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Basic reproduction ratio of a mosquito-borne disease in heterogeneous environment

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Abstract

To explore the influence of spatial heterogeneity on mosquito-borne diseases, we formulate a reaction–diffusion model with general incidence rates. The basic reproduction ratio $\mathcal{R}_0$ for this model is introduced and the threshold dynamics in terms of $\mathcal{R}_0$ are obtained. In the case where the model is spatially homogeneous, the global asymptotic stability of the endemic equilibrium is proved when $\mathcal{R}_0 > 1$. Under appropriate conditions, we establish the asymptotic profiles of $\mathcal{R}_0$ in the case of small or large diffusion rates, and investigate the monotonicity of $\mathcal{R}_0$ with respect to the heterogeneous diffusion coefficients. Numerically, the proposed model is applied to study the dengue fever transmission. Via performing simulations on the impacts of certain factors on $\mathcal{R}_0$ and disease dynamics, we find some novel and interesting phenomena which can provide valuable information for the targeted implementation of disease control measures.

Keywords Mosquito-borne disease model · Spatial heterogeneity · Basic reproduction ratio · Threshold dynamics · Asymptotic profiles and Monotonicity

Mathematics Subject Classification 35B40 · 35K57 · 37N25 · 92D30

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1 Introduction

Mosquito-borne disease is an insect-borne disease transmitted by mosquitoes. Some such common diseases include malaria, chikungunya and dengue fever. Dengue fever, for example, is a mosquito-borne disease caused by five different serotypes of dengue virus, of which *Aedes aegypti* and *Aedes albopictus* are the main vectors Zhang and Wang (2020). As reported in Bhatt et al. (2013), about 390 million people around the world are at risk of dengue infection annually, which has become one of the public health problems in tropical and subtropical regions. Mosquito-borne disease has turned into a major concern due to environmental change and urbanization. It seems thus imperative to explore the disease transmission.

Mathematical models are a powerful tool for studying the mosquito-borne diseases. So far, most mosquito-borne disease models are usually spatially homogeneous. However, there is growing evidence that many factors, such as the natural landscape, urbanity, and human and vector activities, can cause spatial heterogeneity (Allen et al. 2008; Ruan 2007; Wang et al. 2021b; Wu and Zhao 2019). In fact, mosquito-borne diseases are often affected by the spatial structure of environment, and the environment for disease spreading has heterogeneity (Chen and Shi 2021). In addition, the environmental temperature has a great influence on mosquito-borne disease dynamics, and temperature can be characterized by spatial heterogeneity (Vaidya et al. 2019). On the other hand, it is known that the mosquito-borne disease transmission can be significantly shaped by the mobility of humans and mosquitoes. In particular, as mentioned in Pakhare et al. (2016), the flight ability of mosquitoes affects the disease transmission ability, and even from the numerical results of Wang et al. (2021a), it can also be seen that, in some circumstances, the mosquito flight has a greater impact on the speed of disease transmission. Therefore, it is reasonable and necessary to incorporate the spatial heterogeneity and the diffusion of humans and mosquitoes into the mosquito-borne diseases modeling. Moreover, epidemic models with constant diffusion rate in which susceptible and infected populations have the same diffusion ability have been extensively investigated (Wu and Zhao 2019; Zhu et al. 2020). Although the mosquito-borne disease models with different heterogeneous diffusion rates may be more in line with biological significance, there are few theoretical studies on such models (Cai et al. 2019; De Araujo et al. 2016; Fang et al. 2020; Li and Zhao 2021; Magal et al. 2018, 2019; Zhang and Wang 2022). Note that a fair amount of mosquito-borne disease models mainly adopted bilinear (Wu and Zhao 2019) or standard incidence (Wang and Zhao 2011). In some cases, however, the general incidence rates are better to give a sensible qualitative description for disease dynamics (Capasso and Serio 1978).

In epidemiology, the basic reproduction ratio $\mathcal{R}_0$ is one of the most important concepts, and it is a crucial threshold of disease outbreak or not Wang and Zhao (2012). For autonomous epidemic models, a general method to compute $\mathcal{R}_0$ was proposed by Diekmann et al. (1990) with the aid of next generation operator approach . For ordinary differential compartmental models, an approach to compute $\mathcal{R}_0$ was established by Van den Driessche and Watmough (2002). For general reaction–diffusion (R–D) models with compartmental structure, Wang and Zhao (2012) developed a theory of $\mathcal{R}_0$ applying the concept of principal eigenvalue. There are many researches on $\mathcal{R}_0$ of various infectious disease models, and readers can refer to Bacaër and Guernaoui

(2006), Inaba (2012), Liang et al. (2017) and references therein. According to Wang and Zhao (2012), $\mathcal{R}_0$ of R–D epidemic models with constant coefficients in a bounded region has the same form as that of the corresponding kinetic system. Nevertheless, for R–D models with spatial heterogeneity, $\mathcal{R}_0$ is inevitably associated with diffusion rate(s).

A diffusive SIS model with spatial heterogeneity was considered and the variational expression of $\mathcal{R}_0$ was given by Allen et al. Allen et al. (2008). Furthermore, the authors studied the asymptotic profiles and monotonicity of $\mathcal{R}_0$ in terms of the diffusion rate of the infected. Chen and Shi (2021) generalized the results of asymptotic behaviors for $\mathcal{R}_0$ on large or small diffusion rates to a more general R–D compartmental models. Zhang and Zhao (2021) discussed the asymptotic behavior of $\mathcal{R}_0$ for periodic R–D systems. In Magal et al. (2019), Magal et al. explored a vector-host R–D model with spatial heterogeneity and analyzed the asymptotic profiles and monotonicity of $\mathcal{R}_0$ with respect to (w.r.t) constant diffusion rates of humans and mosquitoes. A spatial SEIRS model in heterogeneous environment was considered by Song et al. (2019), and the properties of $\mathcal{R}_0$ were studied. In Gao (2020), Gao investigated the affect of dispersal on infection size and monotonicity of $\mathcal{R}_0$ for patch models. Although the asymptotic profiles and monotonicity of $\mathcal{R}_0$ w.r.t constant diffusion rate(s) have been intensively investigated, there are few literatures to focus on the properties of $\mathcal{R}_0$ for mosquito-borne disease models when the diffusion rates are spatially heterogeneous.

The present paper aims to explore the effects of spatial heterogeneity and non-constant dispersal rates on $\mathcal{R}_0$ and disease dynamics by developing a mosquito-borne disease model, thereby improving our understanding of the transmission mechanism of mosquito-borne diseases and proposing targeted disease control measures.

The remainder of the paper is organized as follows. In Sect. 2, we establish our R–D model that incorporates environmental heterogeneity, non-constant dispersal rates and general incidence rates. In Sect. 3, we present the main results of this article, including the well-posedness, threshold dynamics and asymptotic profiles and monotonicity of $\mathcal{R}_0$, and reveal the corresponding biological interpretations. Section 4 performs some numerical simulations to substantiate the theoretical results and explores the impacts of several factors on $\mathcal{R}_0$ and disease dynamics. Section 5 states a brief discussion to conclude the paper. The proof of the main results is given in Sect. 6.

2 Mathematical model

Motivated by the aforementioned analysis, this work aims to consider the following R–D model with spatial heterogeneity and general incidence rates:

$$\begin{cases} \partial_t S_h = \nabla \cdot [D_S(x)\nabla S_h] + \Lambda(x) - f_1(x, S_h, I_v) - \mu_1(x)S_h, \\ \partial_t I_h = \nabla \cdot [D_I(x)\nabla I_h] + f_1(x, S_h, I_v) - [\mu_1(x) + e_1(x) + \alpha_1(x)]I_h, \\ \partial_t R_h = \nabla \cdot [D_R(x)\nabla R_h] + \alpha_1(x)I_h - \mu_1(x)R_h, \\ \partial_t S_v = \nabla \cdot [d_S(x)\nabla S_v] + M(x) - f_2(x, S_v, I_h) - \mu_2(x)S_v, \\ \partial_t I_v = \nabla \cdot [d_I(x)\nabla I_v] + f_2(x, S_v, I_h) - [\mu_2(x) + e_2(x)]I_v, \end{cases} \tag{1}$$

where $t \geq 0$, $x \in \Omega$ and Ω is a bounded domain of $\mathbb{R}^l$ (l is any positive integer) with smooth boundary $\partial\Omega$. $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_l)$ is the gradient operator. $S_h(t,x)$, $I_h(t,x)$ and $R_h(t,x)$ are the spatial densities of susceptible, infected and recovered individuals, and $S_v(t,x)$ and $I_v(t,x)$ are the spatial densities of susceptible and infected mosquitoes at time t and location x, respectively. The diffusion rates of humans and mosquitoes at x are denoted by $D_S(x)$, $D_I(x)$, $D_R(x)$ and $d_S(x)$, $d_I(x)$, respectively. $f_1(x, S_h, I_v)$ and $f_2(x, S_v, I_h)$ signify the virus transmission functions. The recruitment of humans and mosquitoes at x are represented by $\Lambda(x)$ and $M(x)$ respectively. $\mu_1(x)$ ($\mu_2(x)$) and $e_1(x)$ ($e_2(x)$) indicate the natural and disease-induced death rates of humans (mosquitoes) at x, respectively. $\alpha_1(x)$ denotes the recovery rate of humans at x. For simplicity, let $\gamma_1(\cdot) := \mu_1(\cdot) + e_1(\cdot) + \alpha_1(\cdot)$ and $\gamma_1(\cdot) := \mu_2(\cdot) + e_2(\cdot)$, and impose the initial and homogeneous Neumann boundary conditions of (1) as follows:

$$\begin{cases} S_h^0(x) \geq 0, \ \ I_h^0(x) \geq 0, \ \ R_h^0(x) \geq 0, \ \ S_v^0(x) \geq 0, \ \ I_v^0(x) \geq 0, & x \in \Omega, \\ \partial_{\mathbf{n}} S_h = \partial_{\mathbf{n}} I_h = \partial_{\mathbf{n}} R_h = \partial_{\mathbf{n}} S_v = \partial_{\mathbf{n}} I_v = 0, & t > 0, \ \ x \in \partial\Omega, \end{cases}$$

wherein $S_h^0(\cdot) := S_h(0,\cdot)$, $I_h^0(\cdot) := I_h(0,\cdot)$, $R_h^0(\cdot) := R_h(0,\cdot)$, $S_v^0(\cdot) := S_v(0,\cdot)$ and $I_v^0(\cdot) := I_v(0,\cdot)$ are smooth functions which are not identically zero, and $\mathbf{n}$ denotes the outward normal unit vector on $\partial\Omega$. By the decoupling, it is sufficient to discuss the following system:

$$\begin{cases} \partial_t S_h = \nabla \cdot [D_S(x)\nabla S_h] + \Lambda(x) - f_1(x, S_h, I_v) - \mu_1(x) S_h, & t > 0, \ \ x \in \Omega, \\ \partial_t I_h = \nabla \cdot [D_I(x)\nabla I_h] + f_1(x, S_h, I_v) - \gamma_1(x) I_h, & t > 0, \ \ x \in \Omega, \\ \partial_t S_v = \nabla \cdot [d_S(x)\nabla S_v] + M(x) - f_2(x, S_v, I_h) - \mu_2(x) S_v, & t > 0, \ \ x \in \Omega, \\ \partial_t I_v = \nabla \cdot [d_I(x)\nabla I_v] + f_2(x, S_v, I_h) - \gamma_2(x) I_v, & t > 0, \ \ x \in \Omega, \\ S_h^0(x) \geq 0, \ \ I_h^0(x) \geq 0, \ \ S_v^0(x) \geq 0, \ \ I_v^0(x) \geq 0, & x \in \Omega, \\ \partial_{\mathbf{n}} S_h = \partial_{\mathbf{n}} I_h = \partial_{\mathbf{n}} S_v = \partial_{\mathbf{n}} I_v = 0, & t > 0, \ \ x \in \partial\Omega. \end{cases} \tag{2}$$

Throughout this paper, we make the following assumptions:

(P1) All coefficients of (1) are spatially heterogeneous and positive; The diffusion rates are $C^{1+\nu}(\bar{\Omega})$ and other parameters are Hölder continuous functions in $C^{\nu}(\bar{\Omega})$ with $\nu \in (0, 1)$;
(P2) $f_i(x, S, I) \in C^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}_+)$, and $\partial_S f_i(x, S, I)$ and $\partial_I f_i(x, S, I)$ are positive for all $x \in \Omega$, $S, I > 0$; $f_i(x, S, I) = 0$ if and only if (iff) $SI = 0$; $\partial_I^2 f_i(x, S, I) \leq 0$ for all $x \in \Omega$, $S, I \geq 0$, $i = 1, 2$.

Remark 1 Some frequently used incidence rates satisfy (P2). For example,

(i) The bilinear incidence rate $f_i(\cdot, S, I) = k_i(\cdot) SI$, $k_i(\cdot) > 0$, $i = 1, 2$ (see Wu and Zhao 2019);
(ii) The saturated incidence rate $f_i(\cdot, S, I) = \frac{k_i(\cdot)SI}{1+\rho_i(\cdot)I}$, $k_i(\cdot)$, $\rho_i(\cdot) > 0$, $i = 1, 2$ (see Capasso and Serio 1978; Heesterbeek and Metz 1993);

(iii) The mixture of bilinear and saturated incidence rates $f_1(\cdot, S, I) = k_1(\cdot)SI$ and $f_2(\cdot, S, I) = \frac{k_2(\cdot)SI}{1+\rho_2(\cdot)I}$ or $f_1(\cdot, S, I) = \frac{k_1(\cdot)SI}{1+\rho_1(\cdot)I}$ and $f_2(\cdot, S, I) = k_2(\cdot)SI, k_i(\cdot)$, $\rho_i(\cdot) > 0, i = 1, 2$.

Inspired by above analysis, we in this paper first study the threshold dynamics of (2). From epidemiological perspective, by investigating the properties of $\mathcal{R}_0$, we can clarify the effects of population mobility on the persistence of epidemic. Thus, we further investigate the limit of $\mathcal{R}_0$ as $D_I(\cdot)$ and $d_I(\cdot)$ go arbitrarily small or large, and discuss the monotonicity of $\mathcal{R}_0$ w.r.t $D_I(\cdot)$ and $d_I(\cdot)$ under appropriate conditions. Furthermore, through numerically exploring the influence of some important factors (such as heterogeneity, diffusion rates, parameter sensitivity, temperature) on $\mathcal{R}_0$ and dynamics of (2), we discover some interesting and crucial phenomena, which not only help the disease control, but also improve the understanding of the impacts of spatial heterogeneity and human and mosquito movement on disease dynamics.

It should be pointed out that since mosquito-borne disease contains two infection pathways, the complexity of model (2) leads to several difficulties: (i) Since the diffusion rates of (2) are not necessarily equal, we cannot study the ultimate boundedness by adding the equations of (2). To overcome the challenging, motivated by literatures (Dung 1997, 1998; Wu and Zou 2018), we apply a well known induction method to solve the issue; (ii) In general, the global asymptotic stability of disease-free steady state when $\mathcal{R}_0 = 1$ are rarely addressed. Fortunately, by constructing an appropriate Lyapunov functional and employing the LaSalle's invariance principle and the methods of Shu et al. (2021), we obtain the global asymptotic stability when $\mathcal{R}_0 = 1$; (iii) Although the ideas of Chen and Shi (2021), Magal et al. (2019) cannot be directly used to deal with the properties of $\mathcal{R}_0$, which is mainly because the diffusion rates of (2) depend on spatial variable, we can address it by improving the approaches (Song et al. 2019). In fact, the process is more complicated due to the general incidence rates but the results are more profound. This also implies that the conclusions of this paper can be applied to some other deterministic models.

3 Main results

In this section, we present the main results of this work, whose proofs are given in Sect. 6.

3.1 Well-posedness

To proceed, we first give some definitions. Let $X := C(\bar{\Omega}, \mathbb{R}^4)$ be endowed with the supreme norm, and $X^+ := C(\bar{\Omega}, \mathbb{R}^4_+)$ be the positive cone of X. In what follows, denote $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$, $p \in \mathbb{Z}^+$ (the set of positive integers) and $\|\cdot\| := \|\cdot\|_{L^\infty(\Omega)}$. Set $h^* := \max\{h(x) : x \in \bar{\Omega}\}$ and $h_* := \min\{h(x) : x \in \bar{\Omega}\}$, here $h(\cdot)$ represents the coefficients of (2).

Define $A_1\psi := \nabla \cdot [D_S(\cdot)\nabla\psi]$, $A_2\psi := \nabla \cdot [D_I(\cdot)\nabla\psi]$, $A_3\psi := \nabla \cdot [d_S(\cdot)\nabla\psi]$ and $A_4\psi := \nabla \cdot [d_I(\cdot)\nabla\psi]$ for A_i: $O(A_i) \to C(\bar{\Omega}, \mathbb{R})$ where $O(A_i) = \{\psi \in \cap_{p\in\mathbb{Z}^+} W^2_p(\Omega) : \partial_{\mathbf{n}}\psi = 0 \text{ on } \partial\Omega, A_i\psi \in C(\bar{\Omega}, \mathbb{R})\}$. Let

$\mathbb{A} := \mathrm{diag}\{A_1, A_2, A_3, A_4\}$. Then $\mathbb{A}$ is the infinitesimal generator of the C_0-semigroup $\{e^{t\mathbb{A}}\}_{t\geq 0}$ in X. Moreover, define the nonlinear operator $\mathbb{F} : X \to X$ by

$$\mathbb{F}(\varrho)(\cdot) = \begin{pmatrix} \Lambda(\cdot) - f_1(\cdot, \varrho_1, \varrho_4) - \mu_1(\cdot)\varrho_1 \\ f_1(\cdot, \varrho_1, \varrho_4) - \gamma_1(\cdot)\varrho_2 \\ M(\cdot) - f_2(\cdot, \varrho_3, \varrho_2) - \mu_2(\cdot)\varrho_3 \\ f_2(\cdot, \varrho_3, \varrho_2) - \gamma_2(\cdot)\varrho_4 \end{pmatrix}, \quad \varrho = (\varrho_1, \varrho_2, \varrho_3, \varrho_4)^T \in X^+,$$

here T denotes the transposition. Thus, system (2) is transformed into the following abstract differential system:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(t, \cdot; \mathbf{u}_0) = \mathbb{A}\mathbf{u}(t, \cdot; \mathbf{u}_0) + \mathbb{F}(\mathbf{u}(t, \cdot; \mathbf{u}_0)), \\ \mathbf{u}(0, \cdot; \mathbf{u}_0) = \mathbf{u}_0, \end{cases} \tag{3}$$

wherein $\mathbf{u} = (u_1, u_2, u_3, u_4)^T := (S_h, I_h, S_v, I_v)^T$ and $\mathbf{u}_0 := (S_h^0, I_h^0, S_v^0, I_v^0)^T$. Then the well-posedness statements are as follows:

Theorem 1 *For any $\mathbf{u}_0(\cdot) \in X^+$, system (3) possesses a unique nonnegative solution $\mathbf{u}(t, \cdot; \mathbf{u}_0)$ on $[0, \infty) \times \bar{\Omega}$. Furthermore, the solution semiflow $\Phi(t)\mathbf{u}_0 := \mathbf{u}(t, \cdot; \mathbf{u}_0)$ admits a global compact attractor in X^+.*

3.2 Threshold dynamics

To derive the basic reproduction ratio of (2). Set $Y := C(\bar{\Omega}, \mathbb{R}^2)$ and $Y^+ := C(\bar{\Omega}, \mathbb{R}^2_+)$. By the proof of Theorem (1), the disease-free steady state of (2) is $E_0 = (H(x), 0, W(x), 0)$. Linearizing system (2) at E_0 to obtain

$$\begin{cases} \partial_t \bar{I}_h = \nabla \cdot [D_I(x)\nabla \bar{I}_h] + k_1(x, H)\bar{I}_v - \gamma_1(x)\bar{I}_h, & t > 0, \ x \in \Omega, \\ \partial_t \bar{I}_v = \nabla \cdot [d_I(x)\nabla \bar{I}_v] + k_2(x, W)\bar{I}_h - \gamma_2(x)\bar{I}_v, & t > 0, \ x \in \Omega, \\ \partial_{\mathbf{n}} \bar{I}_h = \partial_{\mathbf{n}} \bar{I}_v = 0, & t > 0, \ x \in \partial\Omega, \end{cases} \tag{4}$$

where $k_1(x, H) := \partial_{I_v} f_1(x, H, 0)$ and $k_2(x, W) := \partial_{I_h} f_2(x, W, 0)$. For $\upsilon := (\upsilon_1, \upsilon_2)^T \in Y^+$, define the operators $\mathbf{F}, \mathbf{B} : Y \to Y$ by

$$\mathbf{F}(x)\upsilon = \begin{pmatrix} k_1(x, H)\upsilon_2 \\ k_2(x, W)\upsilon_1 \end{pmatrix}, \quad -\mathbf{B}(x)\upsilon = \begin{pmatrix} \nabla \cdot [D_I(x)\nabla \upsilon_1] - \gamma_1(x)\upsilon_1 \\ \nabla \cdot [d_I(x)\nabla \upsilon_2] - \gamma_2(x)\upsilon_2 \end{pmatrix}.$$

Then $\mathbf{F}$ is positive in Y^+ in the sense that $\mathbf{F}(x)Y^+ \subset Y^+$. Let $\hat{T}(t)$ be the semigroup generated by $\mathrm{d}\upsilon/\mathrm{d}t = -\mathbf{B}(x)\upsilon$ subject to the Neumann boundary condition. One can see that $\hat{T}(t)$ is a positive C_0-semigroup in Y^+. Suppose $\upsilon(x)$ is the initial density distribution of infected humans and mosquitoes at x. Then $\hat{T}(t)\upsilon(x)$ represents the density distribution for infected at t. Thus, $\int_0^\infty \mathbf{F}(x)\hat{T}(t)\upsilon(x)\mathrm{d}t$ means the density distribution of the accumulative new infections. Let $\mathfrak{L}[\upsilon](x) := \int_0^\infty \mathbf{F}(x)\hat{T}(t)\upsilon(x)\mathrm{d}t$. By employing the next generation operator approach in Wang and Zhao (2012) (see also Diekmann et al. 1990), the spectral radius of $\mathfrak{L}$ is defined as the basic reproduction ratio of (2), i.e., $\mathcal{R}_0 := r(\mathfrak{L})$.

Hence, there are the global dynamic results of (2) in terms of $\mathcal{R}_0$ as follows:

Theorem 2 *Assume that (P1)–(P2) hold. Then*

(i) *If $\mathcal{R}_0 \le 1$, then the disease-free steady state E_0 is globally asymptotically stable, and unstable if $\mathcal{R}_0 > 1$;*
(ii) *If $\mathcal{R}_0 > 1$, then there exists a $\delta_0 > 0$, such that the solution of (2) satisfies*

$$\liminf_{t\to\infty} \|(S_h(t,x), I_h(t,x), S_v(t,x), I_v(t,x)) - (H(x), 0, W(x), 0)\| > \delta_0 \tag{5}$$

uniformly for $x \in \bar{\Omega}$. Moreover, system (2) admits at least one endemic steady state.

Remark 2 Biologically, the conclusions of Theorem 2 show that the mosquito-borne disease will disappear when $\mathcal{R}_0 \le 1$ and will persist when $\mathcal{R}_0 > 1$.

When all coefficients of (2) are positive constants, system (2) is rewritten as

$$\begin{cases} \partial_t S_h = D_S \Delta S_h + \Lambda - f_1(S_h, I_v) - \mu_1 S_h, & t > 0,\ x \in \Omega, \\ \partial_t I_h = D_I \Delta I_h + f_1(S_h, I_v) - \gamma_1 I_h, & t > 0,\ x \in \Omega, \\ \partial_t S_v = d_S \Delta S_v + M - f_2(S_v, I_h) - \mu_2 S_v, & t > 0,\ x \in \Omega, \\ \partial_t I_v = d_I \Delta I_v + f_2(S_v, I_h) - \gamma_2 I_v, & t > 0,\ x \in \Omega, \\ \partial_{\mathbf{n}} S_h = \partial_{\mathbf{n}} I_h = \partial_{\mathbf{n}} S_v = \partial_{\mathbf{n}} I_v = 0, & t > 0,\ x \in \partial\Omega. \end{cases} \tag{6}$$

If $\mathcal{R}_0 > 1$, then Theorem 2 (ii) indicates that system (6) admits an endemic equilibrium $E_1^* = (S_h^*, I_h^*, S_v^*, I_v^*)$. Thus, the global stability results of E_1^* read as follows:

Theorem 3 *Assume that (P1)–(P2) hold, and $f_1(S_h, I_v) = S_h g_1(I_v)$ and $f_2(S_v, I_h) = S_v g_2(I_h)$ for some positive function $g_i(\cdot)$, $i = 1, 2$. If $\mathcal{R}_0 > 1$, then the endemic equilibrium E_1^* is globally asymptotically stable.*

Remark 3 It is not difficult to see that the function $f_i(\cdot,\cdot)$ can take the forms of (i), (ii) and (iii) in Remark 1.

