RISK-NEUTRAL VALUATION UNDER FUNDING COSTS AND COLLATERALIZATION

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Abstract

The expected cash flows approach leading to an extended version of the classical risk-neutral valuation formula was proposed in the recent works by Pallavicini et al. [6] and Brigo et al. [3] who studied the problem of valuation of contracts under differential funding costs and collateralization. The crucial difference between the approach developed in [3, 6] and the present note is that the pricing formula via conditional expectation of discounted adjusted cash flows is postulated in [3, 6] and it is justified using financial arguments, whereas in this note the formula is established in more general set-up and it is shown to be a consequence of the standard replication-based arguments.

Keywords: risk-neutral valuation, hedging, funding costs, collateral, margin agreement

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1 Simple Trading Model

We first provide an informal explanation how the cash flows adjustments are motivated in [3, 6]. To show how the adjusted cash flows originate, we assume that the hedger buys a call option on an equity asset S_T with strike K. In other words, he enters into a contract (A, C) where $A_t = \mathbb{1}_{\{t=T\}}(S_T - K)^+$. We assume that collateral is re-hypothecated and, as in [6], we denote the cash in the collateral account at time t by C_t. When $C_t > 0$ then the cash collateral is received as a guarantee and remunerated by the hedger at the rate c^b and when C_t is negative, then the cash collateral is posted by the hedger and remunerated by the counterparty at the rate c^l.

We analyze the hedger's operations with the treasury, the repo market and the counterparty, in order to fund his trade. The following steps in each small interval $[t, t + dt]$ are presented from the point of view of the hedger buying the option. Let us first describe the hedger's initial trades at time t:

1. The hedger wishes to buy a call option with maturity T whose current price is $V_t < 0$. The price is negative for the hedger since he is the buyer and thus has to pay the price at time t. He borrows $-V_t$ cash from the treasury and buys the call option.
2. To hedge the call option he bought, he takes the synthetic short position $\xi_t < 0$ in the stock in the repo market. Specifically, he borrows $H_t = -\xi_t S_t$ from the treasury and lends cash at repo market and gets $-\xi_t > 0$ shares of the stock as collateral.
3. He immediately sells the stock just obtained from the repo to the market, getting back H_t in cash and gives H_t back to the treasury. Hence his outstanding debt at time t to the treasury is V_t.
4. He posts (when $C_t = -C_t^- < 0$) or receives (when $C_t = C_t^+ > 0$) the cash collateral, which can be rehypothecated in other transactions.

We now proceed to the description of the hedger's terminal trades at time $t + dt$:

1. To close the repo, the hedger needs to buy and deliver $-\xi_t$ stock. For this trade, he needs $-\xi_t S_{t+dt}$ cash and thus he borrows that amount of cash from the treasury. He buys $-\xi_t$ stock and gives the stock back to close the repo. Hence he gets back the cash H_t deposited at time t with interest $h_t H_t\, dt$, that is, $(1 + h_t\, dt)H_t$ cash. Thus the net value of these trades is
$$(1 + h_t\, dt)H_t + \xi_t S_{t+dt} = -(1 + h_t\, dt)\xi_t S_t + \xi_t\, S_{t+dt} = \xi_t\, dS_t - h_t \xi_t S_t\, dt. \tag{1.1}$$
2. He sells the call option in the market and thus gets $-V_{t+dt}$ cash (recall that $V_{t+dt} < 0$ for the buyer of the option). He pays back his outstanding debt V_t to the treasury plus interest $f_t V_t\, dt$ using the cash $-V_{t+dt}$ just obtained, the net effect being
$$-V_{t+dt} + V_t(1 + f_t\, dt) = -dV_t + f_t V_t\, dt. \tag{1.2}$$
3. He either pays the interest $c_t^b C_t^+\, dt$ or receives the interest $c_t^l C_t^-\, dt$ on the collateral amount C_t and pays the interest $f_t C_t^-\, dt$ to the treasury or receives the interest $f_t C_t^+\, dt$ from the treasury. Hence the net cash flow at time $t + dt$ due to the margin account equals
$$d\widehat{C}_t := (c_t^l - f_t)C_t^-\, dt - (c_t^b - f_t)C_t^+\, dt. \tag{1.3}$$
4. The total value of cash flows from equations (1.1)–(1.3) equals
$$\xi_t\, (dS_t - h_t S_t\, dt) - dV_t + f_t V_t\, dt + d\widehat{C}_t = 0 \tag{1.4}$$
where the equality is a consequence of the self-financing property, which underpins the hedger's trades. Equalities (1.3) and (1.4) will be formally established in Lemma 2.1.

