mathbb{R}, m \ge 0$$

are upper and lower solutions of system (4.1), respectively.

Proof. Without loss of generality, we assume that $\hat{z} = 0$. Note that the derivatives of Q at N is $Q'(N)$. According to the stability of equilibrium points, it is obvious that $0 < Q'(0) < 1$, $0 < Q'(\beta) < 1$, and $Q'(\alpha) > 1$. Then there exist $0 < \theta_1 < 1$ and $0 < \delta \ll 1$ such that $Q'(N) < \theta_1 < 1, N \in [-\delta,\delta]$, $Q'(N) < \theta_1 < 1, N \in [\beta - \delta,\beta + \delta]$, and $\theta_2 = \max\{Q'(N) \mid N \in [-1,\beta + 1]\} > 1$. Set $\delta_0 = \delta/2$, since $W(-\infty) = 0$, $W(+\infty) = \beta$, there exists $M_1 > 0$ such that $W(\xi) \in [-\delta_0,\delta_0]$, $\xi \le -M_1$, and $W(\xi) \in [\beta - \delta_0,\beta + \delta_0]$, $\xi \ge M_1$, that is,

$$| W(\xi) + \delta_0 - 0 |\le \delta, \xi \le -M_1, \qquad | W(\xi) + \delta_0 - \beta |\le \delta, \xi \ge M_1.$$

Denote $D_m^\pm(z) := W_{m+1}^\pm(z) - Q\left[W_m^\pm\right](z + c)$. Let $z_m = \varepsilon\left(1 - e^{-\sigma m}\right), \forall m \ge 0$. Then $\{z_m\}_{m\ge 0}$ is increasing and between 0 and 1, where the positive number $\sigma \in (0, -\ln\theta_1)$.

$$\begin{aligned}
D_m^+(z) &= W_{m+1}^+(z) - Q\left[W_m^+\right](z+c) \\
&= W(z+z_{m+1}) + \delta_0 e^{-\sigma(m+1)} - Q[W(z+c+z_m) + \delta_0 e^{-\sigma m}] \\
&= W(z+z_{m+1}) - W(z+z_m) + \delta_0 e^{-\sigma(m+1)} \\
&- Q[W(z+c+z_m) + \delta_0 e^{-\sigma m}] + Q[W(z+c+z_m)] \\
&= W(z+z_{m+1}) - W(z+z_m) + \delta_0 e^{-\sigma(m+1)} \\
&- \delta_0 e^{-\sigma m} \int_0^1 Q'[W(z+c+z_m) + s\delta_0 e^{-\sigma m}] ds.
\end{aligned}$$

For any $m \geq 0$, we discuss the following three cases.
Case (1): $z > M_1 - z_m - c$. By the monotonicity of W, we have

$$\begin{aligned}
D_m^+(z) &\geq \delta_0 e^{-\sigma(m+1)} - \delta_0 e^{-\sigma m} \int_0^1 Q'[W(z+c+z_m) + s\delta_0 e^{-\sigma m}] ds \\
&= \delta_0 e^{-\sigma m}\left(e^{-\sigma} - \theta_1\right) \geq 0.
\end{aligned}$$

Case (2): $z < -M_1 - z_m - c$. Clearly, $| W(z+z_m+c) + \delta_0 | \leq \delta$. By a similar analysis, we have

$$\begin{aligned}
D_m^+(z) &\geq \delta_0 e^{-\sigma(m+1)} - \delta_0 e^{-\sigma m} \int_0^1 Q'[W(z+c+z_m) + s\delta_0 e^{-\sigma m}] ds \\
&= \delta_0 e^{-\sigma m}\left(e^{-\sigma} - \theta_1\right) \geq 0.
\end{aligned}$$

Case (3): $z \in [-M_1 - z_m - c, M_1 - z_m - c]$, that is, $z + z_m \in [-M_1 - c, M_1 - c]$. From $0 < (z+z_{m+1}) - (z+z_m) = z_{m+1} - z_m = \varepsilon e^{-\sigma m}\left(1 - e^{-\sigma}\right) < 1$, we have $z + z_{m+1} < z + z_m + 1 < M_1 - c + 1$.

According to the uniform continuity of W and Lemma 5 in [4], we know $W \in C^1(\mathbb{R}, \mathbb{R})$, $W'(z)$ is uniformly continuous, and $W'(z) > 0$. Since W is strictly increasing in compact set $[-M_1 - |c| - 1, M_1 + |c| + 1]$, there exists $\theta_3 > 0$ such that

$$W(Z_2) - W(Z_1) \geq \theta_3(Z_2 - Z_1),\, Z_2 > Z_1, \forall Z_1, Z_2 \in [-M_1 - |c| - 1, M_1 + |c| + 1]. \quad (4.3)$$

Thus, we have

$$W(z+z_{m+1}) - W(z+z_m) \geq \theta_3(z_{m+1} - z_m). \quad (4.4)$$

It follows from (4.4) that

$$\begin{aligned}
D_m^+(z) &\geq \theta_3(z_{m+1} - z_m) + \delta_0 e^{-\sigma(m+1)} - \theta_2 \delta_0 e^{-\sigma m} \\
&= \theta_3 \varepsilon e^{-\sigma m}\left(1 - e^{-\sigma}\right) + \delta_0 e^{-\sigma(m+1)} - \theta_2 \delta_0 e^{-\sigma m} \\
&= e^{-\sigma m}\left[\varepsilon\theta_3\left(1 - e^{-\sigma}\right) - \delta_0(\theta_2 - e^{-\sigma})\right] \geq 0
\end{aligned}$$

provided that we choose δ_0 sufficiently small such that $\delta_0 \le \tau\varepsilon := \frac{\varepsilon\theta_3(1-e^{-\sigma})}{\theta_2-e^{-\sigma}}$.