3.3 Asymptotic profiles in terms of $\mathcal{R}_0$

In Sects. 3.3 and 3.4, without loss of generality, we assume that $D_I(x) = D_{I0} \cdot \bar{D}_I(x)$, $d_I(x) = d_{I0} \cdot \bar{d}_I(x)$, where D_{I0}, d_{I0} and $\bar{D}_I(x), \bar{d}_I(x)$ are positive constants and $C^{1+\nu}$ functions in $\bar{\Omega}$, respectively. Actually, D_{I0} and d_{I0} can be interpreted as diffusion magnitude of infected humans and mosquitoes.

Denote $\bar{k}(\cdot) := k_1(\cdot, H)k_2(\cdot, W)$ and $\bar{\gamma}(\cdot) := \gamma_1(\cdot)\gamma_2(\cdot)$. We have the following asymptotic profiles statements:

Theorem 4 *Assume that (P1)–(P2) hold. Then*

(i) *Fix* $d_I(\cdot) > 0$. *If* $\gamma_1(\cdot) \in C^2(\bar{\Omega})$ *and* $\partial_{\mathbf{n}}\gamma_1(\cdot) = 0$ *on* $\partial\Omega$, *then* $\mathcal{R}_0 \to \frac{1}{\lambda_1}$ *as* $D_{I0} \to 0$, *where* λ_1 *is the smallest eigenvalue of the problem*

$$\begin{cases} -\nabla \cdot [d_I(x)\nabla\phi_2^*] + \gamma_2(x)\phi_2^* = \lambda_1^2 \dfrac{\bar{k}(x)}{\gamma_1(x)}\phi_2^*, & x \in \Omega, \\ \partial_{\mathbf{n}}\phi_2^* = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

Moreover, if $\gamma_2(\cdot) \equiv \gamma_2^0$ *and* $k_1(\cdot, H) \equiv k_1^0$ *are positive constants in* Ω, *then* $\mathcal{R}_0 \to \sqrt{\frac{k_1^0 \int_\Omega k_2(x,W)\mathrm{d}x}{\gamma_2^0 \int_\Omega \gamma_1(x)\mathrm{d}x}}$ *as* $D_{I0} \to \infty$;

(ii) *Fix* $D_I(\cdot) > 0$. *If* $\gamma_2(\cdot) \in C^2(\bar{\Omega})$ *and* $\partial_{\mathbf{n}}\gamma_2(\cdot) = 0$ *on* $\partial\Omega$, *then* $\mathcal{R}_0 \to \frac{1}{\lambda_2}$ *as* $d_{I0} \to 0$, *where* λ_2 *is the smallest eigenvalue of the problem*

$$\begin{cases} -\nabla \cdot [D_I(x)\nabla\phi_1^*] + \gamma_1(x)\phi_1^* = \lambda_2^2 \dfrac{\bar{k}(x)}{\gamma_2(x)}\phi_1^*, & x \in \Omega, \\ \partial_{\mathbf{n}}\phi_1^* = 0, & x \in \partial\Omega. \end{cases} \tag{8}$$

Moreover, if $\gamma_1(\cdot) \equiv \gamma_1^0$ *and* $k_2(\cdot, W) \equiv k_2^0$ *are positive constants in* Ω, *then* $\mathcal{R}_0 \to \sqrt{\frac{k_2^0 \int_\Omega k_1(x,H)\mathrm{d}x}{\gamma_1^0 \int_\Omega \gamma_2(x)\mathrm{d}x}}$ *as* $d_{I0} \to \infty$;

(iii) *As* $D_{I0} \to 0$ *and* $d_{I0} \to 0$, *then* $\mathcal{R}_0 \to \mathcal{R}_0^{\mathrm{loc}} := \max\left\{\sqrt{\bar{k}(x)/\bar{\gamma}(x)} : x \in \bar{\Omega}\right\}$;

(iv) *As* $D_{I0} \to \infty$ *and* $d_{I0} \to 0$, *then* $\mathcal{R}_0 \to \mathcal{R}_0^{\mathrm{a1}} := \sqrt{\frac{\int_\Omega \bar{k}(x)\gamma_2^{-1}(x)\mathrm{d}x}{\int_\Omega \gamma_1(x)\mathrm{d}x}}$; *As* $D_{I0} \to 0$ *and* $d_{I0} \to \infty$, *then* $\mathcal{R}_0 \to \mathcal{R}_0^{\mathrm{a2}} := \sqrt{\frac{\int_\Omega \bar{k}(x)\gamma_1^{-1}(x)\mathrm{d}x}{\int_\Omega \gamma_2(x)\mathrm{d}x}}$;

(v) *As* $D_{I0} \to \infty$ *and* $d_{I0} \to \infty$, *then* $\mathcal{R}_0 \to \mathcal{R}_0^{\mathrm{a}} := \sqrt{\frac{\int_\Omega k_1(x,H)\mathrm{d}x \int_\Omega k_2(x,W)\mathrm{d}x}{\int_\Omega \gamma_1(x)\mathrm{d}x \int_\Omega \gamma_2(x)\mathrm{d}x}}$.

Remark 4 (a) For Theorem 4 (i), multiplying the first equation of (7) by ϕ_2^* and then integrating by parts over Ω, we get

$$\lambda_1 = \inf_{\phi_2^* \in W_2^1(\Omega), \phi_2^* \neq 0} \left\{\sqrt{\frac{\int_\Omega d_I(x)|\nabla\phi_2^*|^2\mathrm{d}x + \int_\Omega \gamma_2(x)(\phi_2^*)^2\mathrm{d}x}{\int_\Omega \bar{k}(x)\gamma_1^{-1}(x)(\phi_2^*)^2\mathrm{d}x}}\right\}$$

which is due to the variational method in Cantrell and Cosner (2003). Thus,

$$\mathcal{R}_0 \to \frac{1}{\lambda_1} = \sup_{\phi_2^* \in W_2^1(\Omega), \phi_2^* \neq 0} \left\{\sqrt{\frac{\int_\Omega \bar{k}(x)\gamma_1^{-1}(x)(\phi_2^*)^2\mathrm{d}x}{\int_\Omega d_I(x)|\nabla\phi_2^*|^2\mathrm{d}x + \int_\Omega \gamma_2(x)(\phi_2^*)^2\mathrm{d}x}}\right\},$$

as D_{I0} goes arbitrarily small for $d_I(x) > 0$. If γ_2 and k_1 are constants, then $\mathcal{R}_0$ tends to $\mathcal{R}_0^{\mathrm{a}}$ (the product of the average of k_1 and the average of k_2 divided by the product of the average of γ_1 and the average of γ_2) as D_{I0} goes arbitrarily large. For Theorem 4 (ii), by (8), one similarly gets

$$\mathcal{R}_0 \to \frac{1}{\lambda_2} = \sup_{\phi_1^* \in W_2^1(\Omega), \phi_2^* \neq 0} \left\{ \sqrt{\frac{\int_\Omega \bar{k}(x)\gamma_2^{-1}(x)(\phi_1^*)^2 \mathrm{d}x}{\int_\Omega D_I(x)|\nabla \phi_1^*|^2 \mathrm{d}x + \int_\Omega \gamma_1(x)(\phi_1^*)^2 \mathrm{d}x}} \right\},$$

as d_{I0} goes arbitrarily small for $D_I(x) > 0$. If γ_1 and k_2 are constants, then $\mathcal{R}_0$ tends to $\mathcal{R}_0^{\mathrm{a}}$ as d_{I0} goes arbitrarily large; When both D_{I0} and d_{I0} go arbitrarily small, $\mathcal{R}_0$ tends to the maximum local basic reproduction ratio $\mathcal{R}_0^{\mathrm{loc}}$ as shown in Theorem 4 (iii) (see, e.g., Allen et al. 2008); When both D_{I0} and d_{I0} go arbitrarily large, $\mathcal{R}_0$ tends to $\mathcal{R}_0^{\mathrm{a}}$ as given by Theorem 4 (v);

(b) It can be seen from (i), (ii) and (iv) in Theorem 4 that, if D_I and d_I have different scales, then $\mathcal{R}_0$ tends to different forms. More precisely, when d_I goes arbitrarily small for fixed D_I, $\mathcal{R}_0$ is a monotone nonincreasing function of D_I which implies that $\mathcal{R}_0$ depends on the infected human mobility; When D_I goes arbitrarily small for fixed d_I, $\mathcal{R}_0$ is a monotone nonincreasing function of d_I which means that $\mathcal{R}_0$ depends on the infected mosquito flight ability; When D_I and d_I go arbitrarily large and small, respectively, $\mathcal{R}_0$ tends to $\mathcal{R}_0^{\mathrm{a1}}$ (the average of $\bar{k}\gamma_2^{-1}$ divided by the average of γ_1^{-1}); When D_I and d_I go arbitrarily small and large, respectively, $\mathcal{R}_0$ tends to $\mathcal{R}_0^{\mathrm{a2}}$ (the average of $\bar{k}\gamma_1^{-1}$ divided by the average of γ_2^{-1});

(c) To verify the conditions of Theorem 4 (i). Let $\alpha_1(x) = 1 + 0.3\cos x$, $\mu_1(x) = 0.5 - 0.1\cos x$ and $e_1(x) = 1 - 0.1\cos x$, $x \in \Omega := (0, \pi)$. Then $\gamma_1(x) = 2.5 + 0.1\cos x$ and $\gamma_1'(x) = -0.1\sin x$ which yields that $\gamma_1'(0) = \gamma_1'(\pi) = 0$. The conditions of (ii) can be similarly testified.

3.4 Monotonicity in terms of $\mathcal{R}_0$

In this subsection, for convenience, we assume the domain Ω is one-dimensional. Then the conclusions about the monotonicity of $\mathcal{R}_0$ read as follows:

Theorem 5 *Assume that (P1)–(P2) hold, and $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are positive constants in Ω, and $\bar{D}_I(\cdot)$, $\bar{d}_I(\cdot) \in C^2(\bar{\Omega})$ satisfying*

$$\frac{\bar{D}_I'(x)}{\bar{D}_I(x)} = \frac{\bar{d}_I'(x)}{\bar{d}_I(x)}, \quad \text{for all } x \in \Omega,$$

where $'$ denotes the first derivative w.r.t x. Then

(i) *If $k_2(\cdot, W)$ is a constant and $\bar{d}_I''(\cdot)d_{I0} < \gamma_2$ in Ω, then $\mathcal{R}_0$ is a monotone non-increasing function of D_{I0}. Furthermore, if $k_1(\cdot, H)$ is non-constant, then $\mathcal{R}_0$ decreases monotonically in respect of D_{I0};*

(ii) *If $k_1(\cdot, H)$ is a constant and $\bar{D}_I''(\cdot)D_{I0} < \gamma_1$ in Ω, then $\mathcal{R}_0$ is a monotone non-increasing function of d_{I0}. Furthermore, if $k_2(\cdot, W)$ is non-constant, then $\mathcal{R}_0$ decreases monotonically w.r.t d_{I0};*

where $''$ denotes the second derivative w.r.t x.

Remark 5 (a) The results of Theorem 5 show that the rapid movement of infected humans or mosquitoes is conducive to disease control under certain conditions, perhaps because this will reduce the contact with susceptible individuals or obtain more medical resources;

(b) The hypothesis $\bar{D}_I'(\cdot)/\bar{D}_I(\cdot) = \bar{d}_I'(\cdot)/\bar{d}_I(\cdot)$, mathematically, is a technical condition. In the biological sense, inspired by the explanation in Zhou and Xiao (2018), this indicates that the movement strategies of infected humans and mosquitoes are proportional in Ω, i.e., $\bar{D}_I'(\cdot)/\bar{D}_I(\cdot)$ and $\bar{d}_I'(\cdot)/\bar{d}_I(\cdot)$ have the same scale. Moreover, the hypothesises $\bar{D}_I''(\cdot)D_{I0} < \gamma_1$ and $\bar{d}_I''(\cdot)d_{I0} < \gamma_2$ are also technical conditions which mean that the second derivatives of D_I and d_I have upper bounds γ_1 and γ_2, respectively;

(c) To verify the conditions of Theorem 5. Let $\bar{D}_I(x) = 2(1+x)$ and $\bar{d}_I(x) = 1+x$, $x \in \Omega := (0,\pi)$ and $D_{I0} = d_{I0} = 1$. Then $\bar{D}_I'(x)/\bar{D}_I(x) = \bar{d}_I'(x)/\bar{d}_I(x) = (1+x)^{-1}$, and $0 = \bar{D}_I''(x)D_{I0} < \gamma_1$ and $0 = \bar{d}_I''(x)d_{I0} < \gamma_2$, $x \in (0,\pi)$.

In what follows, we discuss the monotonicity of $\mathcal{R}_0$ w.r.t $D_I(\cdot)$ and $d_I(\cdot)$. Denote $\tilde{D}_I(\cdot) := D_I''(\cdot)/D_I'(\cdot)$ and $\tilde{d}_I(\cdot) := d_I''(\cdot)/d_I'(\cdot)$, here $D_I'(\cdot) \neq 0$ and $d_I'(\cdot) \neq 0$. There are the following results:

Theorem 6 *Assume that (P1)–(P2) hold, and* $D_I(x), d_I(x) \in C^3(\bar{\Omega})$ *and* $k_1(x,H) \equiv k_2(x,W)$ *for all* $x \in \bar{\Omega}$. *Then*

(i) *If* $\tilde{D}_I(\cdot)\phi_1^2(\cdot)|_{\partial\Omega} \leq 0$ *and* $\tilde{D}_I'(\cdot) \geq 0$ *in* Ω, *then* $\mathcal{R}_0$ *is a monotone nonincreasing function of* D_I;

(ii) *If* $\tilde{d}_I(\cdot)\phi_2^2(\cdot)|_{\partial\Omega} \leq 0$ *and* $\tilde{d}_I'(\cdot) \geq 0$ *in* Ω, *then* $\mathcal{R}_0$ *is a monotone nonincreasing function of* d_I.

Remark 6 (a) Similar to the interpretation of Theorem 5 in Remark 5 (a), the conclusions of Theorem 6 mean that $\mathcal{R}_0$ is a monotonic nonincreasing function of $D_I(x)$ and $d_I(x)$, respectively, under some conditions;

(b) Mathematically, the hypothesises in Theorem 6 are technical conditions. To verify them, letting $D_I(x) = 2+x$ and $d_I(x) = 1+x$, $x \in \Omega := (0,\pi)$. Then $\tilde{D}_I(x)\phi_1^2(x) \equiv 0$, $x \in \{0,\pi\}$ and $\tilde{D}_I'(x) \equiv 0$, $x \in (0,\pi)$.

Above, we investigate the monotonicity of $\mathcal{R}_0$ in terms of diffusion rates. However, this monotonicity does not always hold. The following results illustrate this situation.

Theorem 7 *Assume that (P1)–(P2) hold. Then*

(i) *If*

$$\frac{\int_\Omega k_1(x,H)\mathrm{d}x \int_\Omega k_2(x,W)\mathrm{d}x}{\int_\Omega \gamma_2(x)\mathrm{d}x} > \int_\Omega \frac{\bar{k}(x)}{\gamma_2(x)}\mathrm{d}x, \tag{9}$$

then there are positive constants D_{I0}^*, d_{I0}^1 *and* d_{I0}^2, *such that* $\mathcal{R}_0(D_I^*, d_I^1) < \mathcal{R}_0(D_I^*, d_I^2)$, *wherein* $D_I^* := D_{I0}^*\bar{D}_I(x)$, $d_I^i := d_{I0}^i\bar{d}_I(x)$ *and* $d_{I0}^1 < d_{I0}^2$;

(ii) *If*

$$\frac{\int_\Omega k_1(x,H)\mathrm{d}x \int_\Omega k_2(x,W)\mathrm{d}x}{\int_\Omega \gamma_1(x)\mathrm{d}x} > \int_\Omega \frac{\bar{k}(x)}{\gamma_1(x)}\mathrm{d}x, \tag{10}$$

then there are positive constants D_{I0}^1, D_{I0}^2 *and* d_{I0}^*, *such that* $\mathcal{R}_0(D_I^1, d_I^*) < \mathcal{R}_0(D_I^2, d_I^*)$, *wherein* $D_{I0}^i := D_{I0}^i \bar{D}_I(x)$, $D_{I0}^1 < D_{I0}^2$ *and* $d_I^* := d_{I0}^* \bar{d}_I(x)$.

Remark 7 (a) The conclusions of Theorem 7 indicate that when the coefficients of (2) satisfy (9) or (10), $\mathcal{R}_0$ no longer has monotonicity w.r.t D_I or d_I, which is in sharp contrast with Theorems 5 and 6;

(b) To testify the conditions of Theorem 7. Let $\Omega = (0, 1)$, $k_1(x, H) = \sin(\pi x/2)+\alpha$, $k_2(x, W) = \cos(\pi x/2)+\alpha$ and $\gamma_2(x) = 1$, here α represents any positive constant greater than 1. Then

$$\left(\alpha + \frac{2}{\pi}\right)^2 = \frac{\int_\Omega k_1(x,H)\mathrm{d}x \int_\Omega k_2(x,W)\mathrm{d}x}{\int_\Omega \gamma_2(x)\mathrm{d}x} > \int_\Omega \frac{\bar{k}(x)}{\gamma_2(x)}\mathrm{d}x = \frac{1}{\pi} + \frac{4}{\pi}\alpha + \alpha^2.$$

The verification of condition (10) is analogous;

(c) Combining Theorems 5–7, it can seen that, due to the heterogeneity diffusion of humans and mosquitoes, the monotonicity of $\mathcal{R}_0$ becomes more complicated. In other words, the heterogeneous mobility of individuals has important impacts on mosquito-borne disease dynamics.

4 Numerical simulations

We in this section apply model (1) to the dengue fever transmission and then provide simulations to substantiate the theoretical results and study the effects of some factors on $\mathcal{R}_0$ and disease dynamics. For convenience, we assume $\Omega := (0, \pi)$, $f_1(\cdot, S_h, I_v) = k_1(\cdot)S_h I_v$ and $f_2(\cdot, S_v, I_h) = k_2(\cdot)S_v I_h$, where $k_i(\cdot)$ denotes the disease transmission rate and is positive function in Ω, $i = 1, 2$.

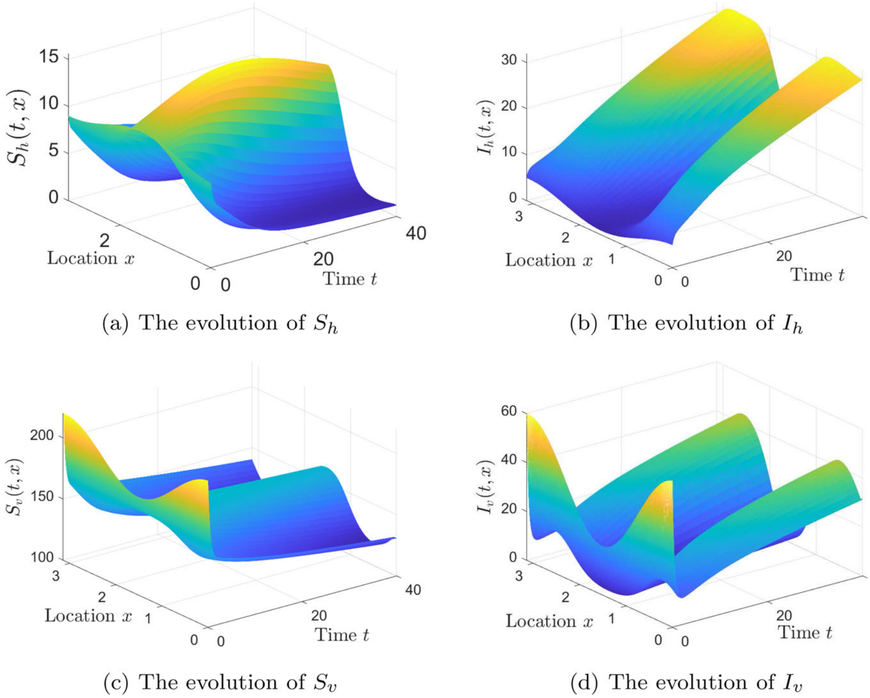
4.1 Long term behavior

In this subsection, numerical simulations are presented to testify the conclusions obtained in Sect. 3.2. All parameters and their definitions as well as their values are listed in Table 1.

To reflect the fact the transmission capacity of dengue fever in urban area (around the center of the spatial domain) is smaller than that in rural area (boundary of the spatial domain) since there are fewer mosquitoes in urban area, we fix $k_1(x) = 0.00682(1.5 + 0.8\sin 3x)$ and $k_2(x) = 0.015(1 + 0.8\sin 3x)$, $x \in (0, \pi)$. In addition, taking $\alpha_1 = 0.00168\ \text{month}^{-1}$ (see Wu and Zhao 2019) and other parameters are determined by Table 1. Then $\mathcal{R}_0 = 1.91 > 1$. Figure 1 shows the corresponding long term behaviors

Table 1 Parameters values in simulation

Parameter	Description	Value	Dimension	Sources
Λ	Recruitment rate of humans	0.5054	$(\mathrm{km}^2\ \mathrm{month})^{-1}$	Assumed
μ_1	Natural death rate of humans	$1/(75 \times 12)$	month^{-1}	Zhao et al. (2020)
e_1	Disease-induced death rate of I_h	0.001	month^{-1}	Wang and Zhao (2011)
α_1	Recovery rate of I_h	30.4/6	month^{-1}	Andraud et al. (2013)
M	Recruitment rate of mosquitoes	346.56	$(\mathrm{km}^2\ \mathrm{month})^{-1}$	Zhang and Wang (2020)
μ_2	Natural death rate of mosquitoes	30.4/14.49	month^{-1}	Zhang and Wang (2020)
e_2	Disease-induced death rate of I_v	0.2	month^{-1}	Wang and Zhao (2011)
D_S	Diffusion rate of S_h	0.1	$\mathrm{km}^2\ \mathrm{month}^{-1}$	Wu and Zhao (2019)
D_I	Diffusion rate of I_h	0.05	$\mathrm{km}^2\ \mathrm{month}^{-1}$	Assumed
d_S	Diffusion rate of S_v	0.0125	$\mathrm{km}^2\ \mathrm{month}^{-1}$	Wu and Zhao (2019)
d_I	Diffusion rate of I_v	0.005	$\mathrm{km}^2\ \mathrm{month}^{-1}$	Assumed



(a) The evolution of S_h (b) The evolution of I_h

(c) The evolution of S_v (d) The evolution of I_v

Fig. 1 The spatial–temporal evolution of system (2) when $\mathcal{R}_0 = 1.91$

of (2) in the case of $\mathcal{R}_0 = 1.91$, fulfilling the following initial conditions:

$$\begin{pmatrix} S_h(0,x) \\ I_h(0,x) \\ S_v(0,x) \\ I_v(0,x) \end{pmatrix} = \begin{pmatrix} 10 - \cos 2x \\ 3 + 2\cos 2x \\ 200 + 20\cos 2x \\ 40 + 20\cos 2x \end{pmatrix}, \quad x \in [0, \pi]. \tag{11}$$

From Fig. 1, one can see that system (2) is persistent when $\mathcal{R}_0 = 1.91$ which is coincident with Theorem 2 (ii). Furthermore, in order to present the shape of I_h and I_v more clearly, the cross section curves at different times are depicted in Fig. 2. It is noted that the densities of infected humans and mosquitoes in urban area are lower than that in rural area from Fig. 2, which suggests that it should pay more attention to the impact of heterogeneity on dengue fever.

To control dengue fever, the human mortality rate is reduced to $0.5e_1$ via increasing the medical resources and improving cure rate, and the mosquito recruitment rate is decreased to $0.5M$ and natural death rate is increased to $3.5\mu_2$ through utilizing bed nets and spraying insecticides. Then $\mathcal{R}_0 = 0.76 < 1$. By Theorem 2 (i), the densities of infected humans and mosquitoes tend to zero, which means that the disease will eventually be eliminated (as shown in Fig. 3).