Assume that $\mathbb{Q}^h$ is a probability measure such that $\mathbb{E}_{\mathbb{Q}^h}(dS_t - h_t S_t\, dt \,|\, \mathcal{F}_t) = 0$ or, more formally, that the process $\tilde{S}_t := (B^h)^{-1} S$ is a local martingale under $\mathbb{Q}^h$ where $dB^h_t = h_t B^h_t\, dt$ with $B^h_0 = 1$. Then we obtain an explicit representation under $\mathbb{Q}^h$ for the price V_t for $t < T$

$$V_t = B^f_t\, \mathbb{E}_{\mathbb{Q}^h}\bigg(-(B^f_T)^{-1}(S_T - K)^+ - \int_t^T (B^f_u)^{-1}\, d\widehat{C}_u \,\bigg|\, \mathcal{F}_t\bigg) \tag{1.5}$$

where $dB^f_t = f_t B^f_t\, dt$ with $B^f_0 = 1$. It is natural to refer to (1.5) as the *extended risk-neutral valuation formula* with funding costs.

To derive another version of the risk-neutral valuation formula, we now assume that there exists a risk-neutral measure $\mathbb{Q}^r$ associated with a locally risk-free bank account numeraire with the risk-free interest rate r_t, so that $\mathbb{E}_{\mathbb{Q}^r}(dS_t - r_t S_t\, dt \,|\, \mathcal{F}_t) = 0$. This is a formal postulate, which is satisfied in a typical model, and it does not mean that funding from the risk-free bank account is available to the hedger or that the model at hand is arbitrage-free in any sense. Then, using (1.4), we obtain

$$dV_t - r_t V_t\, dt = (f_t - r_t)V_t\, dt + \xi_t(dS_t - r_t S_t\, dt) - (h_t - r_t)H_t\, dt + d\widehat{C}_t.$$

Hence the price V_t admits also the following implicit representation under $\mathbb{Q}^r$ called the *risk-nautral valuation formula with adjusted cash flows*

$$\begin{aligned} V_t =& B^r_t\, \mathbb{E}_{\mathbb{Q}^r}\bigg(-(B^r_T)^{-1}(S_T - K)^+ - \int_t^T (B^r_u)^{-1}\, d\widehat{C}_u + \int_t^T (B^r_u)^{-1}(r_u - f_u)V_u\, du \,\bigg|\, \mathcal{F}_t\bigg) \\ &+ B^r_t\, \mathbb{E}_{\mathbb{Q}^r}\bigg(\int_t^T (B^r_u)^{-1}(h_u - r_u)H_u\, du \,\bigg|\, \mathcal{F}_t\bigg) \end{aligned} \tag{1.6}$$

where $B^r_t = r_t B^r_t\, dt$ with $B^r_0 = 1$ and $\widehat{C}_t$ equals (see (1.3))

$$\widehat{C}_t = \int_0^t (c^l_u - f_u)C^-_u\, du - \int_0^t (c^b_u - f_u)C^+_u\, du. \tag{1.7}$$

The formal derivation of (1.5) and (1.6) is given in Proposition 2.1. It is clear that (1.5) and (1.6) are equivalent and they reduce to the classic risk-neutral valuation formula when $f = h = r$.

Note that (1.6) coincides with the result given earlier in [6]. It will be more formally derived in Corollary 2.1 in a more general framework. Observe that the financial interpretation of the process r was employed in the derivation of (1.6) given in [6], but in fact it is not relevant at all for the validity of (1.6).

2 Extended Risk-Neutral Valuation Formulae

We fix a finite trading horizon date $T > 0$ for our model of the financial market. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ models the flow of information available to all traders. For convenience, we assume that the initial σ-field $\mathcal{F}_0$ is trivial. Moreover, all processes introduced in what follows are implicitly assumed to be $\mathbb{F}$-adapted and, as usual, any semimartingale is assumed to be càdlàg. Also, we will assume that any process Y satisfies $\Delta Y_0 := Y_0 - Y_{0-} = 0$. Let us introduce the notation for the prices of all traded assets in our model.