Combining cases (1)-(3), we get that there exists $\sigma > 0$, sufficiently small number $\varepsilon_0 > 0$ and $0 < \delta_0 < \tau\varepsilon$ such that $D_m^+(z) \ge 0, m \ge 0, z \in \mathbb{R}$. Hence $W_m^+(z)$ is an upper solution of system (4.1). By the similar arguments, we can prove $W_m^-(z)$ is a lower solution of (4.1). □

Lemma 4.2 shows that if $W_0(z) < \alpha$ near $-\infty$ and $W_0(z) > \alpha$ near ∞, then $W_m(z)$ will be trapped between translations of the traveling waves. According to the Lemma 4.2, we can get the following conclusions.

Lemma 4.3. *The wave profile W is a Lyapunov stable equilibrium of (4.1).*

Proof. Let ε_0 and $W_m^{\pm}(z)$ be given in Lemma 4.2 with $\hat{z} = 0$. By the uniform continuity of W and the boundedness of $\theta_i, i = 1, 2, 3$, it follows that there exists $K > 0$, independent of ε, such that $\| W_m^{\pm}(z, \varepsilon, \delta_0) - W(z) \| < K\varepsilon, \forall z \in \mathbb{R}, \varepsilon \in (0, \varepsilon_0)$, and $\delta_0 \in (0, \tau\varepsilon)$. For any $\varepsilon \in (0, \varepsilon_0)$, let $0 < \delta^* \le \delta_0 \ll 1$, for any given ψ satisfying $\| \psi - W \| < \delta^*$, we have

$$W_0^-(z, \varepsilon, \delta_0) = W(z) - \delta_0 \le \psi \le W(z) + \delta_0 = W_0^+(z, \varepsilon, \delta_0).$$

By the comparison principle, we have

$$W_m^-(z, \varepsilon, \delta_0) \le W_m(z, \psi) \le W_m^+(z, \varepsilon, \delta_0), \forall z \in \mathbb{R}.$$

Therefore, $\| W_m(\cdot, \psi) - W(\cdot) \| < K\varepsilon, m \ge 0$, which completes the proof. □

Let $\mathcal{X} = BUC(\mathbb{R}^n, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ with the usual supreme norm. Let

$$\mathcal{X}_+ = \left\{\psi \in \mathcal{X} : \quad \psi(x) \ge 0, \forall x \in \mathbb{R}^n\right\}.$$

Hence, $\mathcal{X}_+$ is a closed cone of $\mathcal{X}$ and its induced partial ordering makes $\mathcal{X}$ into a Banach space. Then we use the following lemma to prove the attractiveness and uniqueness of traveling waves.

Lemma 4.4. *(Theorem 2.2.4 in [42], Lemma 3.4 in [30]) Let U be a closed and order convex subset of an ordered Banach space $\mathcal{X}$ with nonempty positive cone, and $f : U \to U$ continuous and monotone. Assume that there exists a monotone homeomorphism h from* $[0, 1]$ *onto a subset of U such that*

1. *for each $s \in [0, 1], h(s)$ is a stable fixed point for $f : U \to U$;*
2. *each forward orbit of f on $[h(0), h(1)]_\chi$ is precompact;*
3. *if $\omega(x) > h(s_0)$ for some $s_0 \in [0, 1)$ and $x \in [h(0), h(1)]_\chi$, then there exists $s_1 \in (s_0, 1)$ such that $\omega(x) \ge h(s_1)$.*

Then for any precompact orbit $\gamma^+(y)$ of f in U with $\omega(y) \cap [h(0), h(1)]_\chi \ne \emptyset$, there exists $s^* \in [0, 1]$ such that $\omega(y) = h(s^*)$.

Based on the above lemmas, we can prove the main result of this section.

Theorem 2. *Let $W(x \cdot e - \mathrm{cm})$ be a monotone traveling wave solution of system (1.3) and $U_m(x, \psi)$ be the solution of (1.2) with initial value $\psi(x) \in \mathcal{X}_{[0,\beta]}$. Then for any nondecreasing $\psi \in \mathcal{X}_{[0,\beta]}$ satisfying (4.2), there exists $s_\psi \in \mathbb{R}$ such that $lim_{m\to\infty} \| U_m(x, \psi) - W(x \cdot e - cm + s_\psi) \| = 0$ uniformly for $x \in \mathbb{R}^n$. And any monotone traveling wave solution of system (1.3) connecting 0 to β is a translation of W.*

Proof. Let $\varepsilon \in (0, \varepsilon_0)$ be given as in Lemma 4.2. By Lemma 4.1 we see that for $\delta_0 > 0$ and any $\psi \in \mathcal{X}_{[0,\beta]}$ satisfying (4.2), there exists $\tilde{z}$ such that for any $z \in \mathbb{R}$, then

$$W_0^- = W(z - \tilde{z}) - \delta_0 \le W_0(z, \psi) = \psi \le W(z + \tilde{z}) + \delta_0 = W_0^+.$$

Thus, Lemma 4.2 and the comparison principle imply that

$$W_m^-(z) \le W_m(z, \psi) \le W_m^+(z), \quad \forall z \in \mathbb{R}, m \in \mathbb{N}^+.$$

Then we have

$$W(z - \tilde{z} - \varepsilon_0) - \delta_0 e^{-\sigma m} \le W_m(z, \psi) \le W(z + \tilde{z} + \varepsilon_0) + \delta_0 e^{-\sigma m}. \tag{4.5}$$

Let $\Pi_m(\psi) := W_m(\cdot, \psi), \forall \psi \in \mathcal{X}, m \in \mathbb{N}^+$, be the solution semiflow determined by (4.1). By (4.5) the forward orbit $\gamma^+(\psi) := \{\Pi_m(\psi) : m \ge 0\}$ is bounded in $\mathcal{X}$ since $\lim_{z\to-\infty} W(z) = 0, \lim_{z\to\infty} W(z) = \beta$. From Ascoli-Arzela theorem, one can get that $\gamma^+(\psi)$ is precompact in $\mathcal{X}$. Thus, the omega limit set $\omega(\psi)$ is nonempty, compact, and invariant.