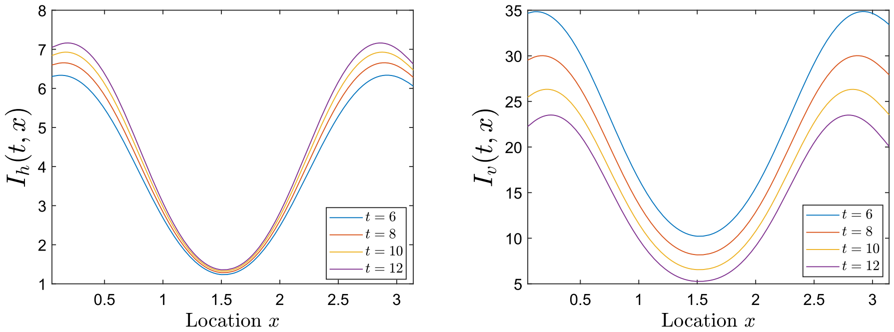
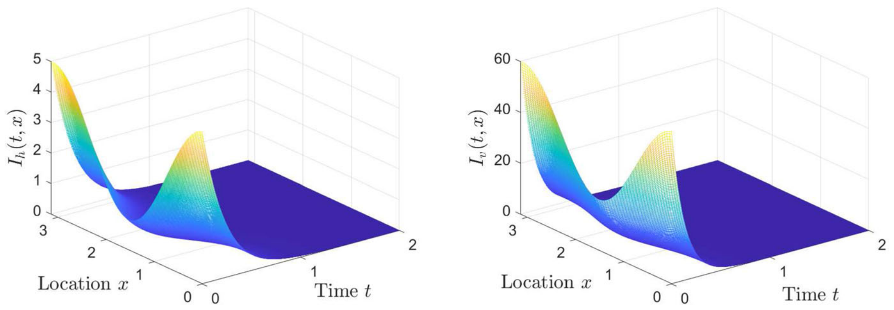


Fig. 2 The cross section curves of I_h and I_v at different times in Fig. 1



(a) The evolution of I_h (b) The evolution of I_v

Fig. 3 The spatial–temporal evolution of infected humans and mosquitoes when $\mathcal{R}_0 = 0.76$

4.2 Sensitivity analysis of $\mathcal{R}_0$ on parameters

To identify which parameters are sensitive to $\mathcal{R}_0$, we perform a sensitivity analysis by evaluating the partial rank correlation coefficients (PRCCs) for all input parameters against $\mathcal{R}_0$ (Marino et al. 2008). In this subsection, we take $k_1 = 0.00682$ and $k_2 = 0.075$ and other parameters are shown in Table 1. In Fig. 4a–i, the abscissa represents a uniform distribution for all input parameters with the minimum and maximum values and the ordinate represents $\mathcal{R}_0$. It can be seen from Fig. 4 that Λ, M, k_1 and k_2 have positive effects on $\mathcal{R}_0$, while α_1, μ_1 and μ_2 have negative effects on $\mathcal{R}_0$. Other parameters e_1 and e_2 have insignificant influences on $\mathcal{R}_0$. The results depicted in Fig. 4j suggest that relative to μ_1, α_1 and μ_2 are more sensitive. Biologically, it is interesting to note that the human recovery rate and the mosquito natural death rate have significant impacts on dengue fever which indicate that inhibiting mosquito reproduction and improving the cure rate are the preferred measures to control dengue fever.

The sensitivity of $\mathcal{R}_0$ has been illustrated in Fig. 4. However, the sensitivity may vary with some parameters (Wang et al. 2019). Based on this, we next discuss the sensitivity to some parameters in different parameter domains. From Fig. 5a, it can be found that $\mathcal{R}_0$ increases with the increase of k_1 when μ_2 is small, but $\mathcal{R}_0$ does not vary significantly when μ_2 is large. Similarly, $\mathcal{R}_0$ increases as μ_2 decreases when k_1 is

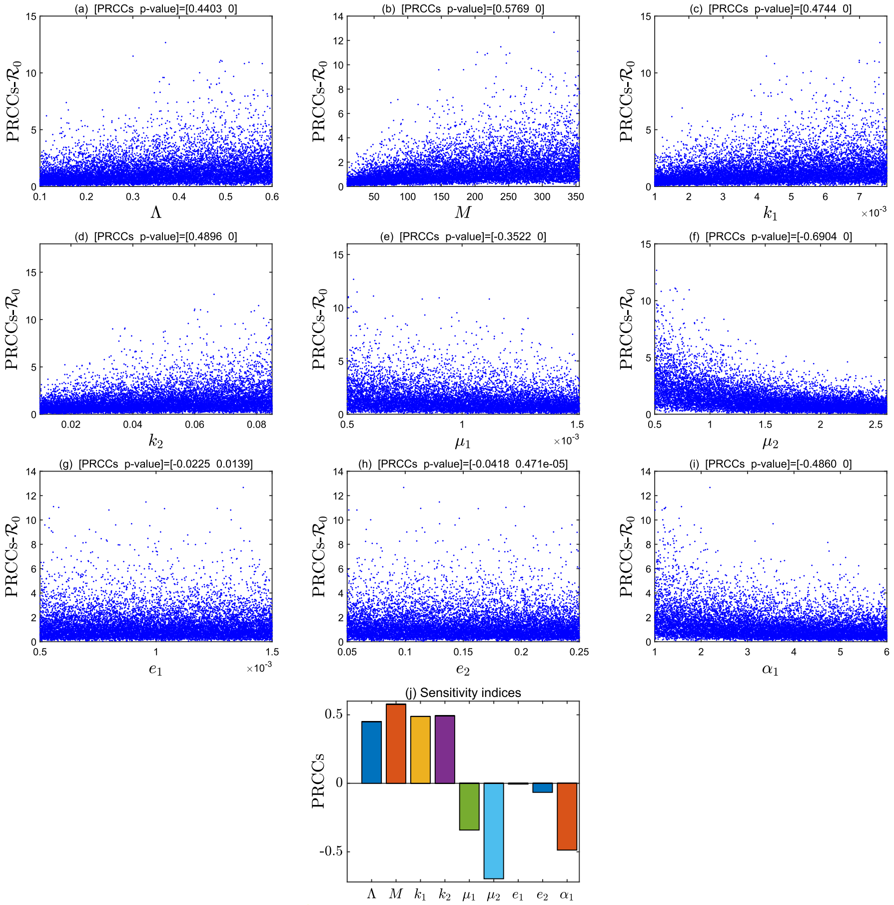


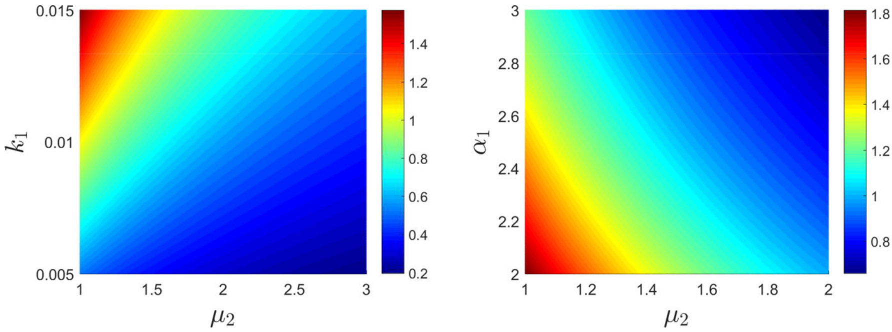
Fig. 4 Sensitivity of $\mathcal{R}_0$ to parameters of system (2)

large, while $\mathcal{R}_0$ does not vary significantly when k_1 is small. In Fig. 5b, $\mathcal{R}_0$ increases with the decrease of α_1 when μ_2 is small, but $\mathcal{R}_0$ does not vary significantly as μ_2 is large. On the other hand, $\mathcal{R}_0$ increases as μ_2 decreases when α_1 is small, while $\mathcal{R}_0$ does not vary significantly when α_1 is large. The phenomena shown in Figs. 4 and 5 picture that sensitivity analysis of $\mathcal{R}_0$ is meaningful so as to do a better prevention against dengue fever.

4.3 The effects of parameters on disease dynamics

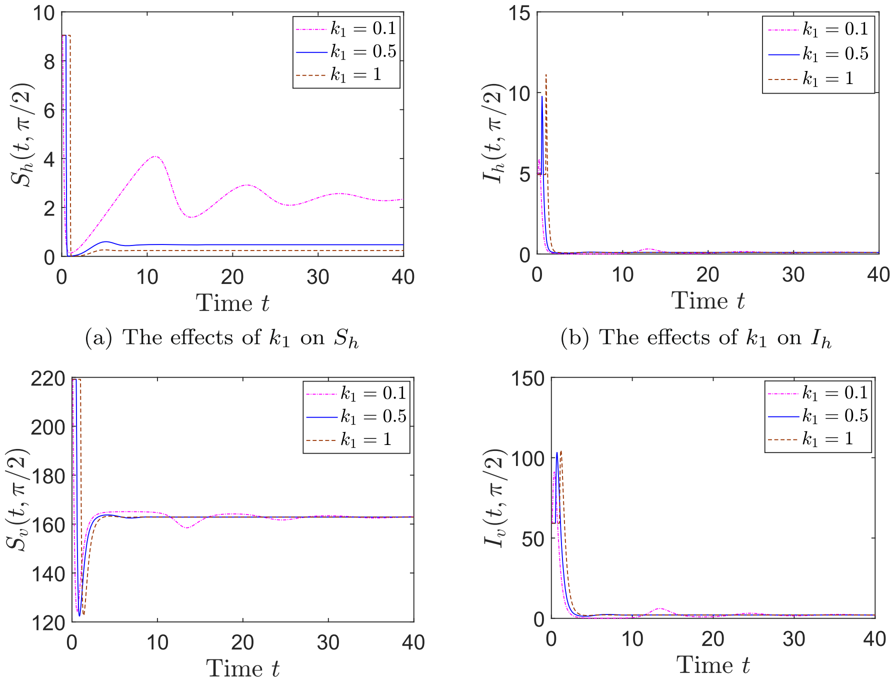
This subsection explores the control measures of dengue fever by studying the effects of transmission rate k_1 and recovery rate α_1 on dynamic behaviors of (2) when $\mathcal{R}_0 > 1$.

Firstly, we fix $k_2 = 0.3$ and spatial location $x = \pi/2$, and vary k_1 from 0.1 to 1. The values of other parameters are shown in Table 1 and the initial values satisfy (11). Figure 6 illustrates the influence of k_1 on solutions. Some noteworthy phenomena are



(a) Sensitivity of $\mathcal{R}_0$ to k_1 and μ_2 (b) Sensitivity of $\mathcal{R}_0$ to α_1 and μ_2

Fig. 5 Sensitivity of $\mathcal{R}_0$ to k_1, α_1 and μ_2



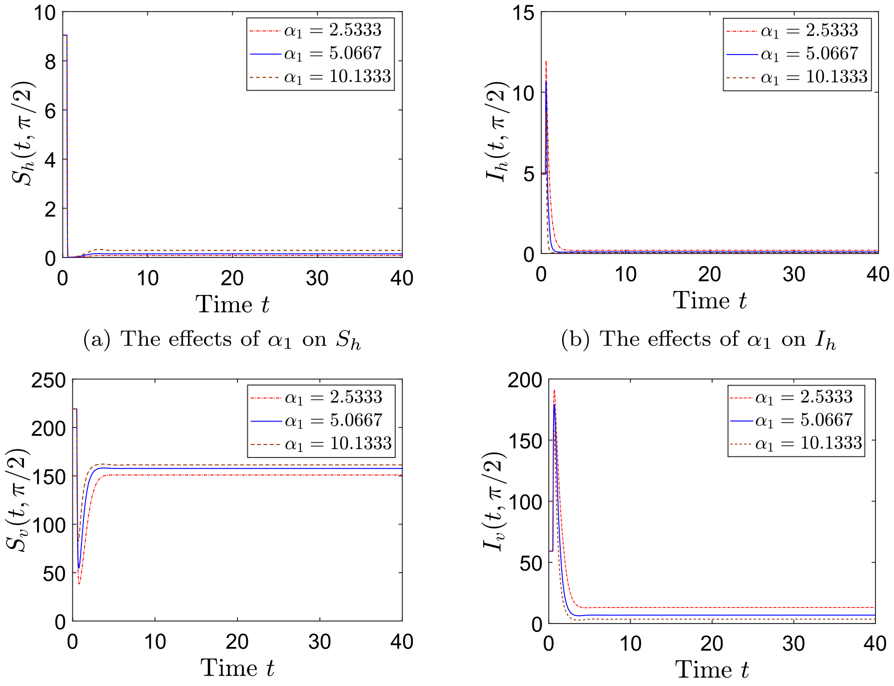
(a) The effects of k_1 on S_h (b) The effects of k_1 on I_h

(c) The effects of k_1 on S_v (d) The effects of k_1 on I_v

Fig. 6 The effects of k_1 on dynamics of humans and mosquitoes when $\mathcal{R}_0 > 1$

found from Fig. 6. More precisely, the increase of k_1 will not only increase the time for the densities of infected humans and mosquitoes to reach the peak, but also lead to the increase of the peak (Fig. 6b, d). These findings suggest that necessary measures, such as spraying insecticides and using bed nets, should be taken at the beginning of dengue fever to reduce the biting rate and thus the disease transmission rate.

Secondly, we let $k_1 = 0.5, k_2 = 1$ and $x = \pi/2$, and make α_1 from 2.5333 month^{-1} to 10.1333 month^{-1}. The values of other coefficients are shown in Table 1. From



(a) The effects of α_1 on S_h

(b) The effects of α_1 on I_h

(c) The effects of α_1 on S_v

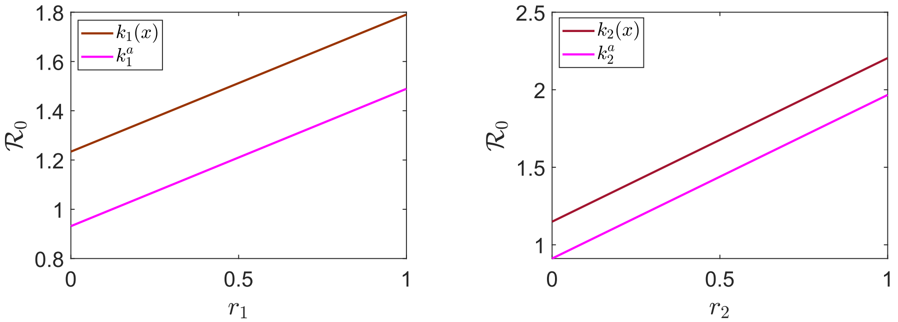
(d) The effects of α_1 on I_v

Fig. 7 The effects of α_1 on dynamics of humans and mosquitoes when $\mathcal{R}_0 > 1$

Fig. 7b, d, the increase of α_1 will not only decrease the time for the densities of infected humans and mosquitoes to reach the peak, but also decrease the peak. In addition, the impact of α_1 becomes significant with the passage of time. It thus seems imperative to improve the cure rate of patients and implement mosquito control measures.

4.4 The effects of spatial heterogeneity on $\mathcal{R}_0$

In this subsection, we discuss the influence of spatial heterogeneity on dengue fever by investigating the dependence of $\mathcal{R}_0$ on disease transmission rate k_i, $i = 1, 2$. To do this, we first let $k_1(x) = 0.00682(1.5 + 0.8 \sin 3x + r_1)$, $r_1 \in [0, 1]$, $k_2 = 0.075$ and other parameters are the same as Table 1. Then the average of $k_1(\cdot)$ on Ω is $k_1^a := \frac{1}{\Omega}\int_\Omega k_1(x)\mathrm{d}x = 0.01139 + 0.00682r_1$. By Fig. 8a, $\mathcal{R}_0$ increases with the increase of r_1, and the pink curve is always below the brown curve, wherein the brown and the pink curves represent the effects of k_1 and k_1^a on $\mathcal{R}_0$. Moreover, we set $k_1 = 0.0682$, $k_2(x) = 0.015(0.5 + 0.3 \cos 3x + r_2)$, $r_2 \in [0, 1]$, and other parameters are given in Table 1. Thus, the average of $k_2(\cdot)$ on Ω is $k_2^a := \frac{1}{\Omega}\int_\Omega k_2(x)\mathrm{d}x = 0.01295 + 0.015r_2$. From Fig. 8b, $\mathcal{R}_0$ is an increasing function of r_2, and the pink curve is always below the brown curve, where the brown and the pink curves denote the effects of k_2 and k_2^a on $\mathcal{R}_0$. Consequently, it can be seen from Fig. 8 that (i) the use of spatial-averaged disease



(a) $k_1(x)$ vs k_1^a (b) $k_2(x)$ vs k_2^a

Fig. 8 The influence of spatial heterogeneity on $\mathcal{R}_0$

transmission rate may underestimate disease infection level; (ii) spatial heterogeneity has an important impact on dengue fever and therefore cannot be ignored in modeling.

4.5 The impacts of temperature on $\mathcal{R}_0$

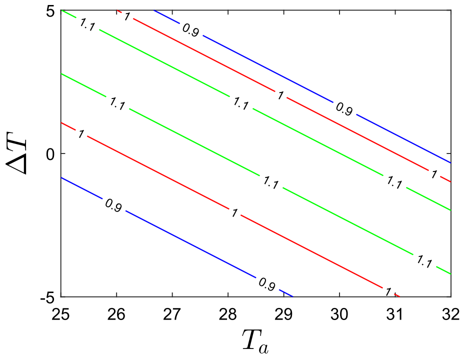
As mentioned in Sect. 1, spatial heterogeneity is closely related to environmental temperature in mosquito-borne diseases. As *Aedes aegypti* is the main vector of dengue fever and its living environment is deeply affected by temperature (Li et al. 2019; Vaidya et al. 2019; Yang et al. 2009), temperature is a factor that cannot be ignored in the study of dengue fever. To explore the effect of temperature, according to Wang and Zhao (2021), Yang et al. (2009), we take the natural death rate of mosquitoes μ_2 as

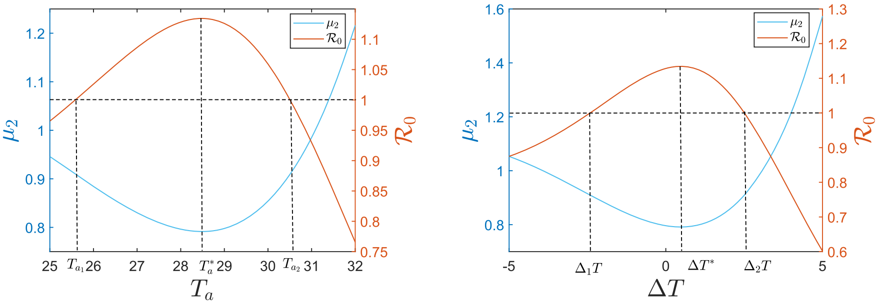
$$\mu_2 = 30.4167(0.8692 - 0.1599\widetilde{T} + 0.01116\widetilde{T}^2 - 3.408 \times 10^{-4}\widetilde{T}^3 + 3.809 \times 10^{-6}\widetilde{T}^4) \text{ month}^{-1},$$

here $\widetilde{T}$ represents the temperature in Celsius. Fix $k_1 = 0.00282$, $k_2 = 0.0085$. And, other parameters are shown in Table 1. Introducing two temperature indexes: T_a (mean temperature) and ΔT (augmenter temperature which is increasing or decreasing temperature from mean temperature), then $\widetilde{T} = T_a + \Delta T$, where ΔT varies from -5 to 5. Thus, by (Wang and Zhao (2012), Theorem 3.4), $\mathcal{R}_0 = \sqrt{k_1 k_2 \Lambda M/(\mu_1 \mu_2 \gamma_1 \gamma_2)}$ since the coefficients of (2) are positive constants.

The dependence of T_a and ΔT on $\mathcal{R}_0$ is depicted in Fig. 9 which is helpful for examining the joint effects of T_a and ΔT. To study the impacts of T_a in more detail, fixing $\Delta T = 0.5$ and changing T_a from 25 to 32. Figure 10a gives that μ_2 decreases as T_a increases when $T_a < T_a^*$, and increases when $T_a > T_a^*$. By Fig. 10a, too high or low average temperature is not conducive to mosquito survival. Hence, $\mathcal{R}_0$ increases as T_a increases when $T_a < T_a^*$, and decreases when $T_a > T_a^*$, and $\mathcal{R}_0 > 1$ when $T_{a_1} < T_a < T_{a_2}$ (see Fig. 10a). To explore the impacts of ΔT in more detail, fixing $T_a = 28.5$ and changing ΔT from -5 to 5. From Fig. 10b, μ_2 decreases with the increase of ΔT as $\Delta T < \Delta T^*$, and increases as $\Delta T > \Delta T^*$. According to

Fig. 9 Contour plot of $\mathcal{R}_0$: mean temperature T_a versus augmenter temperature $\varDelta T$





(a) Dependence of μ_2 and $\mathcal{R}_0$ on T_a when $\varDelta T = 0.5$

(b) Dependence of μ_2 and $\mathcal{R}_0$ on $\varDelta T$ when $T_a = 28.5$

Fig. 10 The dependence of μ_2 and $\mathcal{R}_0$ on T_a and $\varDelta T$

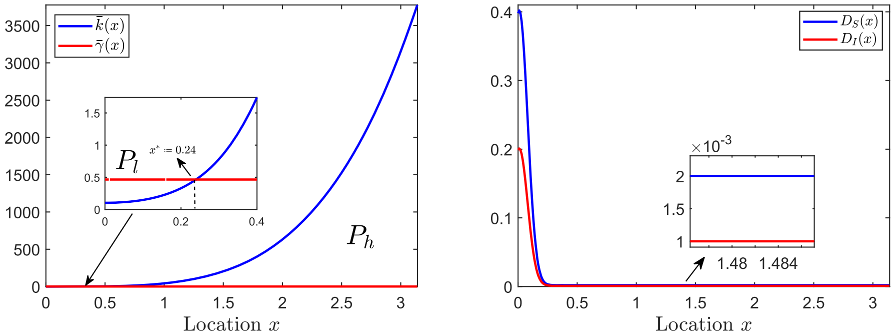
Fig. 10b, too high or low augmenter temperature increase mosquito mortality. Thus, $\mathcal{R}_0$ increases with the increase of $\varDelta T$ as $\varDelta T < \varDelta T^*$, and decreases as $\varDelta T > \varDelta T^*$, and $\mathcal{R}_0 > 1$ as $\Delta_1 T < T_a < \Delta_2 T$ (see Fig. 10b). In conclusion, too high or low average (or augmenter) temperature can inhibit the mosquito survival which reduces $\mathcal{R}_0$ and henceforth helping to reduce the risk level of dengue fever.

4.6 The impacts of individual mobility on disease dynamics

We in this subsection investigate the effects of heterogeneous and homogeneous diffusion rates of human and mosquito on $\mathcal{R}_0$ and dengue fever. Assuming $k_1(x) = 0.00682(2.5x^2+0.1)$, $k_2(x) = 0.02(1.5x^2+0.1)$, $x \in (0, \pi)$, and $\alpha_1 = 0.2\,\text{month}^{-1}$, and other parameters are determined by Table 1. Motivated by the ideas of Allen et al. (2008), we define

$$P_h := \{x \in \varOmega \mid \bar{k}(x) > \bar{\gamma}(x)\} \text{ and } P_l := \{x \in \varOmega \mid \bar{k}(x) < \bar{\gamma}(x)\}$$

as high- and low-risk sites, respectively, wherein $\bar{k}_1(\cdot) = k_1(\cdot)k_2(\cdot)\varLambda M/(\mu_1\mu_2)$ and $\bar{\gamma}(\cdot) = \gamma_1(\cdot)\gamma_2(\cdot)$. By Fig. 11a, $P_h = (0, 0.24)$ and $P_l = (0.24, \pi)$.