Traded risky assets. We denote by $(S^1, \dots, S^d)$ the collection of the *prices* of a family of d risky assets, which do not pay dividends. We assume that the processes $S^1, S^2, \dots, S^d$ are continuous semimartingalea.

Treasury rates. The *lending* (respectively, *borrowing*) *cash account* B^l (respectively, B^b) is used for unsecured lending (respectively, borrowing) of cash from the treasury. When the borrowing and lending cash rates are equal, we denote the single cash account by B^f. We assume that $dB^l_t = f^l_t B^l_t\, dt$, $dB^b_t = f^b_t B^b_t\, dt$ and $dB^f_t = f_t B^f_t\, dt$.

Repo market. We denote by $B^{i,l}$ (respectively $B^{i,b}$) the *lending* (respectively *borrowing*) *repo account* associated with the ith risky asset. In the special case when $B^{i,l} = B^{i,b}$, we will use the notation B^i. We assume that $dB^{i,l}_t = h^{i,l}_t B^{i,l}_t\, dt$, $dB^{i,b}_t = h^{i,b}_t B^{i,b}_t\, dt$ and $dB^i_t = h^i_t B^i_t\, dt$.

2.1 Valuation in a Linear Model with Funding Costs

We will now examine a special case of the linear model with funding costs introduced in [1]. We assume that we have the *cash funding account* B^f and d risky assets traded in the repo market with the *asset prices* S^i and the corresponding *as funding account* B^i for $i = 1, 2, \ldots, d$. Recall from [1] that the trading constraint $\psi^i_t B^i_t + \xi^i_t S^i_t = 0$ means that the positions in stock are funded using exclusively the account B^i with the repo rate h^i. The value process of a trading strategy φ thus equals

$$V^p_t(\varphi) := \psi^f_t B^f_t + \sum_{i=1}^{d} (\psi^i_t B^i_t + \xi^i_t S^i_t) = F_t + \sum_{i=1}^{d} H^i_t$$

where F stands for the *funding* part and $H = \sum_{i=1}^d H^i$ represents the *hedging* part of the value process $V^p(\varphi)$, specifically,

$$F_t := \psi^f_t B^f_t + \sum_{i=1}^{d} \psi^i_t B^i_t, \quad H^i_t := \xi^i_t S^i_t = -\psi^i_t B^i_t.$$

Under the postulate that $\psi^i_t B^i_t + \xi^i_t S^i_t = 0$, we also have

$$V^p_t(\varphi) = \psi^f_t B^f_t$$

and thus $V^p(\varphi)$ coincides with the *cash funding* process $F^f := \psi^f B^f$.

Assuming that A represents the cash flows stream (also known as the dividends stream) of a given contract, the self-financing condition inclusive of the stream of cash flows A reads

$$\begin{aligned} dV^p_t(\varphi) &= \psi^f_t\, dB^f_t + \sum_{i=1}^{d} (\psi^i_t\, dB^i_t + \xi^i_t\, dS^i_t) + dA_t \\ &= \psi^f_t\, dB^f_t - \sum_{i=1}^{d} \xi^i_t S^i_t (B^i_t)^{-1}\, dB^i_t + \sum_{i=1}^{d} \xi^i_t\, dS^i_t + dA_t \end{aligned}$$

where the second equality is the consequence of the trading constraint.

Let us now consider the case of a collateralized contract (A, C). The *margin account* C is assumed to be any adapted process of finite variation such that $C_T = 0$. We do not consider C to be a part of the hedger's trading strategy, but rather as a part of cash flows of a contract. This is motivated by two reasons. First, the margin account C will always be present no matter whether a given contract is hedged or not. Second, the process C is assumed here to be exogenously given, so it does not depend on the hedger's trading strategy. Therefore, the value process of a trading strategy φ is still defined as the sum $F + H$, rather than $F + H + C$.

At the same time, we assume that the cash collateral C is rehypothecated, that is, it is used for the hedger's trading purposes and thus it is implicit in F and H through the self-financing condition stated in Definition 2.1. As in [1], the process $V^p(\varphi) := F + H$ is the *value process* of the hedger's trading strategy, whereas the process $V(\varphi) := V^p(\varphi) - C = F + H - C$ represents the hedger's *wealth* under rehypothecation (although the symbols F and H were not used in [1]). Due to the terminal condition $C_T = 0$, we always have that $V_T(\varphi) = V^p_T(\varphi)$, but $V_t(\varphi) \neq V^p_t(\varphi)$ for $t < T$, in general.