Letting $z_0 = \tilde{z} + \varepsilon_0$ and $m \to \infty$ in (4.5). We obtain $\omega(\psi) \subset I := [W(\cdot - z_0) \ \ W(\cdot + z_0)]\chi$. Let $h(s) = W(\cdot + s), \forall s \in [-z_0, z_0]$. Then h is a monotone homeomorphism from $[-z_0, z_0]$ onto a subset $\hat{I} \subset I$. Let $V = \mathcal{X}_{[0,\beta]}$. Then $\Pi_m : V \to V$ is a monotone autonomous semiflow. By Lemma 4.3, each $h(s)$ is a stable equilibrium for Π_m. Hence, each $\phi \in \hat{I}$ satisfies (4.2), and $\gamma^+(\phi)$ is precompact.

In order to verify condition 3 of Lemma 4.4, we assume that for some $s_0 \in [-z_0, z_0), \phi_0 \in \hat{I}$ and $W(\cdot + s_0) < \phi(\cdot)$ for all $\phi \in \omega(\phi_0)$, that is, $W(\cdot + s_0) < \omega(\phi_0)$. By the strong maximum principle, we obtain $W(z + s_0) < \Pi_m(\phi)(z), \forall z \in \mathbb{R}, m \in \mathbb{N}$. By the invariance of $\omega(\phi_0)$, there is $W(z + s_0) < \phi(z), \forall \phi \in \omega(\phi_0), z \in \mathbb{R}$. By the compactness of $\omega(\phi_0)$, there exists $s_1 \in (s_0, z_0)$ such that $s_1 - s_0 < \varepsilon_0$, and $W(z + s_1) < \phi(z), \forall z \in [-z_1, z_1], \phi \in \omega(\phi_0)$. Let $\varepsilon_1 = \frac{s_1 - s_0}{2} < \varepsilon_0$. Then there exist $\delta_0 \in (0, \tau\varepsilon_1)$ and $\tilde{z}$ such that for any $z \in \mathbb{R}$, the functions $W_m^{\pm}(z, \varepsilon_1, \delta_0)$ are upper and lower solutions of system (4.1), respectively. By the uniform continuity of W' and $\lim_{z\to\infty} W'(z) = 0$, one can find a large positive number $z_1 \in (z_0, \infty)$ such that $\hat{\delta} := \sup_{|z| \ge z_1 - z_0} \| W'(z) \| \le \frac{\delta_0}{4}$. For any fixed $\phi \in \omega(\phi_0)$, there exists a sequence $\{m_j\}$ such that $m_j \to \infty$ as $j \to \infty$, and $\lim_{j\to\infty} \Pi_{m_j}(\phi_0) = \phi$. Fix m_j such that $\left\| \Pi_{m_j}(\phi_0) - \phi \right\| \leqslant \hat{\delta}(s_1 - s_0)$. Due to $W(z + s_1) < \phi(z), \forall z \in [-z_1, z_1]$ and $W(z + s_0) - W(z + s_1) \le \phi(z) - W(z + s_1), \forall z \in \mathbb{R}$, we have

$$\begin{aligned}\Pi_{m_j}(\phi_0)(z) - W(z+s_1) &= \Pi_{m_j}(\phi_0)(z) - \phi(z) + \phi(z) - W(z+s_1)\\ &\geq -\hat{\delta}(s_1-s_0) - \sup_{|z|\geq z_1} \| W(z+s_0) - W(z+s_1) \|\\ &\geq -\hat{\delta}(s_1-s_0) - \sup_{|z|\geq z_1} \| W'(z) \| (s_1-s_0)\\ &\geq -\hat{\delta}(s_1-s_0) - \hat{\delta}(s_1-s_0)\\ &\geq -\varepsilon_1\delta_0 \geq -\delta_0,\end{aligned}$$

where $\varepsilon_1 = \frac{s_1-s_0}{2} < \varepsilon_0 < 1$. By the construction of $W_m^-(z)$, we obtain

$$\Pi_{m_j}(\phi_0)(z) \geq W(z+s_1) - \delta_0 = W_0^-(z).$$

It follows that

$$\begin{aligned}\Pi_m\left(\Pi_{m_j}(\phi_0)(z)\right) \geq W_m^-(z) &= W\left(z+s_1-\varepsilon_1\left(1-e^{-\sigma m}\right)\right) - \delta_0 e^{-\sigma m}\\ &\geq W(z+s_1-\varepsilon_1) - \delta_0 e^{-\sigma m}\\ &= W\left(z+s_1-\frac{s_1-s_0}{2}\right) - \delta_0 e^{-\sigma m}\\ &= W\left(z+\frac{s_1+s_0}{2}\right) - \delta_0 e^{-\sigma m}, \quad \forall z\in\mathbb{R}, m\in\mathbb{N}^+.\end{aligned}$$

Let $m = m_i - m_j$ and $m_i \to \infty$, then $\phi(\cdot) \geq W\left(\cdot + \frac{s_1+s_0}{2}\right)$. Denote $s_2 = \frac{s_1+s_0}{2}$, then $s_2 \in (s_0, s_1) \subseteq [s_0, z_0]$ and $W(\cdot + s_2) \leq \phi(\cdot)$. By the arbitrariness of $\phi \in \omega(\phi_0)$, it means that $\phi(\cdot + s_2) \leq \omega(\phi_0)$.