(a) Graphs of $\bar{k}(x)$ and $\bar{\gamma}(x)$ (b) Graphs of $D_S(x)$ and $D_I(x)$

Fig. 11 Graphs of $(\bar{k}(x), \bar{\gamma}(x))$ and $(D_S(x), D_I(x))$ in $(0, \pi)$

To compare the effects of different dispersal mechanisms on dengue fever, we consider model (2) with constant diffusion coefficients as follows

$$\begin{cases} \partial_t S_h = \hat{D}_S \Delta S_h + \Lambda(x) - k_1(x) S_h I_v - \mu_1(x) S_h, & t > 0,\ x \in \Omega, \\ \partial_t I_h = \hat{D}_I \Delta I_h + k_1(x) S_h I_v - \gamma_1(x) I_h, & t > 0,\ x \in \Omega, \\ \partial_t S_v = \hat{d}_S \Delta S_v + M(x) - k_2(x) S_v I_h - \mu_2(x) S_v, & t > 0,\ x \in \Omega, \\ \partial_t I_v = \hat{d}_I \Delta I_v + k_2(x) S_v I_h - \gamma_2(x) I_v, & t > 0,\ x \in \Omega, \\ \partial_{\mathbf{n}} S_h = \partial_{\mathbf{n}} I_h = \partial_{\mathbf{n}} S_v = \partial_{\mathbf{n}} I_v = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{12}$$

fulfilling the initial conditions (11), where $\hat{D}_S$, $\hat{D}_I$, $\hat{d}_S$ and $\hat{d}_I$ are the average of $D_S(\cdot)$, $D_I(\cdot)$, $d_S(\cdot)$ and $d_I(\cdot)$ on Ω, respectively, that is,

$$\hat{D}_S = \frac{1}{|\Omega|}\int_\Omega D_S(x)\mathrm{d}x,\ \ \hat{D}_I = \frac{1}{|\Omega|}\int_\Omega D_I(x)\mathrm{d}x,$$
$$\hat{d}_S = \frac{1}{|\Omega|}\int_\Omega d_S(x)\mathrm{d}x,\ \ \hat{d}_I = \frac{1}{|\Omega|}\int_\Omega d_I(x)\mathrm{d}x.$$

First, we suppose the diffusion rates of human for model (2) are

$$D_S(x) = 0.4e^{-100(x-0.01)^2} + 0.002,\ \ D_I = 0.2e^{-100(x-0.01)^2} + 0.001,\ \ x \in (0, \pi),$$

which indicate that people move fast in low-risk areas and slowly in high-risk areas, as shown in Fig. 11b, since people have their own dispersal strategies based on their cognition (Wang et al. 2022). Moreover, assume the diffusion rates of mosquito for (2) are

$$d_S(x) = 0.0125 - 0.00125\cos 3x,\ \ d_I(x) = 0.005 - 0.0005\cos 3x,\ \ x \in (0, \pi).$$

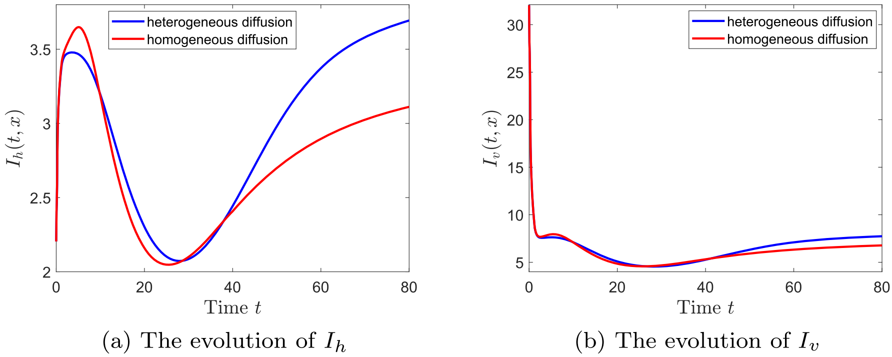


Fig. 12 Evolutions of infected humans and mosquitoes of (2) and (12) at location $x = 1$

Then the diffusion coefficients of model (12) are

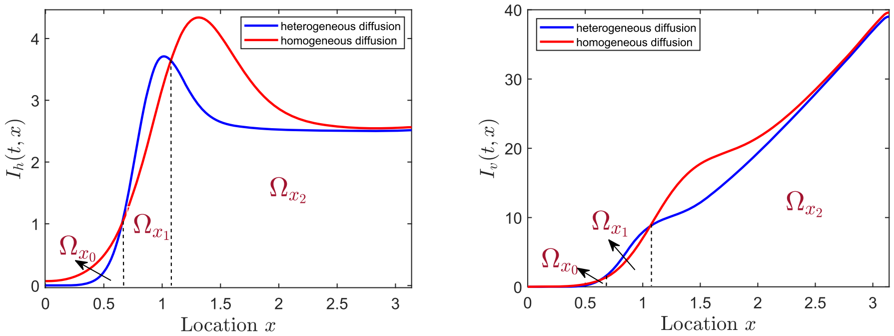
$$\hat{D}_S = 0.01455, \quad \hat{D}_I = 0.00728, \quad \hat{d}_S = 0.0125, \quad \hat{d}_I = 0.005.$$

Therefore, the basic reproduction ratio $\mathcal{R}_0$ of (2) and (12) are 1.48 and 1.40, respectively, which suggests that the infection scale may be underestimated if the spatial-averaged diffusion rate is used in modeling. Fig. 12 shows the evolution trend of infected individuals under homogeneous and heterogeneous diffusion mechanisms at location $x = 1$. It can be found from 12 that, with the passage of time, the densities of infected humans and mosquitoes in model (12) are ultimately smaller than that in model (2), which indicate that the use of homogeneous diffusion can underestimate the risk level of dengue fever to a certain extent. Furthermore, the density distribution of infected individuals under the two diffusion mechanisms at time $t = 80$ is shown in Fig. 13. On the one hand, by Fig. 13a, one finds that the density of infected humans in models (2) and (12) reach the peak at different locations, and the peak of infected human density in (12) is greater than that in (2). On the other hand, as can be seen from Fig. 13, in areas Ω_{x_0} and Ω_{x_2}, the densities of infected individuals in (12) are higher than that in (2). These are because people have cognition that they move slowly in high-risk sites and fast in low-risk sites, which can reduce the contact probability with mosquitoes. However, in area Ω_{x_1}, the densities of infected individuals in (12) is lower than that in (2), which implies that the heterogeneous movement may increase the infection level.

Next, to study the impact of different mobility strategy of infected humans on $\mathcal{R}_0$. Fix $d_I(x) = 0.005 - 0.0005\cos 3x$ and assume the diffusion rate of I_h in model (2) is

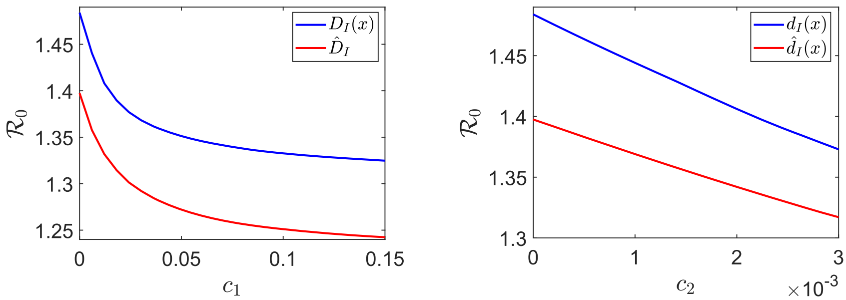
$$D_I = 0.2e^{-100(x-0.01)^2} + 0.001 + c_1, \quad x \in (0, \pi),$$

where c_1 changes from 0 to 0.15. Hence, $\hat{D}_I = 0.00728 + c_1$ and $\hat{d}_I = 0.005$ in model (12). From Fig. 14a, one finds that $\mathcal{R}_0$ is a decreasing function of c_1, and the red curve is always below the blue curve, wherein the blue and the red curves represent



(a) The distribution of I_h (b) The distribution of I_v

Fig. 13 Distribution of infected humans and mosquitoes of (2) and (12) at time $t = 80$



(a) $D_I(x)$ vs $\hat{D}_I$ (b) $d_I(x)$ vs $\hat{d}_I$

Fig. 14 Dependence of $\mathcal{R}_0$ on diffusion rates

the effects of heterogeneous diffusion $D_I(x)$ and homogeneous diffusion $\hat{D}_I$ on $\mathcal{R}_0$, respectively.

To discuss the influence of different flight strategy of infected mosquitoes on $\mathcal{R}_0$. Let $D_I = 0.2e^{-100(x-0.01)^2} + 0.001$ and assume the diffusion rate of I_v in model (2) is

$$d_I = 0.005 - 0.0005 \cos 3x + c_2, \quad x \in (0, \pi),$$

where c_2 varies from 0 to 0.003. Then $\hat{D}_I = 0.00728$ and $\hat{d}_I = 0.005 + c_2$ in model (12). By Fig. 14b, it follows that $\mathcal{R}_0$ decreases with the increase of c_2, and the red curve is always below the blue curve, where the blue and the red curves represent the effects of heterogeneous diffusion $d_I(x)$ and homogeneous diffusion $\hat{d}_I$ on $\mathcal{R}_0$, respectively. It can be summarized from Fig. 14 that the heterogeneous diffusion of humans and mosquitoes may increase the transmission risk of dengue fever.

In summary, it can be concluded from Figs. 12, 13 and 14 that (i) the use of homogeneous diffusion mechanism may underestimate the disease risk; (ii) the cognitive movement of populations contributes to dengue fever control to a certain extent. Accordingly, it is plausible and necessary to take the heterogeneous diffusion into account in the mosquito-borne disease modeling.

5 Discussion

To study the effect of spatial heterogeneity on mosquito-borne diseases, in this work, we investigated a R–D model with general incidence rates. We introduced the basic reproduction ratio $\mathcal{R}_0$ of the model and then established the threshold dynamic results in terms of $\mathcal{R}_0$. Specifically, the disease-free steady state E_0 is globally asymptotically stable when $\mathcal{R}_0 \leq 1$, and unstable when $\mathcal{R}_0 > 1$ (see Theorem 1). It should be pointed out that the global attractivity of E_0 can also be solved by applying the methods of Cui et al. (2017), Shu et al. (2021) as $\mathcal{R}_0 \leq 1$. When $\mathcal{R}_0 > 1$, we proved that the system (2) is uniformly persistent and admits at least one endemic steady state (see Theorem 2). In addition, if all coefficients are constants, then the endemic equilibrium is globally asymptotically stable in the case of $\mathcal{R}_0 > 1$ (see Theorem 3). Moreover, to clarify the effects of human and mosquito mobility on disease persistence, we discussed the asymptotic profiles and monotonicity of $\mathcal{R}_0$ w.r.t heterogeneous diffusion rates (see Theorems 4–7). To our knowledge, there are few studies on the monotonicity and asymptotic behaviors for mosquito-borne disease models with heterogeneity diffusion mechanism.

In the part of simulation, we utilized the model (1) to study dengue fever. Firstly, we investigated the long term dynamics of (2) (see Figs. 1, 2 and 3). In order to detect which parameters are sensitive to $\mathcal{R}_0$, we used the PRCCs for sensitivity analysis on $\mathcal{R}_0$ (see Fig. 4), and found that the human recovery rate and the mosquito natural death rate have crucial effects on dengue fever which implies that inhibiting mosquito reproduction and increasing medical resources to improve cure rate are the preferred measures to control the disease. Since the sensitivity may vary with some parameters, the joint effects of parameters on $\mathcal{R}_0$ were carried out (see Fig. 5). The conclusions show that sensitivity analysis of $\mathcal{R}_0$ can help develop targeted strategies to control dengue fever. Secondly, we analyzed the influence of parameters on dynamics of (2) which suggests that the measures should be taken in time to reduce biting rate at the beginning of the disease and improve recovery rate (see Figs. 6 and 7). By probing the impacts of spatial heterogeneity on $\mathcal{R}_0$, we observe that the disease transmission capacity may be underestimated if the heterogeneity is ignored (see Fig. 8). In Figs. 9 and 10, we explored the effects of temperature on $\mathcal{R}_0$ and observed that it is conducive to reducing the risk of disease transmission by adjusting the environmental temperature to inhibit mosquito survival. Through comparing the effects of heterogeneous and homogeneous diffusion rates on disease dynamics and $\mathcal{R}_0$ (see Figs. 12, 13 and 14), we found that it is crucial to incorporate the heterogeneous diffusion mechanism into the mosquito-borne diseases modeling, and the cognitive movement of populations may contribute to disease control to a certain extent.

As we all know, many infectious diseases, including mosquito-borne diseases, have incubation periods (Wang et al. 2021a; Wu and Zhao 2019). It is reasonable to take the incubation period into account in model (1). One natural question is how the incubation period affects asymptotic behavior and monotonicity of $\mathcal{R}_0$. On the other hand, from Figs. 9 and 10, $\mathcal{R}_0$ is highly associated with environmental temperature, which is closely correlated with seasonality. Thus, another interesting question is how asymptotic profiles and monotonicity of $\mathcal{R}_0$ respond to the change of heterogeneous diffusion rates in a seasonal case. We leave these interesting directions for future investigation.

6 Proofs

6.1 Proof of Theorem 1

To prove Theorem 1, the following lemma is needed. The proof is standard (see for Webb (1985), Proposition 4.16) and so the details are omitted.

Lemma 1 *For any $\mathbf{u}_0 \in X^+$, there exists a positive constant $T_m = T_m(\mathbf{u}_0)$, such that system (3) has a unique nonnegative solution*

$$\mathbf{u}(t,\cdot;\mathbf{u}_0) = e^{t\mathbb{A}}\mathbf{u}_0 + \int_0^t e^{(t-s)\mathbb{A}}\mathbb{F}(\mathbf{u}(s,\cdot;\mathbf{u}_0))\mathrm{d}s,$$

for all $t \in [0, T_m)$, and either $T_m = \infty$ or $\lim_{t\to T_m^-} \|\mathbf{u}(t,\cdot;\mathbf{u}_0)\| = \infty$.

Proof of Theorem 1 For any initial data $\mathbf{u}_0 \in X^+$, it follows from Lemma 1 that system (2) admits a unique nonnegative solution $\mathbf{u}(t,\cdot;\mathbf{u}_0)$ on $[0, T_m)$.

According to the first and third equations of (2), one gets

$$\begin{cases} \partial_t S_h \le \nabla\cdot[D_S(x)\nabla S_h] + \Lambda(x) - \mu_1(x)S_h, & t>0,\ x\in\Omega,\\ \partial_t S_v \le \nabla\cdot[d_S(x)\nabla S_v] + M(x) - \mu_2(x)S_v, & t>0,\ x\in\Omega,\\ \partial_{\mathbf{n}} S_h = \partial_{\mathbf{n}} S_v = 0, & t>0,\ x\in\partial\Omega. \end{cases}$$

Consider the system

$$\begin{cases} \partial_t \tilde{S}_h = \nabla\cdot[D_S(x)\nabla \tilde{S}_h] + \Lambda(x) - \mu_1(x)\tilde{S}_h, & t>0,\ x\in\Omega,\\ \partial_{\mathbf{n}} \tilde{S}_h = 0, & t>0,\ x\in\partial\Omega. \end{cases} \tag{13}$$

Then, by (Zhang et al. (2015), Lemma 2.1), system (13) has a unique positive solution $H(\cdot)$ which is globally attractive in $C(\bar{\Omega},\mathbb{R})$. Using the comparison principle to give

$$\limsup_{t\to\infty} S_h(t,\cdot) \le \limsup_{t\to\infty} \tilde{S}_h(t,\cdot) = H(\cdot) \ \text{ uniformly in } \bar{\Omega},$$

which means that there is a constant $C_1 > 0$, independent of $\mathbf{u}_0$, such that

$$\|S_h(t,\cdot)\| \le C_1, \quad \text{for any } \mathbf{u}_0 \in X^+,\ \ t\in[0,T_m). \tag{14}$$

In the similar fashion, there exists a function $W(\cdot) > 0$ such that

$$\limsup_{t\to\infty} S_v(t,\cdot) \le \limsup_{t\to\infty} \tilde{S}_v(t,\cdot) = W(\cdot) \ \text{ uniformly in } \bar{\Omega},$$

where $W(\cdot)$ is the steady state of the system

$$\begin{cases} \partial_t \tilde{S}_v = \nabla \cdot [d_S(x)\nabla \tilde{S}_v] + M(x) - \mu_2(x)\tilde{S}_v, & t > 0,\ x \in \Omega, \\ \partial_{\mathbf{n}} \tilde{S}_v = 0, & t > 0,\ x \in \partial\Omega. \end{cases} \tag{15}$$

Hence,

$$\|S_v(t,\cdot)\| \le C_2, \quad \text{for any } \mathbf{u}_0 \in X^+,\ t \in [0, T_m), \tag{16}$$

for some $C_2 > 0$ independent of $\mathbf{u}_0$.

From the assumption (P2), one gets $f_1(x, S_h, I_v) \le \partial_{I_v} f_1(x, C_1, 0) I_v$ and $f_2(x, S_v, I_h) \le \partial_{I_h} f_2(x, C_2, 0) I_h$, for all $x \in \Omega$. It then follows from the second and fourth equations of (2) that

$$\begin{cases} \partial_t I_h \le \nabla \cdot [D_I(x)\nabla I_h] + f^*_{1I_v} I_v - \gamma_{1*} I_h, & t > 0,\ x \in \Omega, \\ \partial_t I_v \le \nabla \cdot [d_I(x)\nabla I_v] + f^*_{2I_h} I_h - \gamma_{2*} I_v, & t > 0,\ x \in \Omega, \\ I_h^0(x) \ge 0,\ \ I_v^0(x) \ge 0, & x \in \Omega, \\ \partial_{\mathbf{n}} I_h = \partial_{\mathbf{n}} I_v = 0, & t > 0,\ x \in \partial\Omega, \end{cases}$$

where $f^*_{1I_v} := \max_{x\in\bar{\Omega}} \partial_{I_v} f_1(x, C_1, 0)$ and $f^*_{2I_h} := \max_{x\in\bar{\Omega}} \partial_{I_h} f_2(x, C_2, 0)$. Let $N_h(t) := \int_\Omega [S_h(t,x) + I_h(t,x) + R_h(t,x)]\mathrm{d}x$, $N_v(t) := \int_\Omega [S_v(t,x) + I_v(t,x)]\mathrm{d}x$. Adding the first three equations of (1), and then integrating the resulting equation over Ω to give $\mathrm{d}N_h(t)/\mathrm{d}t \le \Lambda_0 - \mu_{1*} N_h(t)$ where $\Lambda_0 := \int_\Omega \Lambda(x)\mathrm{d}x$. Therefore, there exists a constant $C_3 > 0$, depending on $N_h(0)$, such that

$$\int_\Omega [S_h(t,x) + I_h(t,x) + R_h(t,x)]\mathrm{d}x \le C_3, \quad t \in [0, T_m). \tag{17}$$

Similarly,

$$\int_\Omega [S_v(t,x) + I_v(t,x)]\mathrm{d}x \le C_4, \quad t \in [0, T_m), \tag{18}$$

for some $C_4 > 0$, depending on $N_v(0)$. Then, in view of (Dung (1997), Theorem 1), there is a constant $C_5 > 0$, depending on $I_h^0(\cdot)$ and $I_v^0(\cdot)$, such that

$$\|I_h(t,\cdot)\| + \|I_v(t,\cdot)\| \le C_5, \quad t \in [0, T_m). \tag{19}$$

Together with (14), (16) and (19), the solution of (2) is bounded on $[0, T_m) \times \bar{\Omega}$. Hence, the solution of (2) exists globally on $[0, \infty) \times \bar{\Omega}$.

Next, we show the ultimate boundedness of $I_h(t,\cdot)$ and $I_v(t,\cdot)$. To this end, one needs to verify the following claim. □

Claim. There exists a constant $B_{2^m} > 0$, independent of $\mathbf{u}_0$, such that

$$\limsup_{t\to\infty} (\|I_h(t,\cdot)\|_{2^m} + \|I_v(t,\cdot)\|_{2^m}) \le B_{2^m}, \quad m \in \mathbb{N}, \tag{20}$$

where $\mathbb{N}$ represents the set of natural numbers.

For $m = 0$, the inequality (20) holds thanks to (17) and (18). Assume (20) is true for $m-1$, i.e., there exists a constant $B_{2^{m-1}} > 0$ such that

$$\limsup_{t\to\infty}(\|I_h(t,\cdot)\|_{2^{m-1}} + \|I_v(t,\cdot)\|_{2^{m-1}}) \le B_{2^{m-1}}. \tag{21}$$

Multiplying the second equation of (2) by $I_h^{2^m-1}$ and integrating by parts yield

$$\frac{1}{2^m}\partial_t \int_\Omega I_h^{2^m}\mathrm{d}x = -\frac{2^m-1}{2^{2m-2}}\int_\Omega D_I(x)|\nabla I_h^{2^{m-1}}|^2\mathrm{d}x + \int_\Omega f_1(x,S_h,I_v)I_h^{2^m-1}\mathrm{d}x \\ -\int_\Omega \gamma_1(x)I_h^{2^m}\mathrm{d}x.$$

Following from (14) that there exists a point $\hat{t} > 0$ such that $\int_\Omega f_1(x,S_h,I_v)I_h^{2^m-1}\mathrm{d}x \le (f_{1I_v}^* + 1)\int_\Omega I_v I_h^{2^m-1}\mathrm{d}x$, for any $t \ge \hat{t}$. Using the Young inequality, we have $\int_\Omega I_v I_h^{2^m-1}\mathrm{d}x \le \gamma_2^*(f_{1I_v}^*+1)^{-1}\int_\Omega I_v^{2^m}\mathrm{d}x + K_\varepsilon\int_\Omega I_h^{2^m}\mathrm{d}x$, where $K_\varepsilon = (\varepsilon p)^{-q/p}q^{-1}$, $\varepsilon = \gamma_2^*/[4(f_{1I_v}^*+1)]$, $p = 2^m$ and $q = 2^m/(2^m-1)$. Hence,

$$\frac{1}{2^m}\partial_t\int_\Omega I_h^{2^m}\mathrm{d}x \le -D_m\int_\Omega|\nabla I_h^{2^{m-1}}|^2\mathrm{d}x + \frac{\gamma_2^*}{4}\int_\Omega I_v^{2^m}\mathrm{d}x + K'_\varepsilon\int_\Omega I_h^{2^m}\mathrm{d}x, \tag{22}$$

where $D_m = (2^m-1)D_{I*}/2^{2m-2}$ and $K'_\varepsilon = K_\varepsilon(f_{1I_v}^*+1)$.

Similarly,

$$\frac{1}{2^m}\partial_t\int_\Omega I_v^{2^m}\mathrm{d}x \le -d_m\int_\Omega|\nabla I_v^{2^{m-1}}|^2\mathrm{d}x + \frac{\gamma_1^*}{4}\int_\Omega I_h^{2^m}\mathrm{d}x + K'_{\varepsilon'}\int_\Omega I_v^{2^m}\mathrm{d}x, \tag{23}$$

wherein $d_m = (2^m-1)d_{I*}/2^{2m-2}$, $K'_{\varepsilon'} = K_{\varepsilon'}(f_{2I_h}^*+1)$ and $\varepsilon' = \gamma_1^*/[4(f_{2I_h}^*+1)]$.

Denote $I^{2^m} := I_h^{2^m} + I_v^{2^m}$. Adding (22) and (23) to get

$$\frac{1}{2^m}\partial_t\int_\Omega I^{2^m}\mathrm{d}x \le -D_m\int_\Omega|\nabla I_h^{2^{m-1}}|^2\mathrm{d}x - d_m\int_\Omega|\nabla I_v^{2^{m-1}}|^2\mathrm{d}x \\ + \left(K'_\varepsilon + \frac{\gamma_1^*}{4}\right)\int_\Omega I_h^{2^m}\mathrm{d}x + \left(K'_{\varepsilon'} + \frac{\gamma_2^*}{4}\right)\int_\Omega I_v^{2^m}\mathrm{d}x.$$

Let $\chi_1 := D_m/(2K'_\varepsilon + \gamma_1^*/2)$ and $\chi_2 := d_m/(2K'_{\varepsilon'} + \gamma_2^*/2)$. By the interpolation inequality, there are constants $C_{\chi_1}, C_{\chi_2} > 0$ such that

$$-D_m\int_\Omega|\nabla I_h^{2^{m-1}}|^2\mathrm{d}x \le -\left(2K'_\varepsilon + \frac{\gamma_1^*}{2}\right)\int_\Omega I_h^{2^m}\mathrm{d}x + C_{\chi_1}\left(2K'_\varepsilon + \frac{\gamma_1^*}{2}\right)\left(\int_\Omega I_h^{2^{m-1}}\mathrm{d}x\right)^2,$$

and

$$-d_m\int_\Omega|\nabla I_v^{2^{m-1}}|^2\mathrm{d}x \le -\left(2K'_{\varepsilon'} + \frac{\gamma_2^*}{2}\right)\int_\Omega I_v^{2^m}\mathrm{d}x + C_{\chi_2}\left(2K'_{\varepsilon'} + \frac{\gamma_2^*}{2}\right)\left(\int_\Omega I_v^{2^{m-1}}\mathrm{d}x\right)^2.$$

Therefore,

$$\frac{1}{2^m}\partial_t \int_\Omega I^{2^m}\mathrm{d}x \le -\gamma_* \int_\Omega I^{2^m}\mathrm{d}x + 2\gamma^* \left[\left(\int_\Omega I_h^{2^{m-1}}\mathrm{d}x\right)^2 + \left(\int_\Omega I_v^{2^{m-1}}\mathrm{d}x\right)^2\right], \tag{24}$$

where $t \ge \hat{t}$, $\gamma_* = \min\{K'_\varepsilon + \gamma_1^*/4, K'_{\varepsilon'} + \gamma_2^*/4\}$ and $\gamma^* = \max\{C_{\chi_1}(K'_\varepsilon + \gamma_1^*/4), C_{\chi_2}(K'_{\varepsilon'} + \gamma_2^*/4)\}$. From (21), we get

$$\limsup_{t\to\infty}\left[\left(\int_\Omega I_h^{2^{m-1}}\mathrm{d}x\right)^2 + \left(\int_\Omega I_v^{2^{m-1}}\mathrm{d}x\right)^2\right] \le 2B_{2^{m-1}}^{2^m}.$$

Substituting the above inequality into (24) to get

$$\limsup_{t\to\infty}\left[\left(\int_\Omega I_h^{2^m}\mathrm{d}x\right)^{\frac{1}{2^m}} + \left(\int_\Omega I_v^{2^m}\mathrm{d}x\right)^{\frac{1}{2^m}}\right] \le B_{2^m} := 2\cdot \sqrt[2^m]{\frac{4\gamma^*}{\gamma_*}}B_{2^{m-1}},$$

which implies that the claim holds. The embedding theorem $L^q \hookrightarrow L^p$, $q \ge p \ge 1$ yields that there is a constant $B_p > 0$ independent $\mathbf{u}_0$ such that

$$\limsup_{t\to\infty}(\|I_h(t,\cdot)\|_p + \|I_v(t,\cdot)\|_p) \le B_p, \quad \text{for any } p \in \mathbb{Z}^+.$$

It then follows from (Dung (1997), Theorem 1) (or see Dung 1998, Theorem 2.6 , Wu and Zou 2018, Lemma 2.4) that $I_h(t,\cdot)$ and $I_v(t,\cdot)$ are ultimately bounded, i.e., there is a constant $C_6 > 0$ such that

$$\limsup_{t\to\infty}(\|I_h(t,\cdot)\| + \|I_v(t,\cdot)\|) \le C_6. \tag{25}$$

Moreover, $S_h(t,\cdot)$ and $S_v(t,\cdot)$ are ultimately bounded owing to (14) and (16). Thus, system (2) is point dissipative and $\Phi(t)$ is compact according to the proof of Corollary 3.6 in Dung (1998). By means of Theorem 3.4.8 in Hale (1988), $\Phi(t)$ admits a compact global attractor in X^+. This finishes the proof. □

6.2 Proof of Theorem 2 (i)

Before proving Theorem 2, we first give some preliminaries.