By Lemma 4.4, there exists $s_\psi \in [-z_0, z_0]$ such that $\omega(\psi) = h(s_\psi) = W(\cdot + s_\psi)$. Then $\lim_{m\to\infty} \Pi_m(\psi) = W(\cdot + s_\psi)$. Since $U_m(x,\psi) = W_m(x\cdot e - cm, \psi) = \Pi_m(\psi)(x\cdot e - cm)$, we have $\lim_{m\to\infty} \| U_m(x,\psi) - W(x\cdot e - cm + s_\psi) \| = 0$ uniformly for $x \in \mathbb{R}^n$.

Let $\tilde{W}(x\cdot e - \tilde{c}m)$ be a traveling wave solution of system (1.3) connecting 0 to β. Clearly, $\tilde{W}$ satisfies (4.2) in Lemma 4.1. Then there is $\tilde{s}_\psi \in \mathbb{R}$ such that $\lim_{m\to\infty} \| \tilde{W}(x\cdot e - \tilde{c}m) - W(x\cdot e - cm + \tilde{s}_\psi) \| = 0$. Set $\tilde{x} = x - cme$, then $\lim_{m\to\infty} \| \tilde{W}(\tilde{x}\cdot e + (c-\tilde{c})m) - W(\tilde{x}\cdot e + \tilde{s}_\psi) \| = 0$. Since $\tilde{W}(-\infty) = 0$, $\tilde{W}(\infty) = \beta$, and $W(\cdot)$ is strictly increasing on $\mathbb{R}$, we then have $\tilde{c} = c$, and hence, $\tilde{W}(\cdot) = W(\cdot + \tilde{s}_\psi)$. □

In fact, for any $\psi \in \mathcal{X}_{[0,\beta]}$ satisfying (4.2), there exist two nondecreasing functions $\psi_1(x), \psi_2(x) \in \mathcal{X}_{[0,\beta]}$ satisfying (4.2) such that $\psi_1(x) \leq \psi(x) \leq \psi_2(x)$. Then there exists $Z_0 > 0$ such that $\check{W}(z) - \varepsilon \leq \psi_1(z) \leq \psi(z) \leq \psi_2(z) \leq \hat{W}(z) + \varepsilon$ holds for all $|z| \geq Z_0$. By the uniqueness up to translation and monotonicity of W, there exists $\tilde{z} > 0$ such that $W(z-\tilde{z}) - \varepsilon \leq \psi \leq W(z+\tilde{z}) + \varepsilon$. This is consistent with Lemma 4.1. We replace the above theorem with the following theorem.

Theorem 3. *Let $W(x\cdot e - \mathrm{cm})$ be a monotone traveling wave solution of system (1.3) and $U_m(x,\psi)$ be the solution of (1.2) with initial value $\psi(x) \in \mathcal{X}_{[0,\beta]}$. Then for any $\psi \in \mathcal{X}_{[0,\beta]}$ satisfying (4.2), there exists $s_\psi \in \mathbb{R}$ such that $lim_{m\to\infty} \| U_m(x,\psi) - W(x\cdot e - cm + s_\psi) \| = 0$*

uniformly for $x \in \mathbb{R}^n$. And any monotone traveling wave solution of system (1.3) connecting 0 *to* β *is a translation of* W.

5. Numerical simulations

In this section, we consider the model (1.2) in one dimension that is $n = 1$, $A = d$, $e = 1$, and we consider the simplest case $f(u) = u(1-u)(u-s)$ and $g(N) = kN$ where $k > 0$ is a constant, or the case where g is the Beverton-Holt function

$$g(N) = \mu N/(\kappa + N). \tag{5.1}$$

We truncate the infinite domain $\mathbb{R}$ to finite domain $[-L, L]$, where L is sufficiently large. Then we present some numerical simulations to illustrate our theoretical results.

By Theorem 3, system (1.3) admits a unique monotone bistable traveling wave up to translation, which is globally stable with phase shift. In order to simulate this result, we choose $L = 50$, $s = 0.5$, $a = 3$, $g(N) = 1.3N$, and the initial function $N_0(x)$.

We have selected four functions that satisfy (4.2) as initial functions:

$$f_1(x) = \begin{cases} 0, & -50 \leq x \leq 0; \\ 0.04x, & 0 < x < 10; \\ 0.4, & 10 \leq x \leq 50. \end{cases} \tag{5.2}$$

$$f_2(x) = \begin{cases} 0, & -50 \leq x \leq 0; \\ 0.06x, & 0 < x < 10; \\ 0.6, & 10 \leq x \leq 50. \end{cases} \tag{5.3}$$