Lemma 2 *Assume that (P1)–(P2) hold. Then the eigenvalue problem*

$$\begin{cases} -\nabla\cdot[D_I(x)\nabla\phi_1] + \gamma_1(x)\phi_1 = \kappa k_1(x,H)\phi_2, & x\in\Omega,\\ -\nabla\cdot[d_I(x)\nabla\phi_2] + \gamma_2(x)\phi_2 = \kappa k_2(x,W)\phi_1, & x\in\Omega,\\ \partial_{\mathbf{n}}\phi_1 = \partial_{\mathbf{n}}\phi_2 = 0, & x\in\partial\Omega, \end{cases} \tag{26}$$

admits a unique positive eigenvalue κ_0 with a positive eigenfunction. Moreover, the basic reproduction ratio $\mathcal{R}_0$ of (2) fulfills $\mathcal{R}_0 = 1/\kappa_0$.

Proof We apply the idea of (Mitidieri and Sweers (1995), Theorem 5.1) to prove the existence of κ_0. By the definitions of operators $\mathbf{B}$ and $\mathbf{F}$ in Sect. 3.2, problem (26) can be rewritten as

$$\begin{cases} \mathbf{B}\phi = \kappa \mathbf{F}\phi, & x \in \Omega, \\ \partial_{\mathbf{n}}\phi = 0, & x \in \partial\Omega, \end{cases} \tag{27}$$

where $\phi := (\phi_1, \phi_2)^T$. According to Mitidieri and Sweers (1995), operator $\mathbf{B}$ is cooperative and fully coupled. Since $\mathbf{B}$ and $\mathbf{F}$ are invertible and strictly positive, respectively, we consider the eigenvalue problem

$$\Theta = \kappa \mathbf{B}^{-1}\mathbf{F}\Theta. \tag{28}$$

Following from (Sweers (1992), Lemma 1.4) that $\mathbf{B}^{-1}$ is positive, irreducible and compact operator, which leads to $\mathbf{B}^{-1}\mathbf{F}$ being positive, irreducible and compact. Applying the Krein-Rutman theorem Krein and Rutman (1962) thereby yields that problem (28) has a unique positive eigenvalue $\tilde{\kappa}_0$ with a positive eigenfunction $\tilde{\Theta}$. More precisely, $\tilde{\kappa}_0 = (r(\mathbf{B}^{-1}\mathbf{F}))^{-1}$ (the inverse of the spectral radius of $\mathbf{B}^{-1}\mathbf{F}$) and $\tilde{\Theta} = \tilde{\kappa}_0\mathbf{B}^{-1}\mathbf{F}\tilde{\Theta}$. Hence, from (Mitidieri and Sweers (1995), Theorem 5.1), problem (27) admits a positive eigenvalue $\kappa_0 = \tilde{\kappa}_0$ with a positive eigenfunction $\phi = \tilde{\kappa}_0\mathbf{B}^{-1}\mathbf{F}\tilde{\Theta}$.

To prove $\mathcal{R}_0 = 1/\kappa_0$, inspired by the arguments of Theorem 3.2 in Wang and Zhao (2012), it is necessary to show that κ_0 is unique. By inspection of (Song et al. (2019), Lemma 2.2), we assume that there exists another eigenvalue $\hat{\kappa}_0 > 0$ with a positive eigenfunction $(\hat{\phi}_1, \hat{\phi}_2)^T$, such that

$$\begin{cases} -\nabla \cdot [D_I(x)\nabla\hat{\phi}_1] + \gamma_1(x)\hat{\phi}_1 = \hat{\kappa}_0 k_2(x, W)\hat{\phi}_2, & x \in \Omega, \\ -\nabla \cdot [d_I(x)\nabla\hat{\phi}_2] + \gamma_2(x)\hat{\phi}_2 = \hat{\kappa}_0 k_1(x, H)\hat{\phi}_1, & x \in \Omega, \\ \partial_{\mathbf{n}}\hat{\phi}_1 = \partial_{\mathbf{n}}\hat{\phi}_2 = 0, & x \in \partial\Omega. \end{cases} \tag{29}$$

Multiplying the two equations of (26) and (29) by $\hat{\phi}_1, \hat{\phi}_2$ and ϕ_1, ϕ_2 respectively, and then integrating by parts over Ω, one obtains

$$\begin{cases} \int_\Omega D_I(x)\nabla\phi_1\nabla\hat{\phi}_1 \mathrm{d}x + \int_\Omega \gamma_1(x)\phi_1\hat{\phi}_1 \mathrm{d}x = \kappa_0 \int_\Omega k_1(x, H)\phi_2\hat{\phi}_1 \mathrm{d}x, \\ \int_\Omega d_I(x)\nabla\phi_2\nabla\hat{\phi}_2 \mathrm{d}x + \int_\Omega \gamma_2(x)\phi_2\hat{\phi}_2 \mathrm{d}x = \kappa_0 \int_\Omega k_2(x, W)\phi_1\hat{\phi}_2 \mathrm{d}x, \end{cases}$$

and

$$\begin{cases} \int_\Omega D_I(x)\nabla\phi_1\nabla\hat{\phi}_1 \mathrm{d}x + \int_\Omega \gamma_1(x)\phi_1\hat{\phi}_1 \mathrm{d}x = \hat{\kappa}_0 \int_\Omega k_2(x, W)\phi_1\hat{\phi}_2 \mathrm{d}x, \\ \int_\Omega d_I(x)\nabla\phi_2\nabla\hat{\phi}_2 \mathrm{d}x + \int_\Omega \gamma_2(x)\phi_2\hat{\phi}_2 \mathrm{d}x = \hat{\kappa}_0 \int_\Omega k_1(x, H)\phi_2\hat{\phi}_1 \mathrm{d}x. \end{cases}$$

Hence,

$$(\kappa_0 - \hat{\kappa}_0) \int_\Omega [k_1(x, H)\phi_2\hat{\phi}_1 + k_2(x, W)\phi_1\hat{\phi}_2]\mathrm{d}x = 0.$$

Then $\kappa_0 = \hat{\kappa}_0$ due to the positivity of k_i, ϕ_i and $\hat{\phi}_i$ on $\bar{\Omega}$, $i = 1, 2$, which means that κ_0 is unique. Combining the uniqueness of κ_0 and (Wang and Zhao (2012), Theorem 3.2), one has $\mathcal{R}_0 = 1/\kappa_0$. This ends the proof. □

Let $(\bar{I}_h(t, \cdot), \bar{I}_v(t, \cdot)) = e^{-\kappa t}(\psi_1(\cdot), \psi_2(\cdot))$ in (4). Then $(\psi_1, \psi_2)^T$ satisfies

$$\begin{cases} \nabla \cdot [D_I(x)\nabla\psi_1] + k_1(x, H)\psi_2 - \gamma_1(x)\psi_1 + \kappa\psi_1 = 0, & x \in \Omega, \\ \nabla \cdot [d_I(x)\nabla\psi_2] + k_2(x, W)\psi_1 - \gamma_2(x)\psi_2 + \kappa\psi_2 = 0, & x \in \Omega, \\ \partial_{\mathbf{n}}\psi_1 = \partial_{\mathbf{n}}\psi_2 = 0, & x \in \partial\Omega. \end{cases} \tag{30}$$

According to the Krein-Rutman theorem, problem (30) has a unique principal eigenvalue κ_1, i.e., κ_1 is real and simple with positive eigenfunction $(\psi_1, \psi_2)^T$ and the real parts of other eigenvalues are strictly greater than κ_1.

Lemma 3 $1 - \mathcal{R}_0$ *has the same sign as* κ_1*, i.e.,* $sign(1 - \mathcal{R}_0) = sign(\kappa_1)$.

Proof Let $(\hat{\psi}_1, \hat{\psi}_2)^T$ be a positive eigenfunction of the corresponding eigenvalue κ_1 of the adjoint problem of (30). Thus, $(\hat{\psi}_1, \hat{\psi}_2)^T$ fulfills

$$\begin{cases} \nabla \cdot [D_I(x)\nabla\hat{\psi}_1] + k_2(x, W)\hat{\psi}_2 - \gamma_1(x)\hat{\psi}_1 + \kappa_1\hat{\psi}_1 = 0, & x \in \Omega, \\ \nabla \cdot [d_I(x)\nabla\hat{\psi}_2] + k_1(x, H)\hat{\psi}_1 - \gamma_2(x)\hat{\psi}_2 + \kappa_1\hat{\psi}_2 = 0, & x \in \Omega, \\ \partial_{\mathbf{n}}\hat{\psi}_1 = \partial_{\mathbf{n}}\hat{\psi}_2 = 0, & x \in \partial\Omega. \end{cases} \tag{31}$$

Multiplying the first equations of (26) and (31) by $\hat{\psi}_1$ and ϕ_1 respectively, integrating by parts and then subtracting the two equations to yield

$$\kappa_1 \int_\Omega \hat{\psi}_1\phi_1\mathrm{d}x = \frac{1}{\mathcal{R}_0}\int_\Omega k_1(x, H)\phi_2\hat{\psi}_1\mathrm{d}x - \int_\Omega k_2(x, W)\phi_1\hat{\psi}_2\mathrm{d}x.$$

In the similar fashion, we get

$$\kappa_1 \int_\Omega \hat{\psi}_2\phi_2\mathrm{d}x = \frac{1}{\mathcal{R}_0}\int_\Omega k_2(x, W)\phi_1\hat{\psi}_2\mathrm{d}x - \int_\Omega k_1(x, H)\phi_2\hat{\psi}_1\mathrm{d}x.$$

Thus, by adding above two equalities, it follows that

$$\kappa_1 \int_\Omega (\hat{\psi}_1\phi_1 + \hat{\psi}_2\phi_2)\mathrm{d}x = \left(\frac{1}{\mathcal{R}_0} - 1\right)\int_\Omega [k_1(x, H)\phi_2\hat{\psi}_1 + k_2(x, W)\phi_1\hat{\psi}_2]\mathrm{d}x.$$

Thanks to the positivity of ϕ_i and $\hat{\psi}_i$, $i = 1, 2$, $sign(1 - \mathcal{R}_0) = sign(\kappa_1)$. This completes the proof. □

By Lemma 4.2 in Zhang et al. (2015) (or see Wu and Zhao 2019), we have the following result:

Lemma 4 *Let* $\mathbf{u} = (S_h, I_h, S_v, I_v)^T$ *be the solution of* (2) *satisfying* $\mathbf{u}_0 \in X^+$.

(i) *If there exist some* $\tilde{t}_0 \geq 0$*, such that* $I_h(\tilde{t}_0, \cdot) \not\equiv 0$ *and* $I_v(\tilde{t}_0, \cdot) \not\equiv 0$*, then* $I_h(t, x) > 0$ *and* $I_v(t, x) > 0$*, for any* $t > \tilde{t}_0$ *and* $x \in \bar{\Omega}$*;*

(ii) *For any* $\mathbf{u}_0 \in X^+$*, then* $S_h(t, x) > 0$*,* $S_v(t, x) > 0$ *and there exists a positive constant* ζ_0*, independent of* $\mathbf{u}_0$*, such that*

$$\liminf_{t\to\infty} S_h(t, x) > \zeta_0, \quad \liminf_{t\to\infty} S_v(t, x) > \zeta_0, \quad \text{uniformly for } x \in \bar{\Omega}.$$

Proof Following from system (2) that

$$\begin{cases} \partial_t I_h \geq \nabla \cdot [D_I(x)\nabla I_h] - \gamma_1(x) I_h, & t > 0, \ x \in \Omega, \\ \partial_t I_v \geq \nabla \cdot [d_I(x)\nabla I_v] - \gamma_2(x) I_v, & t > 0, \ x \in \Omega, \\ \partial_{\mathbf{n}} I_h = \partial_{\mathbf{n}} I_v = 0, & t > 0, \ x \in \partial\Omega. \end{cases}$$

Since $I_h(\tilde{t}_0, \cdot) \not\equiv 0$ and $I_v(\tilde{t}_0, \cdot) \not\equiv 0$, with the help of the maximum principle, it follows that $I_h(t, x) > 0$ and $I_v(t, x) > 0$, for any $t > \tilde{t}_0$, $x \in \bar{\Omega}$.

In view of the proof of Theorem 1, there is a constant $C_9 > 0$ such that $|I_h(t, x)| + |I_v(t, x)| \leq C_9$, for any $t > 0$, $x \in \bar{\Omega}$. Denote

$$f^*_{1S_h} := \max_{x\in\bar{\Omega},\, 0<S_h\leq C_1} \partial_{S_h} f_1(x, S_h, C_9), \quad f^*_{2S_v} := \max_{x\in\bar{\Omega},\, 0<S_v\leq C_2} \partial_{S_v} f_2(x, S_v, C_9).$$

Then $f_1(\cdot, S_h, I_v) \leq f^*_{1S_h} S_h$ and $f_2(\cdot, S_v, I_h) \leq f^*_{2S_v} S_v$ by the assumption (P2). Assume that $(\hat{S}_h(\cdot, \cdot), \hat{S}_v(\cdot, \cdot))^T$ is a solution of the problem

$$\begin{cases} \partial_t \hat{S}_h = \nabla \cdot [D_S(x)\nabla \hat{S}_h] + \Lambda(x) - [f^*_{1S_h} + \mu_1(x)]\hat{S}_h, & t > 0, \ x \in \Omega, \\ \partial_t \hat{S}_v = \nabla \cdot [d_S(x)\nabla \hat{S}_v] + M(x) - [f^*_{2S_v} + \mu_2(x)]\hat{S}_v, & t > 0, \ x \in \Omega, \\ \hat{S}_h(0, x) = S^0_h(x), \ \ \hat{S}_v(0, x) = S^0_v(x), & x \in \Omega, \\ \partial_{\mathbf{n}} \hat{S}_h = \partial_{\mathbf{n}} \hat{S}_v = 0, & t > 0, \ x \in \partial\Omega. \end{cases} \tag{32}$$

By the comparison principle, we have $S_h(t, x) \geq \hat{S}_h(t, x) > 0$ and $S_v(t, x) \geq \hat{S}_v(t, x) > 0$, for any $t > 0$, $x \in \bar{\Omega}$. Notice that system (32) possesses a unique steady state, denoted by $(\hat{H}(\cdot), \hat{W}(\cdot))$, from the proof of Theorem 1. Thus,

$$\liminf_{t\to\infty} S_h(t, x) \geq \inf_{x\in\bar{\Omega}} \hat{H}(x) := \zeta^1_0 \text{ and } \liminf_{t\to\infty} S_v(t, x) \geq \inf_{x\in\bar{\Omega}} \hat{W}(x) := \zeta^2_0.$$

Choose $\zeta_0 = \min\{\zeta^1_0, \zeta^2_0\}$. This ends the proof. □

In the following, we start to prove the Theorem 2 (i).

Proof of Theorem 2 (i). To deal with the global asymptotic stability of E_0 when $\mathcal{R}_0 \le 1$, we divided it into three steps.
Step 1 We show that $(I_h(t,\cdot), I_v(t,\cdot))^T \to (0,0)^T$ as $t \to \infty$ uniformly in Ω.

Define the set $P := X^+ \cap X_0$, where

$$X_0 := \left\{\mathbf{u}_0 \in X \mid \|S_h^0\| + \|I_h^0\| + \|S_v^0\| + \|I_v^0\| \le C_*\right\},$$

here $C_* = C_1 + C_2 + C_5$. Let $\Phi(t)\mathbf{u}_0 = (S_h(t,\cdot), I_h(t,\cdot), S_v(t,\cdot), I_v(t,\cdot))^T$ be the unique solution of (2) with $\mathbf{u}_0 \in P$. By the Sobolev inequalities and L^p estimates, for any $\beta \in (0,1)$, there is a constant $C_7 > 0$ such that

$$\|(S_h, I_h, S_v, I_v)^T\|_{C^{\frac{\beta}{2},\beta}([t^*-1,t^*+1]\times\bar{\Omega})} \le C_7\|(S_h, I_h, S_v, I_v)^T\|_{L^\infty([t^*-1,t^*+1]\times\bar{\Omega})},$$

for each $t^* \ge 1$. It then follows from (14), (16) and (19) that there is a constant $C_8 > 0$ such that

$$\|(S_h, I_h, S_v, I_v)^T\|_{C^\beta(\bar{\Omega})} \le C_7C_8, \quad \text{for any } t \ge 1.$$

Hence, $\Phi(t)$ is compact, and for each $\mathbf{u}_0 \in P$, the orbit of $\Phi(t)\mathbf{u}_0$ under the dynamical system generated by (2) has a compact closure in P.

To study the convergence of $(I_h(t,\cdot), I_v(t,\cdot))^T$ as $t \to \infty$, we define the following Lyapunov functional:

$$\mathcal{V}[\mathbf{u}](t) = \int_\Omega (I_h\hat{\psi}_1 + I_v\hat{\psi}_2)\mathrm{d}x, \quad \mathbf{u} = (S_h, I_h, S_v, I_v)^T \in P,$$

wherein $(\hat{\psi}_1, \hat{\psi}_2)^T$ is a positive eigenfunction corresponding to κ_1 of (31). By direct calculating and applying the assumption (P2), we obtain

$$\begin{aligned}
\dot{\mathcal{V}}[\mathbf{u}](t) &= \int_\Omega \left(\hat{\psi}_1\partial_t I_h + \hat{\psi}_2\partial_t I_v\right)\mathrm{d}x \\
&= \int_\Omega \{[\nabla\cdot(D_I\nabla I_h) + f_1(x, S_h, I_v) - \gamma_1 I_h]\hat{\psi}_1 \\
&\quad + [\nabla\cdot(d_I\nabla I_v) + f_2(x, S_v, I_h) - \gamma_2 I_v]\hat{\psi}_2\}\mathrm{d}x \\
&= \int_\Omega \{-[k_2(x,W)\hat{\psi}_2 + \kappa_1\hat{\psi}_1]I_h - [k_1(x,H)\hat{\psi}_1 + \kappa_1\hat{\psi}_2]I_v\}\mathrm{d}x \\
&\quad + \int_\Omega f_1(x, S_h, I_v)\hat{\psi}_1 + f_2(x, S_v, I_h)\hat{\psi}_2\mathrm{d}x \\
&= -\kappa_1\int_\Omega (\hat{\psi}_1 I_h + \hat{\psi}_2 I_v)\mathrm{d}x - \int_\Omega [k_1(x,H)I_v - f_1(x, S_h, I_v)]\hat{\psi}_1\mathrm{d}x \\
&\quad - \int_\Omega [k_2(x,W)I_h - f_2(x, S_v, I_h)]\hat{\psi}_2\mathrm{d}x \\
&\le -\kappa_1\int_\Omega (\hat{\psi}_1 I_h + \hat{\psi}_2 I_v)\mathrm{d}x - \int_\Omega [k_1(x,H) - \partial_{I_v} f_1(x, S_h, 0)]I_v\hat{\psi}_1\mathrm{d}x
\end{aligned}$$

$$- \int_{\Omega} [k_2(x, W) - \partial_{I_h} f_2(x, S_v, 0)] I_h \hat{\psi}_2 \mathrm{d}x,$$

where $\dot{\mathcal{V}}[\mathbf{u}](t) = \mathrm{d}\mathcal{V}[\mathbf{u}](t)/\mathrm{d}t$. Since $\mathcal{R}_0 \le 1$, $\kappa_1 \ge 0$ due to Lemma 3. From (P2) and the facts $\mathbf{u} \in P$, $S_h(t, \cdot) \le H(\cdot)$, $S_v(t, \cdot) \le W(\cdot)$, we have $k_1(\cdot, H) \ge \partial_{I_v} f_1(\cdot, S_h, 0)$ and $k_2(\cdot, W) \ge \partial_{I_h} f_2(\cdot, S_v, 0)$. Then $\dot{\mathcal{V}}[\mathbf{u}](t) \le 0$ because of the positivity of $\hat{\psi}_i$. Let $\dot{\mathcal{V}}[\mathbf{u_0}] := \dot{\mathcal{V}}[\mathbf{u}](t) \mid_{t=0}$ and $\mathbf{E} := \{\mathbf{u}_0 \in P \mid \dot{\mathcal{V}}[\mathbf{u_0}] \equiv 0\}$.

If $\kappa_1 > 0$ (i.e., $\mathcal{R}_0 < 1$), then

$$\begin{aligned}\dot{\mathcal{V}}[\mathbf{u}](t) = -\kappa_1 \int_{\Omega} (\hat{\psi}_1 I_h + \hat{\psi}_2 I_v) \mathrm{d}x - \int_{\Omega} [k_1(x, H) I_v - f_1(x, S_h, I_v)] \hat{\psi}_1 \mathrm{d}x \\ - \int_{\Omega} [k_2(x, W) I_h - f_2(x, S_v, I_h)] \hat{\psi}_2 \mathrm{d}x.\end{aligned}$$

Thus, from $\dot{\mathcal{V}}[\mathbf{u_0}] \equiv 0$, one gets $I_h^0 = 0$ and $I_v^0 = 0$. Then the maximal invariant set of $\mathbf{E}$ is $\mathbf{E}_M = \{\mathbf{u}_0 \in P | I_h^0 = 0, I_v^0 = 0\}$ when $\kappa_1 > 0$.

If $\kappa_1 = 0$ (i.e., $\mathcal{R}_0 = 1$), then

$$\dot{\mathcal{V}}[\mathbf{u}](t) = - \int_{\Omega} [k_1(x, H) I_v - f_1(x, S_h, I_v)] \hat{\psi}_1 \mathrm{d}x - \int_{\Omega} [k_2(x, W) I_h - f_2(x, S_v, I_h)] \hat{\psi}_2 \mathrm{d}x.$$

Following from $\dot{\mathcal{V}}[\mathbf{u_0}] \equiv 0$ that

$$\begin{aligned}0 &= - \int_{\Omega} [k_1(x, H) I_v^0 - f_1(x, S_h^0, I_v^0)] \hat{\psi}_1 \mathrm{d}x - \int_{\Omega} [k_2(x, W) I_h^0 - f_2(x, S_v^0, I_h^0)] \hat{\psi}_2 \mathrm{d}x \\ &\le - \int_{\Omega} [k_1(x, H) - \partial_{I_v} f_1(x, S_h^0, 0)] I_v^0 \hat{\psi}_1 \mathrm{d}x - \int_{\Omega} [k_2(x, W) - \partial_{I_h} f_2(x, S_v^0, 0)] I_h^0 \hat{\psi}_2 \mathrm{d}x \le 0.\end{aligned}$$

Hence,

$$\int_{\Omega} [k_1(x, H) - \partial_{I_v} f_1(x, S_h^0, 0)] I_v^0 \hat{\psi}_1 \mathrm{d}x + \int_{\Omega} [k_2(x, W) - \partial_{I_h} f_2(x, S_v^0, 0)] I_h^0 \hat{\psi}_2 \mathrm{d}x = 0.$$

Note that $k_1(\cdot, H) = \partial_{I_v} f_1(\cdot, H, 0)$ and $k_2(\cdot, W) = \partial_{I_h} f_2(\cdot, W, 0)$. Then, from the above equality, we have

$$\partial_{I_v} f_1(x, H, 0) I_v^0 = \partial_{I_v} f_1(x, S_h^0, 0) I_v^0 \quad \text{and} \quad \partial_{I_h} f_2(x, W, 0) I_h^0 = \partial_{I_h} f_2(x, S_v^0, 0) I_h^0.$$

To show $I_v^0(\cdot) \equiv 0$ in Ω. Suppose not. From $\partial_{I_v} f_1(\cdot, H, 0) I_v^0 = \partial_{I_v} f_1(\cdot, S_h^0, 0) I_v^0$, we assume that there exist two subsets Ω_1 and Ω_2 of Ω satisfying $\Omega_1 \cup \Omega_2 = \Omega$ and $\Omega_1 \cap \Omega_2 = \emptyset$ such that

$$S_h^0(\cdot) = H(\cdot) \text{ and } I_v^0(\cdot) \ne 0 \text{ in } \Omega_1, \text{ and } S_h^0(\cdot) \ne H(\cdot) \text{ and } I_v^0(\cdot) = 0 \text{ in } \Omega_2.$$

Substituting $S_h^0(\cdot) = H(\cdot)$ in Ω_1 into system (2) to yield $f_1(\cdot, H, I_v^0) = 0$ which implies that $I_v^0(\cdot) \equiv 0$ in Ω_1 from the assumption (P2). Then $I_v^0(\cdot) \equiv 0$ in Ω. Similarly, $I_h^0(\cdot) \equiv 0$ in Ω. Thus, the maximal invariant set is $\mathbf{E}_M = \{\mathbf{u}_0 \in P | I_h^0 = 0, I_v^0 = 0\}$

when $\kappa_1 = 0$. Based on above discussions, $\mathbf{E}_M = \{\mathbf{u}_0 \in P | I_h^0 = 0, I_v^0 = 0\}$ when $\kappa_1 \geq 0$ (i.e., $\mathcal{R}_0 \leq 1$).