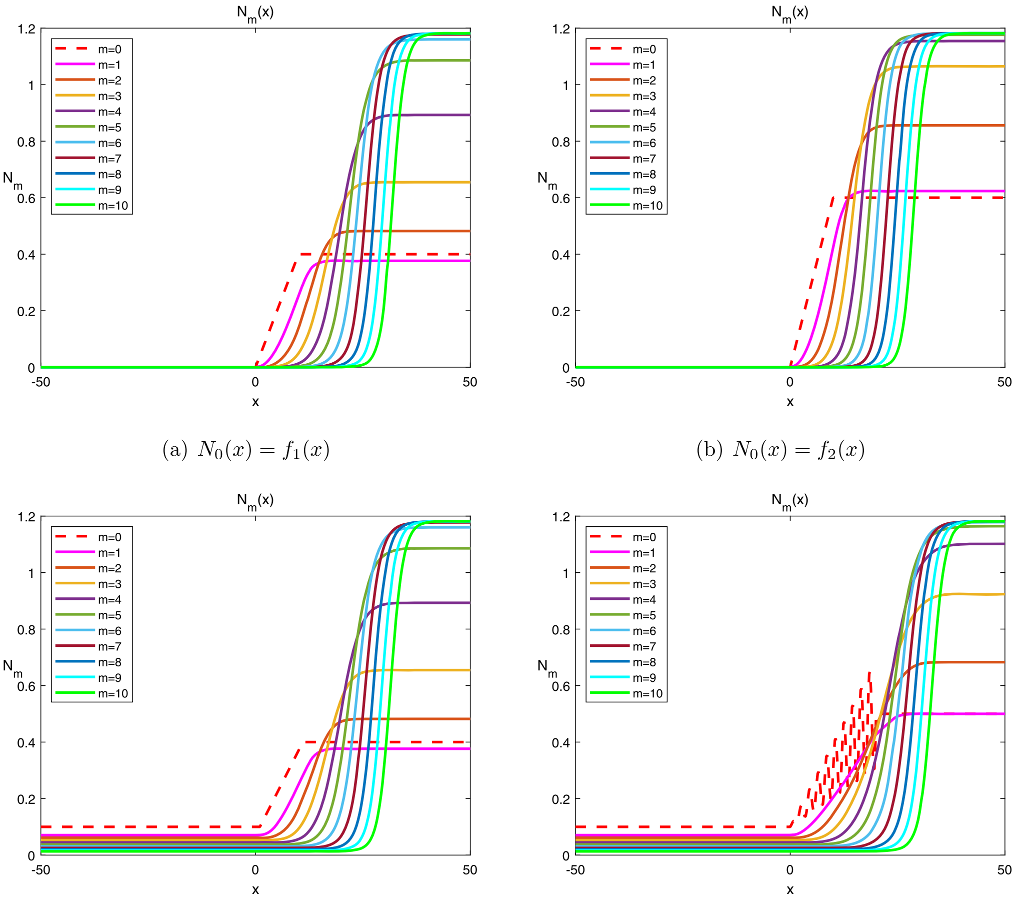
$$f_3(x) = \begin{cases} 0.1, & -50 \leq x \leq 1; \\ 0.03x + 0.07, & 1 < x < 11; \\ 0.4, & 11 \leq x \leq 50. \end{cases} \tag{5.4}$$

$$f_4(x) = \begin{cases} 0, & -50 \leq x \leq 0; \\ 0.01x(sin(\pi x) + 2) + 0.1, & 0 < x < 20; \\ 0.5, & 20 \leq x \leq 50. \end{cases} \tag{5.5}$$

The evolution of the solution is shown in Fig. 3.

The numerical wave profiles and the initial conditions are plotted by solid and dashed lines in Fig. 3(a)-3(d), respectively. We can see that the solution converges quickly to the numerical wave profile, and different initial values evolve the unique wave profile when the other parameters are the same.

According to Remark 1 in Section 3, we obtain the estimation of the spreading speed of the bistable traveling wave solution, that is, $c^*_{0,\beta}(e) \in \left(-c^*_{\alpha,\beta}(-e), c^*_{0,\alpha}(e)\right)$. Otherwise, the bistable traveling wave will be disconnected by the comparison principle. Next, we explain the impact of advection on the spreading speed. We choose $L = 50$, $s = 0.2$, and g is given by (5.1) where $\mu = 1.6$, $\kappa = 0.2$. Let the initial function be $N_0(x) = f_1(x)$. Due to Theorem 3.1 in [38], it can be seen that when the vector modulus of a is sufficiently large, the sign of the wave speed c depends on the angle between the vectors a and e. Fig. 4 shows the evolution of the initial function for the cases where the advection coefficient a takes 5 and -5, respectively.



(a) $N_0(x) = f_1(x)$

(b) $N_0(x) = f_2(x)$

(c) $N_0(x) = f_3(x)$

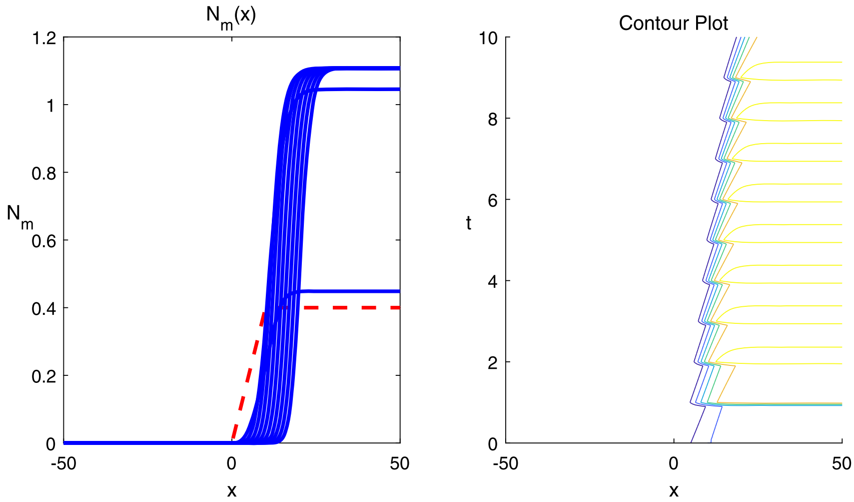
(d) $N_0(x) = f_4(x)$

Fig. 3. A numerical approximation to the graph of $N_m(x)$ for (1.3). Fig. 3 (a)-(d) show that the system evolves from different initial values, and the unique bistable traveling wave solution can be obtained finally. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