Accordingly, the LaSalle's invariance principle for infinite dimensional dynamical systems (Hale 1969, Theorem 1) implies that

$$\|(I_h(t,\cdot), I_v(t,\cdot))^T\| \to (0,0)^T, \quad \text{as } t \to \infty.$$

□

Step 2 We prove that, when $\mathcal{R}_0 \leq 1$,

$$\lim_{t\to\infty} \|(S_h(t,\cdot), S_v(t,\cdot))^T - (H(\cdot), W(\cdot))^T\| = 0 \ \text{ uniformly in } \Omega. \tag{33}$$

We apply the theory of internally chain transitive sets established in Zhao (2017) to show (33). It is obvious that S_h and S_v in (2) are asymptotic to systems (13) and (15). Recall that $H(\cdot)$ and $W(\cdot)$ are global attractive steady states of (13) and (15), respectively. Set $\mathcal{J} := \omega_{X^+}(\mathbf{u}_0)$ be the omega limit of $\mathbf{u}_0 \in X^+$ for $\Phi(t)$. From Step 1 and Lemma 4, it follows that $\mathcal{J} = \mathcal{J}_{h,v} \times \{(0,0)^T\}$ and $\{(0,0)^T\} \notin \mathcal{J}_{h,v}$. Then $\mathcal{J}$ is an internally chain transitive set of $\Phi(t)$ (see Lemma 1.2.1 in Zhao 2017). As $\Phi(\mathcal{J}) = \mathcal{J}$, we get $\Phi(\mathcal{J}) = \tilde{\Phi}(\mathcal{J}_{h,v}) \times \{(0,0)^T\} = \mathcal{J}_{h,v} \times \{(0,0)^T\}$ and so $\tilde{\Phi}(\mathcal{J}_{h,v}) = \mathcal{J}_{h,v}$, where $\tilde{\Phi}(t)$ is the semiflow generated by (13) and (15). Hence, $\mathcal{J}_{h,v}$ is an internally chain transitive set of $\tilde{\Phi}(t)$. Let $(S_h^0(\cdot), S_v^0(\cdot)) = (H(\cdot), W(\cdot))$. Therefore, $\mathcal{J}_{h,v} \cap W^S(\{(S_h^0, S_v^0)^T\}) \neq \emptyset$ owing to the attractivity of $(S_h^0, S_v^0)^T$ and the fact $\mathcal{J}_{h,v} \neq \{(0,0)^T\}$, here $W^S(\{(S_h^0, S_v^0)^T\})$ is the stable set of $\{(S_h^0, S_v^0)^T\}$ for $\tilde{\Phi}(t)$. By (Zhao (2017), Theorem 1.2.1), $\mathcal{J}_{h,v} = \{(S_h^0, S_v^0)^T\}$ which indicates that (33) holds. Combining Step 1 and Step 2, E_0 is globally attractive in the case of $\mathcal{R}_0 \leq 1$.

Step 3 In what follows, we deal with the stability of E_0 when $\mathcal{R}_0 \leq 1$.

Through utilizing the ideas of Cui et al. (2017), Shu et al. (2021), we let $\Psi(t)$ be the solution semigroup generated by the system (4) and $\ell(\Psi)$ be the exponential growth bound of Ψ. With the aid of (Thieme (2009), Theorem 3.14) and (Wang and Zhao (2012), Theorem 3.1) $\ell(\Psi) = -\kappa_1 \leq 0$ which means that there exists a constant $C_9 > 0$ such that $\|\Psi(t)\| \leq C_9$. Set $\mathbf{u}(t,\cdot)$ be a solution of (2) satisfying the following initial value condition

$$\mathbf{u}_0 \in Q_\sigma := \left\{\mathbf{u}_0 \in X^+ \mid \|S_h^0 - H\| + \|I_h^0\| + \|S_v^0 - W\| + \|I_v^0\| \leq \sigma\right\},$$

for any $\sigma > 0$. On account of the proof of Theorem 1, $\mathbf{u}(t,\cdot)$ fulfils

$$\|S_h(t,\cdot)\| + \|I_h(t,\cdot)\| + \|S_v(t,\cdot)\| + \|I_v(t,\cdot)\| \leq C_*, \quad t \geq 0, \tag{34}$$

where $C_* = C_1 + C_2 + C_5$.

Noticing that $H(\cdot)$ and $W(\cdot)$ are the steady states of (13) and (15), respectively, and by means of (2), one has

$$\begin{cases} \partial_t U_1 = \nabla \cdot [D_S(x)\nabla U_1] - f_1(x, S_h, I_v) - \mu_1(x)U_1, & t > 0,\ x \in \Omega, \\ \partial_t V_1 = \nabla \cdot [d_S(x)\nabla V_1] - f_2(x, S_v, I_h) - \mu_2(x)V_1, & t > 0,\ x \in \Omega, \\ S_h^0(x) \in Q_\sigma,\ \ S_v^0(x) \in Q_\sigma, & x \in \Omega, \\ \partial_{\mathbf{n}} U_1 = \partial_{\mathbf{n}} V_1 = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{35}$$

where $U_1(t,\cdot) = S_h(t,\cdot) - H(\cdot)$ and $V_1(t,\cdot) = S_v(t,\cdot) - W(\cdot)$. Thanks to the comparison principle and (Wang and Zhao (2011), (1.10)), we have

$$S_h(t,x) - H(x) \le \sigma e^{-\mu_{1*}t} \ \text{ and } \ S_v(t,x) - W(x) \le \sigma e^{-\mu_{3*}t}, \tag{36}$$

for all $t > 0$ and $x \in \Omega$.

By the assumption (P2) and (34), it follows that

$$f_1(\cdot, S_h, I_v) \le k_1(\cdot, H)I_v + C_*\sigma e^{-\mu_{1*}t},\ \ f_2(\cdot, S_v, I_h) \le k_2(\cdot, W)I_h + C_*\sigma e^{-\mu_{3*}t},$$

in $(0,\infty) \times \Omega$. Then following from (2) that

$$\begin{cases} \partial_t I_h \le \nabla \cdot [D_I(x)\nabla I_h] + k_1(x, H)I_v - \gamma_1(x)I_h + C_*\sigma e^{-\mu_{1*}t}, & t > 0,\ x \in \Omega, \\ \partial_t I_v \le \nabla \cdot [d_I(x)\nabla I_v] + k_2(x, W)I_h - \gamma_2(x)I_v + C_*\sigma e^{-\mu_{3*}t}, & t > 0,\ x \in \Omega, \\ I_h^0(x) \in Q_\sigma,\ \ I_v^0(x) \in Q_\sigma, & x \in \Omega, \\ \partial_{\mathbf{n}} I_h = \partial_{\mathbf{n}} I_v = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

Since $\|\Psi(t)\| \le C_9$, applying the comparison principle yields that

$$\|I_h(t,\cdot)\| \le C_9\sigma + \int_0^t C_9 C_*\sigma e^{-\mu_{1*}\tau}\mathrm{d}\tau \le \sigma C_9\left(1 + \frac{C_*}{\mu_{1*}}\right), \tag{37}$$

and

$$\|I_v(t,\cdot)\| \le C_9\sigma + \int_0^t C_9 C_*\sigma e^{-\mu_{3*}\tau}\mathrm{d}\tau \le \sigma C_9\left(1 + \frac{C_*}{\mu_{3*}}\right),\ \ t > 0. \tag{38}$$

Moreover, similar to the arguments of (35), we have

$$\begin{cases} \partial_t U_2 = \nabla \cdot [D_S(x)\nabla U_2] + f_1(x, S_h, I_v) - \mu_1(x)U_2, & t > 0,\ x \in \Omega, \\ \partial_t V_2 = \nabla \cdot [d_S(x)\nabla V_2] + f_2(x, S_v, I_h) - \mu_2(x)V_2, & t > 0,\ x \in \Omega, \\ S_h^0(x) \in Q_\sigma,\ \ S_v^0(x) \in Q_\sigma, & x \in \Omega, \\ \partial_{\mathbf{n}} U_2 = \partial_{\mathbf{n}} V_2 = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{39}$$

where $U_2(t,\cdot) = H(\cdot) - S_h(t,\cdot)$ and $V_2(t,\cdot) = W(\cdot) - S_v(t,\cdot)$. According to (37)–(38) and the assumption (P2), there are constants $C_{10}, C'_{10} > 0$ such that

$$f_1(x, S_h, I_v) \le C_{10}\sigma \ \text{ and } \ f_2(x, S_v, I_h) \le C'_{10}\sigma, \ \text{ for all } t > 0,\ x \in \Omega.$$

Thus, the comparison principle is applied to system (39) to give

$$H(x) - S_h(t,x) \le \frac{C_{10}\sigma}{\mu_{1*}} \ \text{ and } \ W(x) - S_v(t,x) \le \frac{C'_{10}\sigma}{\mu_{3*}}, \tag{40}$$

for any $t > 0$, $x \in \Omega$.

Consequently, together with (36), (37), (38) and (40), we get

$$\|S_h(t,\cdot) - H(\cdot)\| + \|I_h(t,\cdot)\| + \|S_v(t,\cdot) - W(\cdot)\| + \|I_v(t,\cdot)\| \le C_{11}\sigma,$$

where

$$C_{11} = 2 + C_9\left(2 + \frac{C_*}{\mu_{1*}} + \frac{C_*}{\mu_{3*}}\right) + \frac{C_{10}}{\mu_{1*}} + \frac{C'_{10}}{\mu_{3*}} > 1$$

which implies that the selection of C_{11} does not depend on σ. Accordingly, for any initial value $\mathbf{u}_0 \in Q_\sigma$, the solution $\mathbf{u}(t,\cdot)$ of (2) lies in $C_{11}Q_\sigma$ which establishes the stability of E_0. In view of Step 1–Step 3, we obtain that E_0 is globally asymptotically stable in the case of $\mathcal{R}_0 \le 1$.

To show the instability of E_0 when $\mathcal{R}_0 > 1$. Consider the spectrum problem

$$\begin{cases} \nabla\cdot[D_S(x)\nabla\varpi_1] - k_1(x,H)\varpi_4 - \mu_1(x)\varpi_1 + \kappa\varpi_1 = 0, & x \in \Omega,\\ \nabla\cdot[D_I(x)\nabla\varpi_2] + k_1(x,H)\varpi_4 - \gamma_1(x)\varpi_2 + \kappa\varpi_2 = 0, & x \in \Omega,\\ \nabla\cdot[d_S(x)\nabla\varpi_3] - k_2(x,W)\varpi_2 - \mu_2(x)\varpi_3 + \kappa\varpi_3 = 0, & x \in \Omega,\\ \nabla\cdot[d_I(x)\nabla\varpi_4] + k_2(x,W)\varpi_2 - \gamma_2(x)\varpi_4 + \kappa\varpi_4 = 0, & x \in \Omega,\\ \partial_{\mathbf{n}}\varpi_1 = \partial_{\mathbf{n}}\varpi_2 = \partial_{\mathbf{n}}\varpi_3 = \partial_{\mathbf{n}}\varpi_4 = 0, & x \in \partial\Omega, \end{cases} \tag{41}$$

By inspection of Theorem 5.1.3 in Henry (1981), it is sufficient to prove system (41) has a nontrivial solution satisfying $\mathrm{Re}\kappa < 0$. By Lemma 3, κ_1 is the eigenvalue of (41). Without loss of generality, one can choose (ϖ_2^*, ϖ_4^*) as the corresponding eigenfunction of κ_1. Furthermore, there is a unique (ϖ_1^*, ϖ_3^*) of (41). Then E_0 is unstable. This ends the proof. □

6.3 Proof of Theorem 2 (ii)

Proof of Theorem 2 (ii) In order to use the persistence theory developed by Magal and Zhao (2005) and Zhao (2017), let

$$P_0 := \{\mathbf{u}_0 \in P \,|\, I_h^0 \ne 0 \text{ and } I_v^0 \ne 0\} \text{ and } \partial P_0 := \{\mathbf{u}_0 \in P \,|\, I_h^0 = 0 \text{ or } I_v^0 = 0\},$$

where $\mathbf{u}_0 = (S_h^0, I_h^0, S_v^0, I_v^0)^T$. It is easy to know that $P = P_0 \cup \partial P_0$, and P_0 and ∂P_0 are relatively open and closed subsets of P, respectively. Moreover, P_0 is a convex set. Recall that $\Phi(t)\mathbf{u}_0$ be the unique solution of (2) with $\mathbf{u}_0 \in P$. By Theorem 1, $\Phi(t)$ admits a global compact attractor. To complete the proof, we first prove the following claims. □

Claim 1. $\Phi(t)P_0 \subset P_0$. This is obvious due to the strong maximum principle Evans (1986).

Let U_∂ be the maximum positive invariant set of $\Phi(t)$ in ∂P_0, that is, $U_\partial := \{\mathbf{u}_0 \in P \mid \Phi(t)\mathbf{u}_0 \in \partial P_0\}$. One can verify that $U_\partial = \{\mathbf{u}_0 \in P \mid I_h^0 = I_v^0 = 0\}$. Denote $\omega(\mathbf{u}_0)$ as the omega limit set of $\mathbf{u}_0$ in P. Set

$$\overline{U}_\partial := \bigcup\nolimits_{\{\mathbf{u}_0 \in U_\partial\}} \omega(\mathbf{u}_0).$$

Claim 2. $\overline{U}_\partial = \{E_0\}$.

Indeed, for any $\mathbf{u}_0 \in U_\partial$, from the definition of U_∂, one obtains $I_h(t,x) = I_h(t,x) = 0$, for all $t \geq 0$, $x \in \bar{\Omega}$. Therefore, substituting it into (2) to give

$$\begin{cases} \partial_t S_h = \nabla \cdot [D_S(x)\nabla S_h] + \Lambda(x) - \mu_1(x)S_h, & t > 0,\ x \in \Omega, \\ \partial_t S_v = \nabla \cdot [d_S(x)\nabla S_v] + M(x) - \mu_2(x)S_v, & t > 0,\ x \in \Omega, \\ \partial_{\mathbf{n}} S_h = \partial_{\mathbf{n}} S_v = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

From (13) and (15), it follows that $\overline{U}_\partial = \{E_0\}$, and then $\{E_0\}$ is an isolated and compact invariant set for $\Phi(t)$ restricted in U_∂.

Claim 3. There is a constant $\delta_1 > 0$, independent of $\mathbf{u}_0$, such that

$$\limsup_{t\to\infty} \|\Phi(t)\mathbf{u}_0 - (H(\cdot), 0, W(\cdot), 0)^T\| > \delta_1.$$

Arguing by contradiction, we assume that, for any $\hat{\delta}_1 > 0$, there exists $\hat{\mathbf{u}}_0 = (\hat{S}_h^0, \hat{I}_h^0, \hat{S}_v^0, \hat{I}_v^0)^T$ such that

$$\limsup_{t\to\infty} \|\Phi(t)\hat{\mathbf{u}}_0 - (H(\cdot), 0, W(\cdot), 0)^T\| \leq \hat{\delta}_1, \tag{42}$$

where $\Phi(t)\hat{\mathbf{u}}_0 = (\hat{S}_h(t,\cdot), \hat{I}_h(t,\cdot), \hat{S}_v(t,\cdot), \hat{I}_v(t,\cdot))^T$. Take a $\delta_2 > 0$ small enough. Let $\kappa_1(\delta_2)$ be the principal eigenvalue of the eigenvalue problem

$$\begin{cases} \nabla \cdot [D_I(x)\nabla\psi_1] + \partial_{I_v} f_1(x, H - \delta_2, \delta_2)\psi_2 - \gamma_1(x)\psi_1 + \kappa\psi_1 = 0, & x \in \Omega, \\ \nabla \cdot [d_I(x)\nabla\psi_2] + \partial_{I_h} f_2(x, W - \delta_2, \delta_2)\psi_1 - \gamma_2(x)\psi_2 + \kappa\psi_2 = 0, & x \in \Omega, \\ \partial_{\mathbf{n}}\psi_1 = \partial_{\mathbf{n}}\psi_2 = 0, & x \in \partial\Omega, \end{cases}$$

wherein $(\psi_1, \psi_2)^T$ is the corresponding positive eigenfunction. Since $\mathcal{R}_0 > 1, \kappa_1 < 0$ by Lemma 3, here κ_1 is the eigenvalue of (30). Note that $\kappa_1(\delta_2) \to \kappa_1 < 0$ as $\delta_2 \to 0$. Thus, one can choose a sufficiently small δ_2 such that $\kappa_1(\delta_2) < 0$. According to the arbitrariness of $\hat{\delta}_1$, we let $\hat{\delta}_1 = \delta_2$. By (42), there is a point $t_0^* > 0$ such that

$\hat{S}_h(t,\cdot) \geq H(\cdot) - \delta_2$, $\hat{S}_v(t,\cdot) \geq W(\cdot) - \delta_2$, $\hat{I}_h(t,\cdot) \leq \delta_2$ and $\hat{I}_v(t,\cdot) \leq \delta_2$ in $\bar{\Omega}$, for any $t \geq t_0^*$. Hence, from the assumption (P2), we have

$$f_1(x, \hat{S}_h, \hat{I}_v) \geq f_1(x, H - \delta_2, \hat{I}_v) \geq \partial_{\hat{I}_v} f_1(x, H - \delta_2, \delta_2)\hat{I}_v,$$

and

$$f_2(x, \hat{S}_v, \hat{I}_h) \geq f_2(x, W - \delta_2, \hat{I}_h) \geq \partial_{\hat{I}_h} f_2(x, W - \delta_2, \delta_2)\hat{I}_h,$$

for all $t \geq t_0^*$ and $x \in \bar{\Omega}$.

In addition, it follows from Theorem 1 and the strong maximum principle that $(\hat{S}_h, \hat{I}_h, \hat{S}_v, \hat{I}_v)^T \in Int(X^+)$ (interior of X^+). Then, there exists a constant $\rho_0 > 0$ small enough, such that $\hat{I}_h(t_0^*, \cdot) \geq \rho_0\psi_1(\cdot)$, $\hat{I}_v(t_0^*, \cdot) \geq \rho_0\psi_2(\cdot)$. We can testify that $(\hat{I}_h(t,\cdot), \hat{I}_v(t,\cdot))^T$ is a super-solution of the system

$$\begin{cases} \partial_t \tilde{I}_h = \nabla \cdot [D_I \nabla \tilde{I}_h] + \partial_{\tilde{I}_v} f_1(x, H - \delta_2, \delta_2)\tilde{I}_v - \gamma_1 \tilde{I}_h, & t > t_0^*,\ x \in \Omega, \\ \partial_t \tilde{I}_v = \nabla \cdot [d_I \nabla \tilde{I}_v] + \partial_{\tilde{I}_h} f_2(x, W - \delta_2, \delta_2)\tilde{I}_h - \gamma_2 \tilde{I}_v, & t > t_0^*,\ x \in \Omega, \\ \partial_{\mathbf{n}} \tilde{I}_h = \partial_{\mathbf{n}} \tilde{I}_v = 0, & t > t_0^*,\ x \in \partial\Omega. \\ \tilde{I}_h(t_0^*, x) = \rho_0\psi_1(x),\ \ \tilde{I}_v(t_0^*, x) = \rho_0\psi_2(x), & x \in \Omega. \end{cases}$$

Noticing that $(\rho_0 e^{-\kappa_1(\delta_2)(t-t_0^*)}\psi_1, \rho_0 e^{-\kappa_1(\delta_2)(t-t_0^*)}\psi_2)^T$ is a solution of above system and $\kappa_1\ (\delta_2) < 0$, one has

$$\hat{I}_h(t,\cdot) \geq \rho_0 e^{-\kappa_1(\delta_2)(t-t_0^*)}\psi_1(\cdot) \to \infty \text{ and } \hat{I}_v(t,\cdot) \geq \rho_0 e^{-\kappa_1(\delta_2)(t-t_0^*)}\psi_2(\cdot) \to \infty,$$

as $t \to \infty$, which contradicts (42) and so Claim 3 is valid. Claim 3 indicates that $\{E_0\}$ is an isolated invariant set for $\Phi(t)$ restricted in P, and $W^S(\{E_0\}) \cap P_0 = \emptyset$, where $W^S(\{E_0\})$ is the stable set of $\{E_0\}$ w.r.t $\Phi(t)$.