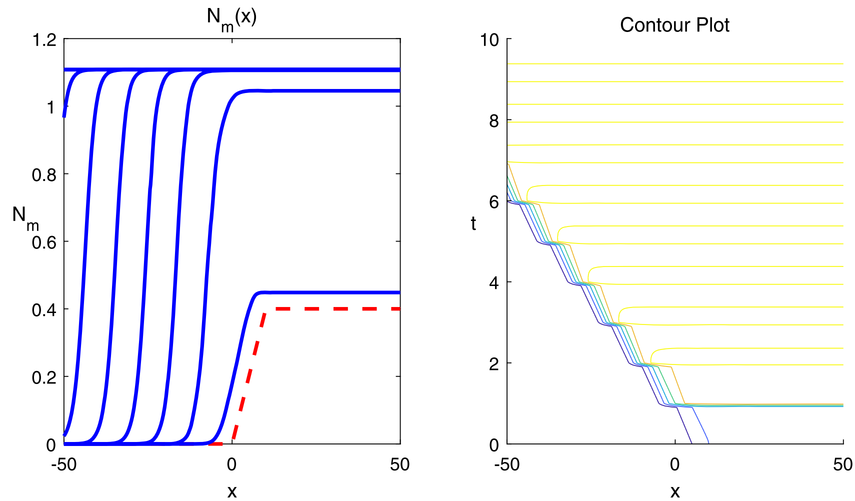
Under the given parameters, we can observe that the traveling wave propagates to the right as $a = 5$, that is, the sign of the wave speed is positive, and the traveling wave propagates to the left as $a = -5$, that is, the wave speed is negative.

By Theorem 1 and Remark 3, system (1.3) admits a traveling wave connecting 0 to β with speed $c^*_{0,\beta}$. Meanwhile, system (1.3) admits a traveling wave connecting β to 0 with speed $c^*_{\beta,0}$. Here we illustrate the effect of the initial function on the monotonicity of the traveling wave solution. We choose $L = 50$, $s = 0.2$, $a = 1$, and g is given by (5.1) where $\mu = 1.6$, $\kappa = 0.2$. Then we choose four initial functions $f_1(x)$, $f_5(x)$, $f_6(x)$, $f_7(x)$ so that they respectively satisfy the conditions in Remark 4, where $f_1(x)$ is given by (5.2), $f_5(x) = f_1(-x)$, $f_6(x) = 0.2sin(\frac{\pi x}{20})$, $x \in (-10, 10)$; $f_6(x) = 0$, $x \in [-50, -10] \cup [10, 50]$, and $f_7(x) = 1.5 f_6(x)$. The evolution of the solution is shown in Fig. 5.

By Theorem 1 and Theorem 3, system (1.3) admits a unique monotone bistable traveling wave up to translation. As can be seen from Fig. 5, the monotonicity of the traveling wave



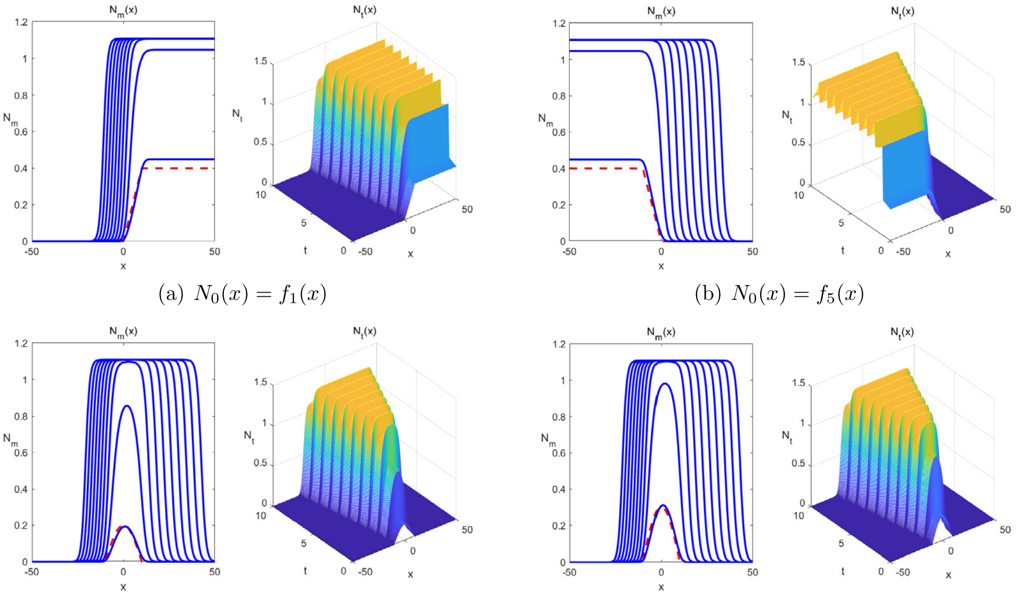
(a) a=5



(b) a=-5

Fig. 4. A numerical approximation to the graph of $N_m(x)$ for (1.3). Figs. 4 (a) and (b) show that the traveling wave propagates in the opposite direction due to advection.