Combining Claims 1-3 and Theorem 1.3.1 in Zhao (2017), $\Phi(t)$ is uniformly persistent for $(P, \partial P_0)$ and then (5) is true. Moreover, Theorem 1.3.7 in Zhao (2017) (see also Magal and Zhao 2005, Theorem 4.7) implies that system (2) has at least one endemic steady state. This completes the proof. □

6.4 Proof of Theorem 3

Proof of Theorem 3 Denote $q(z) = z - 1 - \ln z$ which satisfies that $q(z) \geq 0$ for $z > 0$, and $q(z) = 0$ iff $z = 1$. We first deal with the global attractivity of E_1^* by constructing a Lyapunov functional and applying the LaSalle's invariance principle. Define

$$\mathcal{L}[\mathbf{u}](t) := \frac{1}{f_1(S_h^*, I_v^*)}[\mathcal{L}_1(t) + \mathcal{L}_2(t)] + \frac{1}{f_2(S_v^*, I_h^*)}[\mathcal{L}_3(t) + \mathcal{L}_4(t)],$$

wherein

$$\mathcal{L}_1(t) := \int_\Omega \left(S_h - S_h^* - S_h^* \ln \frac{S_h}{S_h^*}\right) \mathrm{d}x, \quad \mathcal{L}_2(t) := \int_\Omega \left(I_h - I_h^* - I_h^* \ln \frac{I_h}{I_h^*}\right) \mathrm{d}x,$$

and

$$\mathcal{L}_3(t) := \int_\Omega \left(S_v - S_v^* - S_v^* \ln \frac{S_v}{S_v^*}\right) \mathrm{d}x, \quad \mathcal{L}_4(t) := \int_\Omega \left(I_v - I_v^* - I_v^* \ln \frac{I_v}{I_v^*}\right) \mathrm{d}x.$$

Noting that $f_1(S_h^*, I_v^*) + \mu_1 S_h^* = \Lambda$, $\gamma_1 I_h^* = f_1(S_h^*, I_v^*)$, $f_2(S_v^*, I_h^*) + \mu_2 S_v^* = M$ and $\gamma_2 I_v^* = f_2(S_v^*, I_h^*)$, after elementary but tedious computations, we obtain

$$\begin{aligned}
\dot{\mathcal{L}}[\mathbf{u}](t) = &-\frac{\mu_1}{f_1(S_h^*, I_v^*)} \int_\Omega \frac{(S_h - S_h^*)^2}{S_h} \mathrm{d}x - \frac{D_S S_h^*}{f_1(S_h^*, I_v^*)} \int_\Omega \frac{|\nabla S_h|^2}{S_h^2} \mathrm{d}x \\
&- \frac{\mu_2}{f_2(S_v^*, I_h^*)} \int_\Omega \frac{(S_v - S_v^*)^2}{S_v} \mathrm{d}x - \frac{d_S S_v^*}{f_2(S_v^*, I_h^*)} \int_\Omega \frac{|\nabla S_v|^2}{S_v^2} \mathrm{d}x \\
&- \frac{D_I I_h^*}{f_1(S_h^*, I_v^*)} \int_\Omega \frac{|\nabla I_h|^2}{I_h^2} \mathrm{d}x - \frac{d_I I_v^*}{f_2(S_v^*, I_h^*)} \int_\Omega \frac{|\nabla I_v|^2}{I_v^2} \mathrm{d}x + Q,
\end{aligned}$$

where $\dot{\mathcal{L}}[\mathbf{u}](t) = \mathrm{d}\mathcal{L}[\mathbf{u}](t)/\mathrm{d}t$ and

$$\begin{aligned}
Q = &\int_\Omega \left[2 - \frac{S_h^*}{S_h} - \frac{f_1(S_h, I_v) I_h^*}{f_1(S_h^*, I_v^*) I_h} - \frac{I_h}{I_h^*} + \frac{S_h^* f_1(S_h, I_v)}{S_h f_1(S_h^*, I_v^*)}\right] \mathrm{d}x \\
&+ \int_\Omega \left[2 - \frac{S_v^*}{S_v} - \frac{f_2(S_v, I_h) I_v^*}{f_2(S_v^*, I_h^*) I_v} - \frac{I_v}{I_v^*} + \frac{S_v^* f_2(S_v, I_h)}{S_v f_2(S_v^*, I_h^*)}\right] \mathrm{d}x \\
= &\int_\Omega \left[2 - \frac{S_h^*}{S_h} - \frac{f_1(S_h, I_v) I_h^*}{f_1(S_h^*, I_v^*) I_h} - \frac{I_h}{I_h^*} + \frac{S_h^* f_1(S_h, I_v)}{S_h f_1(S_h^*, I_v^*)}\right] \mathrm{d}x \\
&+ \int_\Omega \left[2 - \frac{S_v^*}{S_v} - \frac{f_2(S_v, I_h) I_v^*}{f_2(S_v^*, I_h^*) I_v} - \frac{I_v}{I_v^*} + \frac{S_v^* f_2(S_v, I_h)}{S_v f_2(S_v^*, I_h^*)}\right] \mathrm{d}x \\
&+ \ln \frac{f_1(S_h, I_v) I_h^*}{f_1(S_h^*, I_v^*) I_h} + \ln \frac{S_h f_1(S_h^*, I_v^*)}{S_h^* f_1(S_h, I_v)} + \ln \frac{f_2(S_v, I_h) I_v^*}{f_2(S_v^*, I_h^*) I_v} + \ln \frac{S_v f_1(S_v^*, I_h^*)}{S_v^* f_2(S_v, I_h)} \\
&+ \ln \frac{S_h^*}{S_h} + \ln \frac{S_v^*}{S_v} + \ln \frac{I_h}{I_h^*} + \ln \frac{I_v}{I_v^*} \\
&+ \frac{S_h f_1(S_h^*, I_v^*) I_v}{S_h^* f_1(S_h, I_v) I_v^*} - \frac{S_h f_1(S_h^*, I_v^*) I_v}{S_h^* f_1(S_h, I_v) I_v^*} + \frac{S_v f_2(S_v^*, I_h^*) I_h}{S_v^* f_2(S_v, I_h) I_h^*} - \frac{S_v f_2(S_v^*, I_h^*) I_h}{S_v^* f_2(S_v, I_h) I_h^*} \\
= &-q\left(\frac{S_h^*}{S_h}\right) - q\left(\frac{S_v^*}{S_v}\right) - q\left(\frac{f_1(S_h, I_v) I_h^*}{f_1(S_h^*, I_v^*) I_h}\right) - q\left(\frac{f_2(S_v, I_h) I_v^*}{f_2(S_v^*, I_h^*) I_v}\right) \\
&- q\left(\frac{S_h f_1(S_h^*, I_v^*) I_v}{S_h^* f_1(S_h, I_v) I_v^*}\right) - q\left(\frac{S_v f_2(S_v^*, I_h^*) I_h}{S_v^* f_2(S_v, I_h) I_h^*}\right) + B_1 + B_2,
\end{aligned}$$

wherein

$$B_1=\left[1-\frac{S_h f_1(S_h^*,I_v^*)}{S_h^* f_1(S_h,I_v)}\right]\left[\frac{S_h^* f_1(S_h,I_v)}{S_h f_1(S_h^*,I_v^*)}-\frac{I_v}{I_v^*}\right],$$
$$B_2=\left[1-\frac{S_v f_2(S_v^*,I_h^*)}{S_v^* f_2(S_v,I_h)}\right]\left[\frac{S_v^* f_2(S_v,I_h)}{S_v f_2(S_v^*,I_h^*)}-\frac{I_h}{I_h^*}\right].$$

It follows from $f_i(S,I)=Sg_i(I)$ that B_1 and B_2 can be simplified as

$$B_1=\left[1-\frac{g_1(I_v^*)}{g_1(I_v)}\right]\left[\frac{g_1(I_v)}{g_1(I_v^*)}-\frac{I_v}{I_v^*}\right],\ B_2=\left[1-\frac{g_2(I_h^*)}{g_2(I_h)}\right]\left[\frac{g_2(I_h)}{g_2(I_h^*)}-\frac{I_h}{I_h^*}\right].$$

Similar to the arguments of (Shu et al. 2020, Theorem 5.6), we obtain $B_i\le 0$ since $g_i(I)$ is strictly increasing and concave down w.r.t I by means of the assumption (P2), $i=1,2$. Thus, $\dot{\mathcal{L}}[\mathbf{u}](t)\le 0$ and $\dot{\mathcal{L}}[\mathbf{u}](t)\equiv 0$ iff $S_h(t,\cdot)\equiv S_h^*$, $I_h(t,\cdot)\equiv I_h^*$, $S_v(t,\cdot)\equiv S_v^*$ and $I_v(t,\cdot)\equiv I_v^*$. Denote $\dot{\mathcal{L}}[\mathbf{u}_0]=:\dot{\mathcal{L}}[\mathbf{u}](t)|_{t=0}$. Consequently, the largest compact invariant set is

$$\Gamma=\left\{\mathbf{u}_0\in P\mid \dot{\mathcal{L}}[\mathbf{u}_0]=0\right\}\equiv\{E_1^*\}=\{(S_h^*,I_h^*,S_v^*,I_v^*)\}.$$

Then the LaSalle's invariance principle implies that

$$\lim_{t\to+\infty}(S_h(t,x),I_h(t,x),S_v(t,x),I_v(t,x))=(S_h^*,I_h^*,S_v^*,I_v^*),\quad \text{for all } x\in\bar{\Omega},$$

which means that E_1^* is globally attractive.

In the following, we shall cope with the locally asymptotic stability of E_1^*. With the aid of ideas of Shu et al. (2021), linearizing (6) at E_1^* to gives

$$\begin{cases}\partial_t\bar{S}_h=D_S\Delta\bar{S}_h-g_1(I_v^*)\bar{S}_h-S_h^*g_1'(I_v^*)\bar{I}_v-\mu_1\bar{S}_h, & t>0,\ x\in\Omega,\\ \partial_t\bar{I}_h=D_I\Delta\bar{I}_h+g_1(I_v^*)\bar{S}_h+S_h^*g_1'(I_v^*)\bar{I}_v-\gamma_1\bar{I}_h, & t>0,\ x\in\Omega,\\ \partial_t\bar{S}_v=d_S\Delta\bar{S}_v-g_2(I_h^*)\bar{S}_v-S_v^*g_2'(I_h^*)\bar{I}_h-\mu_2\bar{S}_v, & t>0,\ x\in\Omega,\\ \partial_t\bar{I}_v=d_I\Delta\bar{I}_v+g_2(I_h^*)\bar{S}_v+S_v^*g_2'(I_h^*)\bar{I}_h-\gamma_2\bar{I}_v, & t>0,\ x\in\Omega,\\ \partial_{\mathbf{n}}\bar{S}_h=\partial_{\mathbf{n}}\bar{I}_h=\partial_{\mathbf{n}}\bar{S}_v=\partial_{\mathbf{n}}\bar{I}_v=0, & t>0,\ x\in\partial\Omega.\end{cases}$$

Let $(\bar{S}_h,\bar{I}_h,\bar{S}_v,\bar{I}_v)^T=e^{\kappa t}(w_1,w_2,w_3,w_4)^T:=e^{\kappa t}w$. Then, we have

$$\begin{cases}\Pi(\kappa)w=0, & x\in\Omega,\\ \partial_{\mathbf{n}}w=0, & x\in\partial\Omega,\end{cases}\tag{43}$$

where

$$\Pi(\kappa)=\begin{pmatrix}-g_1(I_v^*)-\mu_1-\kappa+D_S\Delta & 0 & 0 & -S_h^*g_1'(I_v^*)\\ g_1(I_v^*) & -\gamma_1-\kappa+D_I\Delta & 0 & S_h^*g_1'(I_v^*)\\ 0 & -S_v^*g_2'(I_h^*) & -g_2(I_h^*)-\mu_2-\kappa+d_S\Delta & 0\\ 0 & S_v^*g_2'(I_h^*) & g_2(I_h^*) & -\gamma_2-\kappa+d_I\Delta\end{pmatrix}.$$

If any eigenvalue of system (43) has a negative real part, then E_1^* is locally asymptotically stable. Suppose not. Assume that there exists an eigenvalue $\kappa^* \in \mathbb{C}$ (the set of complex numbers) of (43) satisfying $\mathrm{Re}\,\kappa^* \geq 0$ (the real part of κ^*), then the operator $-\Delta$ with homogeneous Neumann boundary condition in Ω admits an eigenvalue $\hbar \geq 0$ such that the determinant of matrix $\Pi(\kappa^*)$ vanishes, that is,

$$\begin{vmatrix} -g_1(I_v^*) - \mu_1 - \kappa^* - D_S\hbar & 0 & 0 & -S_h^* g_1'(I_v^*) \\ g_1(I_v^*) & -\gamma_1 - \kappa^* - D_I\hbar & 0 & S_h^* g_1'(I_v^*) \\ 0 & -S_v^* g_2'(I_h^*) & -g_2(I_h^*) - \mu_2 - \kappa^* - d_S\hbar & 0 \\ 0 & S_v^* g_2'(I_h^*) & g_2(I_h^*) & -\gamma_2 - \kappa^* - d_I\hbar \end{vmatrix} = 0.$$

Through a direct calculations, one obtains

$$\begin{aligned} & g_1(I_v^*)g_2(I_h^*)(\kappa^* + D_I\hbar + \gamma_1)(\kappa^* + d_I\hbar + \gamma_2) \\ & \quad + g_1(I_v^*)(\kappa^* + d_S\hbar + \mu_2)(\kappa^* + D_I\hbar + \gamma_1)(\kappa^* + d_I\hbar + \gamma_2) \\ & \quad + g_2(I_h^*)(\kappa^* + D_S\hbar + \mu_1)(\kappa^* + D_I\hbar + \gamma_1)(\kappa^* + d_I\hbar + \gamma_2) + B_3 \\ & = S_h^* S_v^* g_1'(I_v^*) g_2'(I_h^*)(\kappa^* + D_S\hbar + \mu_1)(\kappa^* + d_S\hbar + \mu_2). \end{aligned}$$

where $B_3 = (\kappa^* + D_S\hbar + \mu_1)(\kappa^* + d_S\hbar + \mu_2)(\kappa^* + D_I\hbar + \gamma_1)(\kappa^* + d_I\hbar + \gamma_2)$ satisfying $|B_3| > 0$. Since $S_h^* g_1(I_v^*) = \gamma_1 I_h^*$ and $S_v^* g_2(I_h^*) = \gamma_2 I_v^*$, one gets $S_h^* S_v^* g_1'(I_v^*) g_2'(I_h^*) \leq \gamma_1\gamma_2$ due to the concavity of $g_i(I)$ w.r.t I in view of (P2), $i = 1, 2$. Then dividing the above equality by B_3 to yield

$$\begin{aligned} A_1(\kappa^*, \hbar) =: & \left| \frac{g_1(I_v^*)g_2(I_h^*)}{(\kappa^* + D_S\hbar + \mu_1)(\kappa^* + d_S\hbar + \mu_2)} + \frac{g_1(I_v^*)}{(\kappa^* + D_S\hbar + \mu_1)} \right. \\ & \left. + \frac{g_2(I_h^*)}{(\kappa^* + d_S\hbar + \mu_2)} + 1 \right| \\ \leq & \left| \frac{\gamma_1\gamma_2}{(\kappa^* + D_I\hbar + \gamma_1)(\kappa^* + d_I\hbar + \gamma_2)} \right| := A_2(\kappa^*, \hbar). \end{aligned}$$

Then $A_1(\kappa^*, \hbar)$ is greater than one, whereas $A_2(\kappa^*, \hbar)$ less than or equal to one owing to $\mathrm{Re}\,\kappa^* \geq 0$ and $\hbar \geq 0$ which is a contradiction. Accordingly, all eigenvalues of system (43) have negative real parts which yields that E_1^* is locally asymptotically stable. Combining the global attractivity of E_1^*, it follows that E_1^* is globally asymptotically stable. This finishes the proof. □

6.5 Proof of Theorem 4

Denote $k_{i*} := \min\{k_i(x, \cdot) : x \in \bar{\Omega}\}$ and $k_i^* := \max\{k_i(x, \cdot) : x \in \bar{\Omega}\}$. The following results are necessary before completing the proof of Theorem 4.

Lemma 5 *For each $D_I(x) > 0$ and $d_I(x) > 0$, $x \in \bar{\Omega}$, then $\mathcal{R}_0$ satisfies*

$$\sqrt{\frac{k_{1*}k_{2*}}{\gamma_1^*\gamma_2^*}} \leq \mathcal{R}_0 \leq \sqrt{\frac{k_1^*k_2^*}{\gamma_{1*}\gamma_{2*}}}. \tag{44}$$

Proof By Lemma 2, $1/\mathcal{R}_0$ is the unique principal eigenvalue of (26). Then

$$\begin{cases} \frac{1}{\mathcal{R}_0} k_1(x,H)\phi_2 = -\nabla \cdot [D_I(x)\nabla\phi_1] + \gamma_1(x)\phi_1, & x \in \Omega, \\ \frac{1}{\mathcal{R}_0} k_2(x,W)\phi_1 = -\nabla \cdot [d_I(x)\nabla\phi_2] + \gamma_2(x)\phi_2, & x \in \Omega, \\ \partial_{\mathbf{n}}\phi_1 = \partial_{\mathbf{n}}\phi_2 = 0, & x \in \partial\Omega. \end{cases} \tag{45}$$

Integrating two equations of (45) over Ω and then multiplying two resulting equalities to give

$$\mathcal{R}_0^2 \int_\Omega \gamma_1(x)\phi_1 \mathrm{d}x \int_\Omega \gamma_2(x)\phi_2 \mathrm{d}x = \int_\Omega k_1(x,H)\phi_2 \mathrm{d}x \int_\Omega k_2(x,W)\phi_1 \mathrm{d}x.$$

Hence, (44) holds due to the positivity of ϕ_i, $i = 1, 2$. This ends the proof. □

Remark 8 The result of Lemma 5 suggests that $\mathcal{R}_0$ is bounded, and if $k_1(\cdot, H)$, $k_2(\cdot, W)$ and $\gamma_i(\cdot)$ are constants, then $\mathcal{R}_0$ is independent of $D_I(\cdot)$ and $d_I(\cdot)$.

Proof of Theorem 4 To address (i). From (29), we have

$$\begin{cases} -D_{I0}\nabla \cdot [\bar{D}_I(x)\nabla\hat{\phi}_1] + \gamma_1(x)\hat{\phi}_1 = \frac{1}{\mathcal{R}_0} k_2(x,W)\hat{\phi}_2, & x \in \Omega, \\ -d_{I0}\nabla \cdot [\bar{d}_I(x)\nabla\hat{\phi}_2] + \gamma_2(x)\hat{\phi}_2 = \frac{1}{\mathcal{R}_0} k_1(x,H)\hat{\phi}_1, & x \in \Omega, \\ \partial_{\mathbf{n}}\hat{\phi}_1 = \partial_{\mathbf{n}}\hat{\phi}_2 = 0, & x \in \partial\Omega. \end{cases} \tag{46}$$

For the case $D_{I0} \to 0$. Choosing $\vartheta \in (0, 1)$, by using the density of $\Sigma := \{u \in C^2(\bar{\Omega}) \mid \partial_{\mathbf{n}} u = 0 \text{ on } \partial\Omega\}$ in $C(\bar{\Omega})$, there exist two positive functions $\hat{k}_1(x), \tilde{k}_1(x) \in \Sigma$ such that

$$\frac{k_1(x,H)}{1+\vartheta} < \hat{k}_1(x) < k_1(x,H) < \tilde{k}_1(x) < \frac{k_1(x,H)}{1-\vartheta}. \tag{47}$$

Let $(\breve{\phi}_1, \breve{\phi}_2) := (\frac{\lambda_1 \hat{k}_1}{\gamma_1}\phi_2^*, \phi_2^*)$ and $(\tilde{\phi}_1, \tilde{\phi}_2) := (\frac{\lambda_1 \tilde{k}_1}{\gamma_1}\phi_2^*, \phi_2^*)$, here ϕ_2^* is the positive eigenfunction of (7). By (47), for any $\vartheta > 0$, there is a constant $\tau > 0$ small enough such that

$$\begin{cases} -D_{I0}\nabla \cdot [\bar{D}_I(x)\nabla\breve{\phi}_1] + \gamma_1(x)\left[1 - \frac{k_1(x,H)}{\hat{k}_1(1+\vartheta)}\right]\breve{\phi}_1 \geq 0, & x \in \Omega, \\ \partial_{\mathbf{n}}\breve{\phi}_1 = 0, & x \in \partial\Omega, \end{cases} \tag{48}$$

and

$$\begin{cases} -D_{I0}\nabla \cdot [\bar{D}_I(x)\nabla\tilde{\phi}_1] + \gamma_1(x)\left[1 - \frac{k_1(x,H)}{\tilde{k}_1(1-\vartheta)}\right]\tilde{\phi}_1 \leq 0, & x \in \Omega, \\ \partial_{\mathbf{n}}\tilde{\phi}_1 = 0, & x \in \partial\Omega, \end{cases} \tag{49}$$

for $0 < D_{I0} < \tau$. Furthermore, it follows from (7) and (48) that

$$-d_{I0}\nabla\cdot[\bar{d}_I(x)\nabla\breve{\phi}_2]+\gamma_2(x)\breve{\phi}_2-\lambda_1 k_2(x,W)\breve{\phi}_1 \geq -d_{I0}\nabla\cdot[\bar{d}_I(x)\nabla\breve{\phi}_2]+\gamma_2(x)\breve{\phi}_2-\lambda_1^2\cdot\frac{\bar{k}(x)}{\gamma_1(x)}\breve{\phi}_2=0,$$

and

$$-D_{I0}\nabla\cdot[\bar{D}_I(x)\nabla\breve{\phi}_1]+\gamma_1(x)\breve{\phi}_1 \geq \gamma_1(x)\frac{k_1(x,H)}{\hat{k}_1(1+\vartheta)}\breve{\phi}_1=\lambda_1\frac{k_1(x,H)}{1+\vartheta}\breve{\phi}_2.$$

Thus,

$$\begin{cases} -D_{I0}\nabla\cdot[\bar{D}_I(x)\nabla\breve{\phi}_1]+\gamma_1(x)\breve{\phi}_1 \geq \lambda_1\dfrac{k_1(x,H)}{1+\vartheta}\breve{\phi}_2, & x\in\Omega,\\ -d_{I0}\nabla\cdot[\bar{d}_I(x)\nabla\breve{\phi}_2]+\gamma_2(x)\breve{\phi}_2-\lambda_1 k_2(x,W)\breve{\phi}_1 \geq 0, & x\in\Omega,\\ \partial_{\mathbf{n}}\breve{\phi}_1=\partial_{\mathbf{n}}\breve{\phi}_2=0, & x\in\partial\Omega. \end{cases} \tag{50}$$

Multiplying the first inequality of (50) and (46) by $\hat{\phi}_1$ and $\breve{\phi}_1$ respectively, and then integrating by parts over Ω to yield

$$\begin{cases} D_{I0}\displaystyle\int_\Omega \bar{D}_I(x)\nabla\breve{\phi}_1\nabla\hat{\phi}_1\mathrm{d}x+\int_\Omega\gamma_1(x)\breve{\phi}_1\hat{\phi}_1\mathrm{d}x \geq \lambda_1\int_\Omega\frac{k_1(x,H)}{1+\vartheta}\breve{\phi}_2\hat{\phi}_1\mathrm{d}x,\\ D_{I0}\displaystyle\int_\Omega \bar{D}_I(x)\nabla\hat{\phi}_1\nabla\breve{\phi}_1+\int_\Omega\gamma_1(x)\hat{\phi}_1\breve{\phi}_1\mathrm{d}x=\frac{1}{\mathcal{R}_0}\int_\Omega k_2(x,W)\hat{\phi}_2\breve{\phi}_1\mathrm{d}x. \end{cases}$$

Hence,

$$\int_\Omega\left[\frac{1}{\mathcal{R}_0}k_2(x,W)\hat{\phi}_2\breve{\phi}_1-\frac{\lambda_1}{1+\vartheta}k_1(x,H)\hat{\phi}_1\breve{\phi}_2\right]\mathrm{d}x \geq 0. \tag{51}$$

In the similar way,

$$\int_\Omega\left[\frac{1}{\mathcal{R}_0}k_1(x,H)\hat{\phi}_1\breve{\phi}_2-\lambda_1 k_2(x,W)\hat{\phi}_2\breve{\phi}_1\right]\mathrm{d}x \geq 0. \tag{52}$$

Together with (51) and (52), we get

$$\frac{1}{\mathcal{R}_0}\int_\Omega k_1(x,H)\hat{\phi}_1\breve{\phi}_2\mathrm{d}x \geq \lambda_1\int_\Omega k_2(x,W)\hat{\phi}_2\breve{\phi}_1 \geq \lambda_1\cdot\frac{\mathcal{R}_0\lambda_1}{1+\vartheta}\int_\Omega k_1(x,H)\hat{\phi}_1\breve{\phi}_2\mathrm{d}x,$$

that is,

$$\left(\frac{1}{\mathcal{R}_0^2}-\frac{\lambda_1^2}{1+\vartheta}\right)\int_\Omega k_1(x,H)\hat{\phi}_1\breve{\phi}_2\mathrm{d}x \geq 0.$$

Then $\mathcal{R}_0 \leq \sqrt{1+\vartheta}/\lambda_1$ owing to $\lambda_1 > 0$. In addition, we can similarly obtain $\mathcal{R}_0 \geq \sqrt{1-\vartheta}/\lambda_1$ with the help of (49). Thus, $\sqrt{1-\vartheta}/\lambda_1 \leq \mathcal{R}_0 \leq \sqrt{1+\vartheta}/\lambda_1$ which induces $\mathcal{R}_0 \to 1/\lambda_1$ as $D_{I0} \to 0$ due to the arbitrariness of ϑ.

For the case $D_{I0} \to \infty$. By means of Lemma 5, passing to a sequence if necessary, there is a constant $\widehat{\mathcal{R}}_0 > 0$ such that $\mathcal{R}_0 \to \widehat{\mathcal{R}}_0$ as $D_{I0} \to \infty$. Without loss of generality, set $\|\phi_1\| + \|\phi_2\| = 1$. By (45) and L^p estimates, $\|\phi_1\|_{W_p^2(\Omega)}$ and $\|\phi_2\|_{W_p^2(\Omega)}$ are uniformly bounded, for any integer $p > 1$. Via applying the Sobolev embedding theorem, $\|\phi_1\|_{C^1(\Omega)}$ and $\|\phi_2\|_{C^1(\Omega)}$ are also uniformly bounded. Then there are positive functions $\bar{\phi}_1$ and $\bar{\phi}_2 \in C^1(\bar{\Omega})$ such that $(\phi_1, \phi_2) \to (\bar{\phi}_1, \bar{\phi}_2)$ in $C^1(\bar{\Omega})$, as $D_{I0} \to \infty$. Then $\bar{\phi}_2$ satisfies

$$\begin{cases} -d_{I0}\nabla \cdot [\bar{d}_I(x)\nabla\bar{\phi}_2] + \gamma_2^0\bar{\phi}_2 - \dfrac{k_2(x, W)}{\widehat{\mathcal{R}}_0}\bar{\phi}_1 = 0, & x \in \Omega, \\ \partial_{\mathbf{n}}\bar{\phi}_2 = 0, & x \in \partial\Omega. \end{cases} \tag{53}$$

Applying the elliptic regularity estimate to the first equation of (45), $\bar{\phi}_1$ is a constant and so $\bar{\phi}_1 = \frac{k_1^0 \int_\Omega \bar{\phi}_2 \mathrm{d}x}{\hat{\mathcal{R}}_0 \int_\Omega \gamma_1(x)\mathrm{d}x}$ since $k_1(x, H) \equiv k_1^0$. Thus,

$$\begin{cases} -d_{I0}\nabla \cdot [\bar{d}_I(x)\nabla\bar{\phi}_2] + \gamma_2^0\bar{\phi}_2 - \dfrac{k_2(x, W)}{\widehat{\mathcal{R}}_0} \cdot \dfrac{k_1^0 \int_\Omega \bar{\phi}_2 \mathrm{d}x}{\widehat{\mathcal{R}}_0 \int_\Omega \gamma_1(x)\mathrm{d}x} = 0, & x \in \Omega, \\ \partial_{\mathbf{n}}\bar{\phi}_2 = 0, & x \in \partial\Omega. \end{cases}$$

Then $\mathcal{R}_0^2 \to \widehat{\mathcal{R}}_0^2 = \frac{k_1^0 \int_\Omega k_2(x,W)\mathrm{d}x}{\gamma_2^0 \int_\Omega \gamma_1(x)\mathrm{d}x}$ as $D_{I0} \to \infty$. The proof of (ii) is analogous.

To prove (iii). By Lemma 5, passing to a sequence if necessary, there is a constant $\widetilde{\mathcal{R}}_0 > 0$ such that $\mathcal{R}_0 \to \widetilde{\mathcal{R}}_0$ as $D_{I0} \to 0$ and $d_{I0} \to 0$. Thus, for any $\tilde{\vartheta} > 0$, there is a sufficiently small constant $\tilde{\tau} > 0$ such that

$$|\mathcal{R}_0 - \widetilde{\mathcal{R}}_0| < \tilde{\vartheta}, \quad \text{for all } D_{I0},\, d_{I0} \in (0, \tilde{\tau}). \tag{54}$$

Consider the eigenvalue problem

$$\begin{cases} \mathbf{B}\varpi - \dfrac{1}{\theta}\mathbf{F}\varpi = \kappa_1(\theta)\varpi, & x \in \Omega, \\ \partial_{\mathbf{n}}\varpi = 0, & x \in \partial\Omega, \end{cases} \tag{55}$$

wherein $\varpi = (\varpi_2, \varpi_4)^T$, $\theta > 0$, $\kappa_1(\theta)$ is the principal eigenvalue of (55), and operators $\mathbf{B}$ and $\mathbf{F}$ are given by Sect. 3.2. From (45), $\mathbf{B}\phi - \frac{1}{R_0}\mathbf{F}\phi = 0$, $x \in \Omega$, $\phi = (\phi_1, \phi_2)^T$. It then follows from (54) that

$$\mathbf{B}\phi - \frac{1}{\widetilde{\mathcal{R}}_0 + \tilde{\vartheta}}\mathbf{F}\phi \geq 0 \geq \mathbf{B}\phi - \frac{1}{\widetilde{\mathcal{R}}_0 - \tilde{\vartheta}}\mathbf{F}\phi, \quad x \in \Omega.$$

Utilizing Proposition 3.4 in Lam and Lou (2016) yields that

$$\kappa_1(\widetilde{\mathcal{R}}_0+\tilde{\vartheta}) \geq \kappa_1(\mathcal{R}_0) = 0 \geq \kappa_1(\widetilde{\mathcal{R}}_0-\tilde{\vartheta}), \tag{56}$$

and, by Theorem 1.4 in Lam and Lou (2016), we get

$$\lim_{D_{I0}\to 0,\, d_{I0}\to 0} \kappa_1(\theta) = \hat{\kappa}_1(\theta) := -\max_{x\in\bar{\Omega}} \mathcal{Q}(V(x)),$$

where

$$V(x) = \begin{pmatrix} -\gamma_1(x) & \frac{k_1(x,H)}{\theta} \\ \frac{k_2(x,W)}{\theta} & -\gamma_2(x) \end{pmatrix},$$

and $\mathcal{Q}(V(x))$ is the principal eigenvalue of cooperative matrix $V(x)$ at x, i.e., $\mathcal{Q}(V(x)) = \frac{-[\gamma_1(x)+\gamma_2(x)]+\sqrt{[\gamma_1(x)+\gamma_2(x)]^2+4[\bar{k}(x)/\theta^2-\bar{\gamma}(x)]}}{2}$. Hence,

$$\hat{\kappa}_1(\theta) = -\max_{x\in\bar{\Omega}} \mathcal{Q}(V(x)) \begin{cases} >0, & \theta>\hat{\theta}, \\ =0, & \theta=\hat{\theta}, \\ <0, & \theta<\hat{\theta}, \end{cases} \quad \text{where } \hat{\theta} := \max_{x\in\bar{\Omega}} \sqrt{\bar{k}(x)/\bar{\gamma}(x)}.$$

Then $sign(\hat{\kappa}_1(\theta)) = sign(\theta-\hat{\theta})$. Combining with (56), one obtains $\hat{\theta}-\tilde{\vartheta} \leq \widetilde{\mathcal{R}}_0 \leq \hat{\theta}+\tilde{\vartheta}$. By the arbitrariness of $\tilde{\vartheta}$, $\mathcal{R}_0 \to \max_{x\in\bar{\Omega}} \sqrt{\bar{k}(x)/\bar{\gamma}(x)}$ when both D_{I0} and d_{I0} tend to zero.

To deal with (iv). Similar to the proof of (ii). Let $\|\phi_1\|+\|\phi_2\|=1$. Passing to a sequence if necessary, $\phi_1 \to \bar{\phi}_1^\star$ in $C^1(\bar{\Omega})$ as $D_{I0}\to\infty$ and $d_{I0}\to 0$, here $\bar{\phi}_1^\star$ is a nonnegative constant. Thus, for any $\vartheta>0$, there is a constant $\bar{\tau}>0$ such that

$$\begin{cases} \frac{k_2(x,W)}{\mathcal{R}_0}(\bar{\phi}_1^\star-\vartheta) < -d_{I0}\nabla\cdot[\bar{d}_I(x)\nabla\phi_2]+\gamma_2(x)\phi_2 < \frac{k_2(x,W)}{\mathcal{R}_0}(\bar{\phi}_1^\star+\vartheta), & x\in\Omega, \\ \partial_{\mathbf{n}}\phi_2 = 0, & x\in\partial\Omega, \end{cases}$$

for any $0<d_{I0}$, $\frac{1}{D_{I0}}<\bar{\tau}$, which implies that $\lim_{d_{I0}\to 0}\phi_2 = \frac{k_2(x,W)\bar{\phi}_1^\star}{\gamma_2(x)\mathcal{R}_0}$. Thus, $\bar{\phi}_1^\star>0$ owing to $\|\phi_1\|+\|\phi_2\|=1$. Integrating the first equation of (45) in Ω, and then letting $D_{I0}\to\infty$ and $d_{I0}\to 0$ to give $\mathcal{R}_0^2 \to \frac{\int_\Omega \bar{k}(x)/\gamma_2(x)\mathrm{d}x}{\int_\Omega \gamma_1(x)\mathrm{d}x}$. Similarly, one can obtain $\mathcal{R}_0^2 \to \frac{\int_\Omega \bar{k}(x)/\gamma_1(x)\mathrm{d}x}{\int_\Omega \gamma_2(x)\mathrm{d}x}$ as $D_{I0}\to 0$ and $d_{I0}\to\infty$. By applying the ideas of (i) and (iv), we can prove (v). This ends the proof. □

6.6 Proof of Theorem 5

Proof of Theorem 5 To prove (i). Since the domain Ω is one-dimensional, we rewrite system (45) as follows

$$\begin{cases} -D_{I0}[\bar{D}_I(x)\phi_{1x}]_x + \gamma_1\phi_1 = \dfrac{1}{\mathcal{R}_0}k_1(x,H)\phi_2, & x \in \Omega, \\ -d_{I0}[\bar{d}_I(x)\phi_{2x}]_x + \gamma_2\phi_2 = \dfrac{1}{\mathcal{R}_0}k_2\phi_1, & x \in \Omega, \\ \phi_{1x} = \phi_{2x} = 0, & x \in \partial\Omega, \end{cases} \tag{57}$$

where u_x and u_{xx} denote the first and second partial derivatives of u w.r.t x, respectively, $u \in \{\phi_1, \phi_2\}$. Similar to the discussion in (Cantrell and Cosner 2003, Proposition 2.20) and (Hess 1991, Lemma 15.1), one obtains that $\mathcal{R}_0$ and the corresponding eigenfunctions $(\phi_1, \phi_2)^T$ are analytic functions of D_{I0} and d_{I0}. Hence, differentiating problem (57) by D_{I0} yields

$$\begin{cases} -D_{I0}[\bar{D}_I(x)\dot{\phi}_{1x}]_x - [\bar{D}_I(x)\phi_{1x}]_x + \gamma_1\dot{\phi}_1 = \dfrac{k_1(x,H)}{\mathcal{R}_0}\dot{\phi}_2 - \dfrac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}k_1(x,H)\phi_2, & x \in \Omega, \\ -d_{I0}[\bar{d}_I(x)\dot{\phi}_{2x}]_x + \gamma_2\dot{\phi}_2 = \dfrac{k_2}{\mathcal{R}_0}\dot{\phi}_1 - \dfrac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}k_2\phi_1, & x \in \Omega, \\ \dot{\phi}_{1x} = \dot{\phi}_{2x} = 0, & x \in \partial\Omega, \end{cases} \tag{58}$$

where $\dot{}$ represents the derivative of D_{I0}. From the second equations of (58) and (57), we get

$$\begin{cases} \dot{\phi}_1 = \dfrac{\mathcal{R}_0}{k_2}\left\{-[d_I(x)\dot{\phi}_{2x}]_x + \gamma_2\dot{\phi}_2 + \dfrac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}k_2\phi_1\right\}, \\ \phi_1 = \dfrac{\mathcal{R}_0}{k_2}\left\{-[d_I(x)\phi_2]_x + \gamma_2\phi_2\right\}. \end{cases} \tag{59}$$

Multiplying the first equation of (58) by ϕ_2 and then integrating by parts in Ω, by (59), we obtain

$$\begin{aligned} &\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}\int_\Omega k_1(x,H)\phi_2^2\mathrm{d}x \\ &= \int_\Omega \left[D_I(x)\dot{\phi}_{1x}\right]_x \phi_2\mathrm{d}x + \int_\Omega [\bar{D}_I(x)\phi_{1x}]_x\phi_2\mathrm{d}x - \gamma_1\int_\Omega \dot{\phi}_1\phi_2 + \frac{1}{\mathcal{R}_0}\int_\Omega k_1(x,H)\dot{\phi}_2\phi_2\mathrm{d}x \\ &= \frac{\mathcal{R}_0}{k_2}\int_\Omega \{[D_I(x)\phi_{2x}]_x - \gamma_1\phi_2\}\left\{-[d_I(x)\dot{\phi}_{2x}]_x + \gamma_2\dot{\phi}_2 + \frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}k_2\phi_1\right\}\mathrm{d}x \\ &\quad + \frac{\mathcal{R}_0}{k_2}\int_\Omega \{-[d_I(x)\phi_{2x}]_x + \gamma_2\phi_2\}[\bar{D}_I(x)\phi_{2x}]_x\mathrm{d}x + \frac{1}{\mathcal{R}_0}\int_\Omega k_1(x,H)\dot{\phi}_2\phi_2\mathrm{d}x \\ &= G_1 + G_2 + G_3, \end{aligned} \tag{60}$$

where

$$G_1 = \frac{\mathcal{R}_0}{k_2}\int_\Omega \{-[d_I(x)\phi_{2x}]_x + \gamma_2\phi_2\}\,[\bar{D}_I(x)\phi_{2x}]_x\mathrm{d}x,$$
$$G_2 = \frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0}\int_\Omega \{[D_I(x)\phi_{2x}]_x - \gamma_1\phi_2\}\,\phi_1\mathrm{d}x,$$
$$G_3 = \frac{1}{\mathcal{R}_0}\int_\Omega k_1(x,H)\phi_2\dot{\phi}_2\mathrm{d}x - \frac{\mathcal{R}_0}{k_2}\int_\Omega \{-[D_I(x)\phi_{2x}]_x + \gamma_1\phi_2\}\{-[d_I(x)\dot{\phi}_{2x}]_x + \gamma_2\dot{\phi}_2\}\mathrm{d}x.$$

From the first equation of (57), one obtains $G_2 = -\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}\int_\Omega k_1(x,H)\phi_2^2\mathrm{d}x$ by integrating by parts. Direct calculating yields that

$$G_1 = -\frac{\mathcal{R}_0 d_{I0}}{k_2}\int_\Omega \left[\bar{D}_I'(x)\bar{d}_I'(x)\phi_{2x}^2 + \bar{D}_I(x)\bar{d}_I(x)\phi_{2xx}^2\right]\mathrm{d}x - \frac{\mathcal{R}_0\gamma_2}{k_2}\int_\Omega \bar{D}_I\phi_{2x}^2\mathrm{d}x$$
$$- \frac{\mathcal{R}_0 d_{I0}}{k_2}\int_\Omega \left[\bar{D}_I(x)\bar{d}_I'(x) + \bar{D}_I'(x)\bar{d}_I(x)\right]\phi_{2x}\phi_{2xx}\mathrm{d}x.$$

Since $\bar{D}_I'(x)\bar{d}_I(x) = \bar{D}_I(x)\bar{d}_I'(x)$ for $x \in \Omega$, and

$$\int_\Omega \bar{D}_I(x)\bar{d}_I'(x)\phi_{2x}\phi_{2xx}\mathrm{d}x = -\frac{1}{2}\int_\Omega [\bar{D}_I(x)\bar{d}_I''(x) + \bar{D}_I'(x)\bar{d}_I'(x)]\phi_{2x}^2\mathrm{d}x,$$
$$\int_\Omega \bar{D}_I'(x)\bar{d}_I(x)\phi_{2x}\phi_{2xx}\mathrm{d}x = -\frac{1}{2}\int_\Omega [\bar{D}_I''(x)\bar{d}_I(x) + \bar{D}_I'(x)\bar{d}_I'(x)]\phi_{2x}^2\mathrm{d}x,$$

we obtain

$$G_1 = \frac{\mathcal{R}_0}{k_2}\left\{\int_\Omega [d_{I0}\bar{d}_I''(x) - \gamma_2]\bar{D}_I(x)\phi_{2x}^2\mathrm{d}x - d_{I0}\int_\Omega \bar{D}_I(x)\bar{d}_I(x)\phi_{2xx}^2\mathrm{d}x\right\}.$$

Multiplying the first equation of (58) by $\dot{\phi}_2$ and integrating by parts over Ω, and then together with (59), one has

$$\frac{1}{\mathcal{R}_0}\int_\Omega k_1(x,H)\phi_2\dot{\phi}_2\mathrm{d}x = \frac{\mathcal{R}_0}{k_2}\int_\Omega \{-D_{I0}[\bar{D}_I(x)\dot{\phi}_{2x}]_x + \gamma_1\dot{\phi}_2\}\{-d_{I0}[\bar{d}_I(x)\nabla\phi_{2x}]_x + \gamma_2\phi_2\}\mathrm{d}x.$$

To show

$$\int_\Omega \{-D_{I0}[\bar{D}_I(x)\dot{\phi}_{2x}]_x + \gamma_1\dot{\phi}_2\}\{-d_{I0}[\bar{d}_I(x)\phi_{2x}]_x + \gamma_2\phi_2\}\mathrm{d}x$$
$$= \int_\Omega [-D_{I0}[\bar{D}_I(x)\phi_{2x}]_x + \gamma_1\phi_2][-d_{I0}[\bar{d}_I(x)\dot{\phi}_{2x}]_x + \gamma_2\dot{\phi}_2]\mathrm{d}x, \qquad (61)$$

we only need to verify

$$\int_\Omega [\bar{D}_I(x)\phi_{2x}]_x[\bar{d}_I(x)\dot{\phi}_{2x}]_x\mathrm{d}x = \int_\Omega [\bar{D}_I(x)\dot{\phi}_{2x}]_x[\bar{d}_I(x)\phi_{2x}]_x\mathrm{d}x,$$

which is equivalent to

$$\int_\Omega [\bar{D}_I'(x)\bar{d}_I(x) - \bar{D}_I(x)\bar{d}_I'(x)]\phi_{2x}\dot{\phi}_{2xx}\mathrm{d}x + \int_\Omega [\bar{D}_I(x)\bar{d}_I'(x) - \bar{D}_I'(x)\bar{d}_I(x)]\dot{\phi}_{2x}\phi_{2xx}\mathrm{d}x = 0.$$

Note that $\bar{D}_I'(\cdot)\bar{d}_I(\cdot) = \bar{D}_I(\cdot)\bar{d}_I'(\cdot)$ in Ω. It thus follows that (61) holds. Then we have $G_3 = 0$. Substituting G_1, G_2 and G_3 into (60) to yield

$$\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^3}\int_\Omega k_1(x,H)\phi_2^2\mathrm{d}x = -\frac{1}{2k_2}\left\{\int_\Omega [\gamma_2 - d_{I0}\bar{d}_I''(x)]\bar{D}_I(x)\phi_{2x}^2\mathrm{d}x + \int_\Omega \bar{D}_I d_I \phi_{2xx}^2\mathrm{d}x\right\}.$$

Thus, $\dot{\mathcal{R}}_0 \le 0$ since $d_{I0}\bar{d}_I'' < \gamma_2$, i.e., $\mathcal{R}_0$ is a monotone nonincreasing function of D_{I0}. In addition, $\dot{\mathcal{R}}_0 = 0$ iff ϕ_2 is a constant in Ω. Following from the second equation of (57) that ϕ_1 is also a constant in Ω. So, $k_1(\cdot, H)$ is constant in Ω by the first equation of (57). In conclusion, $\mathcal{R}_0$ decreases monotonically w.r.t D_{I0} if $k_1(\cdot, H)$ is non-constant in Ω. In the similar fashion, we can deal with (ii). This finishes the proof. □

6.7 Proof of Theorem 6

Proof of Theorem 6 We first show (i). From system (45), one has

$$\begin{cases} -D_I(x)\phi_{1xx} - D_I'(x)\phi_{1x} + \gamma_1(x)\phi_1 = \dfrac{1}{\mathcal{R}_0}k_1(x,H)\phi_2, & x\in\Omega,\\ -d_I(x)\phi_{2xx} - d_I'(x)\phi_{2x} + \gamma_2(x)\phi_2 = \dfrac{1}{\mathcal{R}_0}k_2(x,W)\phi_1, & x\in\Omega,\\ \phi_{1x} = \phi_{2x} = 0, & x\in\partial\Omega. \end{cases} \tag{62}$$

Differentiating system (62) by D_I yields

$$\begin{cases} -\phi_{1xx} - D_I(x)\grave{\phi}_{1xx} - \tilde{D}_I(x)\phi_{1x} - D_I'(x)\grave{\phi}_{1x} + \gamma_1(x)\grave{\phi}_1 = \dfrac{k_1(x,H)}{\mathcal{R}_0}\grave{\phi}_2 - \dfrac{\grave{\mathcal{R}}_0}{\mathcal{R}_0^2}k_1(x,H)\phi_2, & x\in\Omega,\\ -d_I(x)\grave{\phi}_{2xx} - d_I'(x)\grave{\phi}_{2x} + \gamma_2(x)\grave{\phi}_2 = \dfrac{k_2(x,W)}{\mathcal{R}_0}\grave{\phi}_1 - \dfrac{\grave{\mathcal{R}}_0}{\mathcal{R}_0^2}k_2(x,W)\phi_1, & x\in\Omega,\\ \grave{\phi}_{1x} = \grave{\phi}_{2x} = 0, & x\in\partial\Omega, \end{cases} \tag{63}$$

where $\grave{}$ denotes the derivative of D_I and $\tilde{D}_I(\cdot) = D_I''(\cdot)/D_I'(\cdot)$. Multiplying ϕ_1 and $\grave{\phi}_1$ by the first equations of (63) and (62), respectively, and integrating by parts and then subtracting the two equalities, we get

$$\begin{aligned}\frac{\grave{\mathcal{R}}_0}{\mathcal{R}_0^2}\int_\Omega k_1(x,H)\phi_1\phi_2\mathrm{d}x = &-\int_\Omega \phi_{1x}^2\mathrm{d}x + \frac{1}{2}\left\{[\tilde{D}_I(x)\phi_1^2]|_{\partial\Omega} - \int_\Omega \tilde{D}_I'(x)\phi_1^2\mathrm{d}x\right\}\\ &+\frac{1}{\mathcal{R}_0}\int_\Omega k_1(x,H)(\phi_1\grave{\phi}_2 - \grave{\phi}_1\phi_2)\mathrm{d}x.\end{aligned} \tag{64}$$

Similarly,

$$\frac{\grave{\mathcal{R}}_0}{\mathcal{R}_0^2}\int_{\Omega} k_2(x,W)\phi_1\phi_2 \mathrm{d}x = \frac{1}{\mathcal{R}_0}\int_{\Omega} k_2(x,W)(\grave{\phi}_1\phi_2 - \phi_1\grave{\phi}_2)\mathrm{d}x. \tag{65}$$

Adding (64) and (65), and by $k_1(\cdot,H) \equiv k_2(\cdot,W)$ in $\bar{\Omega}$, one has

$$\frac{\grave{\mathcal{R}}_0}{\mathcal{R}_0^2}\int_{\Omega}[k_1(x,H)+k_2(x,W)]\phi_1\phi_2 \mathrm{d}x = -\int_{\Omega}\phi_{1x}^2\mathrm{d}x + \frac{1}{2}\left\{[\tilde{D}_I(x)\phi_1^2]|_{\partial\Omega} - \int_{\Omega}\tilde{D}_I'(x)\phi_1^2\mathrm{d}x\right\} \le 0 \tag{66}$$

since $\tilde{D}_I(x)\phi_1^2(x)|_{\partial\Omega} \le 0$ and $\tilde{D}_I'(x) \ge 0$, $x \in \Omega$. Then $\mathcal{R}_0$ is monotone nonincreasing w.r.t D_I. The proof of (ii) resembles that of (i), so we skip the details. This ends the proof. □

6.8 Proof of Theorem 7

Proof of Theorem 7 By Theorem 4 (iv), we have

$$\mathcal{R}_0 \to \sqrt{\frac{\int_{\Omega}\bar{k}(x)\gamma_2^{-1}(x)\mathrm{d}x}{\int_{\Omega}\gamma_1(x)\mathrm{d}x}}, \quad \text{as } D_{I0}\to\infty \text{ and } d_{I0}\to 0.$$

Then, for any $0 < \epsilon \ll 1$, there is a constant $C_{12} = C_{12}(\epsilon) > 0$ large enough, such that

$$\mathcal{R}_0(D_I, d_I^1) \le (1+\epsilon)\sqrt{\frac{\int_{\Omega}\bar{k}(x)\gamma_2^{-1}(x)\mathrm{d}x}{\int_{\Omega}\gamma_1(x)\mathrm{d}x}}, \quad \text{for } D_{I0},\ \frac{1}{d_{I0}^1} \ge C_{12}.$$

Since

$$\mathcal{R}_0 \to \sqrt{\frac{\int_{\Omega}k_1(x,H)\mathrm{d}x\int_{\Omega}k_2(x,W)\mathrm{d}x}{\int_{\Omega}\gamma_1(x)\mathrm{d}x\int_{\Omega}\gamma_2(x)\mathrm{d}x}}, \quad \text{when } D_{I0}\to\infty \text{ and } d_{I0}\to\infty,$$

there is a constant $C_{13} = C_{13}(\epsilon, D_{I0}) > 0$ large enough, such that

$$\mathcal{R}_0(D_I, d_I^2) \ge (1-\epsilon)\sqrt{\frac{\int_{\Omega}k_1(x,H)\mathrm{d}x\int_{\Omega}k_2(x,W)\mathrm{d}x}{\int_{\Omega}\gamma_1(x)\mathrm{d}x\int_{\Omega}\gamma_2(x)\mathrm{d}x}},$$

for any $d_{I0}^2 \ge C_{13}$. By (9), there is a constant $0 < \epsilon^* \ll 1$ such that

$$(1-\epsilon^*)^2\frac{\int_{\Omega}k_1(x,H)\mathrm{d}x\int_{\Omega}k_2(x,W)\mathrm{d}x}{\int_{\Omega}\gamma_1(x)\mathrm{d}x\int_{\Omega}\gamma_2(x)\mathrm{d}x} > (1+\epsilon^*)^2\frac{\int_{\Omega}\bar{k}(x)/\gamma_2(x)\mathrm{d}x}{\int_{\Omega}\gamma_1(x)\mathrm{d}x}.$$

Let $D_{I0}^* = C_{12}(\epsilon^*)$, $d_{I0}^1 = C_{12}^{-1}(\epsilon^*)$ and $d_{I0}^2 = C_{13}(\epsilon^*, D_I^*)$. Then $\mathcal{R}_0(D_I^*, d_I^1) < \mathcal{R}_0(D_I^*, d_I^2)$. The proof of (ii) is analogous. This completes the proof. □

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