solution depends on the choice of the initial function. Fig. 5(a) shows that a nondecreasing initial function eventually evolves into a nondecreasing traveling wave solution. From Fig. 5(a), it is also seen that the propagation direction is left. Correspondingly, Fig. 5(b) shows a result of opposite monotonicity, and the traveling wave propagates to the right at another speed. Figs. 5(a) and 5(b) are consistent with the statements in (1) and (2) of Remark 4, respectively. Fig. 5(c) shows that the initial function is chosen to be a cosine function with a compact support set from -10 to 10, which ultimately evolves a pair of diverging traveling fronts. It is worth noting that a pair of diverging traveling fronts is consistent with the wavefronts in Figs. 5(a) and 5(b), respectively. In



(a) $N_0(x) = f_1(x)$ (b) $N_0(x) = f_5(x)$

(c) $N_0(x) = f_6(x)$ (d) $N_0(x) = f_7(x)$

Fig. 5. A numerical approximation to the graph of $N_m(x)$. Figures (a)-(d) show the effect of the monotonicity of the initial value on the traveling wave direction.

Fig. 5(d), we select other functions with a compact support set as the initial condition, and finally obtain the same wavefront of Fig. 5(c). Figs. 5(c) and 5(d) are consistent with the statements in (3) of Remark 4.

6. Discussion

We studied impulsive reaction-advection-diffusion equation models with reproductive and diffusion stages in the entire space $\mathbb{R}^n$. This kind of model can be formally regarded as a hybrid system composed of reaction diffusion equation and difference equation. Therefore, different from the standard partial differential equation model, the study of impulsive reaction-advection-diffusion models needs to analyze both differential equation and recursive relationship. The case of monostability model was studied by Lewis and Li in [34], Lin and Wang in [35], Vasilyeva et al. in [36], and Fazly et al. in [38]. Different from the above papers, we chose $f(\cdot)$ as Allee function instead of logistic type, and focused on the existence of traveling wave solutions connecting two stable equilibrium points, which is called bistable traveling waves. We extended the monotone semiflows method established in [10], the method of upper and lower solutions, and the global convergence result for monotone systems in [42] to impulsive reaction-advection-diffusion systems with bistable structure in higher-dimensional habitats. We showed the existence, the global stability of bistable traveling wave solutions, and their uniqueness up to translation. Our results show that the monotonicity of the initial surface determines the monotonicity of the traveling waves. Different initial surfaces with approximate monotonicity converge to a translation of the traveling wave. Numerical simulations verify our theorems.

In addition, our numerical simulations illustrate that the traveling wave solution evolved from an initial condition with compact support seems to be unique and stable. For the model studied in this paper, we speculate that under appropriate conditions, $N_m(x)$ converges to a pair of diverging waves $W\left(x \cdot e - c_{0,\beta}m - z_1\right) + \overline{W}\left(x \cdot e - c_{\beta,0}m - z_2\right) - \beta$ as $m \to \infty$. In this paper, the bistable structure is achieved by considering the Allee effect in the PDE. It is worth noting that considering the Allee effect in a discrete-time mapping can also lead to bistable structure of an impulsive PDE system. For example, we can replace the function g with the so-called Sigmoid Beverton-Holt function,

$$g_{k,\lambda}(N) = \frac{kN^\lambda}{1 + (k-1)N^\lambda}, \quad k > 1, \lambda > 1,$$

which has the characteristic 'S' shape, a slow rise from zero, then a rapid rise, and eventually flattened for large N. Since most habitats are heterogeneous, it will be intriguing and challenging to explore invasion dynamics in the case of spatial heterogeneity. Dispersing populations with discrete generations may have both spatially and temporally chaotic dynamics. Recently, by employing a numerical method, Liang et al. [40] showed that the impulsive reaction-diffusion model generates spatial chaos. How to prove the existence of chaos for the impulsive reaction-diffusion model by using analytical methods? This is an open question. Most studies on impulsive reaction-diffusion models focus on single species. It would be interesting to derive and analyze multi-species models in this direction. We leave these questions for future investigations.

References

[1] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in: Partial Differential Equations and Related Topics, in: Lecture Notes in Math., vol. 446, Springer, Berlin, 1975, pp. 5–49.
[2] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, Adv. Math. 30 (1978) 33–76.
[3] H.F. Weinberger, Long-time behavior of a class of biological models, SIAM J. Math. Anal. 13 (1982) 353–396.
[4] R. Lui, A nonlinear integral operator arising from a model in population genetics, I. Monotone initial data, SIAM J. Math. Anal. 13 (8) (1982) 913–937.
[5] R. Lui, Biological growth and spread modeled by systems of recursions. I. Mathematical theory, Math. Biosci. 93 (2) (1989) 269–295.
[6] X. Liang, Y. Yi, X.-Q. Zhao, Spreading speeds and traveling waves for periodic evolution systems, J. Differ. Equ. 231 (1) (2006) 57–77.
[7] X. Liang, X.-Q. Zhao, Asymptotic speeds of spread and travelling waves for monotone semiflow with applications, Commun. Pure Appl. Math. 60 (2007) 1–40. Erratum: 61 (2008) 137–138.
[8] B. Li, M.A. Lewis, H.F. Weinberger, Existence of traveling waves for integral recursions with nonmonotone growth functions, J. Math. Biol. 58 (3) (2009) 323–338.
[9] X. Liang, X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, J. Funct. Anal. 259 (4) (2010) 857–903.
[10] J. Fang, X.-Q. Zhao, Bistable traveling waves for monotone semiflows with applications, J. Eur. Math. Soc. 17 (2015) 2243–2288.
[11] W.C. Allee, Animal Aggregations: A Study in General Sociology, University of Chicago Press, 1931.
[12] M.A. Lewis, P. Kareiva, Allee dynamics and the spread of invading organisms, Theor. Popul. Biol. 43 (1993) 141–158.
[13] M. Kot, Dispersal data and the spread of invading organisms, Ecology 77 (7) (1996) 2027–2042.
[14] T.H. Keitt, M.A. Lewis, R.D. Holt, Allee effects, invasion pinning, and species borders, Am. Nat. 157 (2001) 203–216.
[15] M.-H. Wang, M. Kot, M.G. Neubert, Integrodifference equations, Allee effects, and invasions, J. Math. Biol. 44 (2002) 150–168.
[16] M. Krkosek, B.M. Connors, M.A. Lewis, R. Poulin, Allee effects may slow the spread of parasites in a coastal marine ecosystem, Am. Nat. 179 (2012) 401–412.

[17] P.C. Fife, Long time behavior of solutions of bistable nonlinear diffusion equations, Arch. Ration. Mech. Anal. 70 (1) (1979) 31–36.
[18] P.C. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Ration. Mech. Anal. 65 (4) (1977) 335–361.
[19] R. Lui, Existence and stability of traveling wave solutions of a nonlinear integral operator, J. Math. Biol. 16 (1983) 199–220.
[20] R. Lui, A nonlinear integral operator arising from a model in population genetics III. Heterozygote inferior case, SIAM J. Math. Anal. 16 (6) (1985) 1180–1206.
[21] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, Adv. Differ. Equ. 2 (1) (1997) 125–160.
[22] W. Shen, Traveling waves in time almost periodic structures governed by bistable nonlinearities: I. Stability and uniqueness, J. Differ. Equ. 159 (1) (1999) 1–54.
[23] H.L. Smith, X.-Q. Zhao, Global asymptotic stability of traveling waves in delayed reaction-diffusion equations, SIAM J. Math. Anal. 31 (3) (2000) 514–534.
[24] C.E. Elmer, E.S. Van Vleck, Traveling waves solutions for bistable differential-difference equations with periodic diffusion, SIAM J. Appl. Math. 61 (2001) 1648–1679.
[25] H.F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, J. Math. Biol. 45 (6) (2002) 511–548.
[26] D. Xu, X.-Q. Zhao, Bistable waves in an epidemic model, J. Dyn. Differ. Equ. 16 (2004) 679–707.
[27] Y. Jin, X.-Q. Zhao, Bistable waves for a class of cooperative reaction-diffusion systems, J. Biol. Dyn. 2 (2) (2008) 196–207.
[28] S.L. Wu, W.T. Li, Global asymptotic stability of bistable traveling fronts in reaction-diffusion systems and their applications to biological models, Chaos Solitons Fractals 40 (2009) 1229–1239.
[29] Y.J. Sun, W.T. Li, Z.C. Wang, Entire solutions in nonlocal dispersal equations with bistable nonlinearity, J. Differ. Equ. 251 (3) (2011) 551–581.
[30] Y. Zhang, X.-Q. Zhao, Bistable travelling waves in competitive recursion systems, J. Differ. Equ. 252 (3) (2012) 2630–2647.
[31] G. Lin, W.T. Li, M.J. Ma, Traveling wave solutions in delayed reaction diffusion systems with applications to multi-species models, Discrete Contin. Dyn. Syst., Ser. B 13 (2) (2010) 393–414.
[32] Y. Zhang, X.-Q. Zhao, Bistable travelling waves for a reaction and diffusion model with seasonal succession, Nonlinearity 26 (3) (2013) 691–709.
[33] Y. Pan, Y. Su, J. Wei, Bistable waves of a recursive system arising from seasonal age-structured population models, Discrete Contin. Dyn. Syst., Ser. B 24 (2) (2019) 511–528.
[34] M.A. Lewis, B. Li, Spreading speed, traveling waves, and minimal domain size in impulsive reaction-diffusion models, Bull. Math. Biol. 74 (10) (2012) 2383–2402.
[35] Y. Lin, Q.R. Wang, Spreading speed and traveling wave solutions in impulsive reaction-diffusion models, Commun. Nonlinear Sci. Numer. Simul. 23 (1–3) (2015) 185–191.
[36] O. Vasilyeva, F. Lutscher, M.A. Lewis, Analysis of spread and persistence for stream insects with winged adult stages, J. Math. Biol. 72 (4) (2016) 851–875.
[37] M. Fazly, M.A. Lewis, H. Wang, On impulsive reaction-diffusion models in higher dimensions, SIAM J. Appl. Math. 77 (1) (2017) 224–246.
[38] M. Fazly, M.A. Lewis, H. Wang, Analysis of propagation for impulsive reaction-diffusion models, SIAM J. Appl. Math. 80 (1) (2020) 521–542.
[39] R. Wu, X.-Q. Zhao, Spatial invasion of a birth pulse population with nonlocal dispersal, SIAM J. Appl. Math. 79 (3) (2019) 1075–1097.
[40] J. Liang, Q. Yan, C. Xiang, S. Tang, A reaction-diffusion population growth equation with multiple pulse perturbations, Commun. Nonlinear Sci. Numer. Simul. 74 (2019) 122–137.
[41] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 2006.
[42] X.-Q. Zhao, Dynamical Systems in Population Biology, second edition, Springer-Verlag, New York, 2017.