In the case of a collateralized contract with the margin process C and remuneration rates c^l and c^b for the margin account, to compute the price and hedge for a collateralized contract, it suffices to replace the cash flow stream A by the process A^C given by the following expression

$$\begin{aligned} A^C_t &:= A_t + C_t + \int_0^t C^-_u (B^{c,l}_u)^{-1}\, dB^{c,l}_u - \int_0^t C^+_u (B^{c,b}_u)^{-1}\, dB^{c,b}_u \\ &= A_t + C_t + \int_0^t c^l_u C^-_u \, du - \int_0^t c^b_u C^+_u \, du. \end{aligned} \tag{2.1}$$

Hence the contract (A, C) can be formally identified with the cash flows stream A^C.

2.1.1 Dynamics of the Value Process of a Trading Strategy

We will now examine the dynamics of the value process of a self-financing trading strategy inclusive of the cash flows stream A^C and the concept of replication of the contract A^C. The following definition summarizes these notions.

Definition 2.1 Assume that a contract has cash flows given by a process A^C of finite variation and a *replicating strategy* φ for the cash flow stream A^C exists, meaning that there exists a trading strategy φ such that $V_T(\varphi) = V^p_T(\varphi) = 0$ where $V(\varphi) = V^p(\varphi) - C$ and the process $V^p(\varphi)$ satisfies the *self-financing condition*

$$dV^p_t(\varphi) = \psi^f_t \, dB^f_t - \sum_{i=1}^d \xi^i_t S^i_t (B^i_t)^{-1}\, dB^i_t + \sum_{i=1}^d \xi^i_t \, dS^i_t + dA^C_t \tag{2.2}$$

Then the *ex-dividend price* equals $\pi_t(A^C) = V_t(\varphi) = V^p_t(\varphi) - C_t$ for all $t \in [0, T]$.

Since $dB^i_t = h^i_t B^i_t \, dt$, the self-financing condition can be represented as follows

$$dV^p_t(\varphi) = f_t V^p_t(\varphi)\, dt + \sum_{i=1}^d \xi^i_t (dS^i_t - h^i_t S^i_t \, dt) + dA^C_t. \tag{2.3}$$

We are in a position to formally establish equations (1.3) and (1.4), which were informally postulated in Section 1

Lemma 2.1 *Assume that a single stock S is traded and the contract (A, C) has null cash flows A before time T. Then the self-financing condition reduces to the following equality for the hedger's wealth process $V(\varphi)$*

$$dV_t(\varphi) = f_t V_t(\varphi)\, dt + \xi_t (dS_t - h_t S_t \, dt) + d\widehat{C}_t \tag{2.4}$$

where $\widehat{C}$ is given by (1.3).

Proof. Equality (2.4) is an immediate consequence of equations (2.1) and (2.3) and the relationship $V(\varphi) = V^p(\varphi) - C$. □

Remark 2.1 We have implicitly assumed that the initial endowment of the hedger equals zero. This assumption can be made here without loss of generality, since we deal with a linear model, but it is no longer true when dealing with a non-linear set-up examined in Section 2.2. Nevertheless, for simplicity of presentation, we will still assume in Section 2.2 that the initial endowment of the hedger equals zero.

2.1.2 Linear BSDE

Recall that replication of a contract with cash flows A^C by a self-financing trading strategy φ means that $V_T(\varphi) = V_T^p(\varphi) = 0$. Moreover, we assume that that the hedger's initial endowment is null. Using (2.3), we thus obtain the following linear BSDE for the portfolio's value and hedging strategy

$$dY_t = \Big(f_t Y_t - \sum_{i=1}^{d} Z_t^i h_t^i S_t^i\Big)\,dt + \sum_{i=1}^{d} Z_t^i\,dS_t^i + dA_t^C \tag{2.5}$$

with the terminal condition $Y_T = 0$. Under mild technical assumptions, the unique solution to this BSDE exists and it is given by an explicit formula (of course, provided that the dynamics of S^i are known). We thus henceforth assume that the processes $V(\varphi)$ and $V^p(\varphi)$ are well defined and we will examine their properties.

2.1.3 Abstract Risk-Neutral Valuation Formulae

Let us define (note that, by convention, we set $B_0^{\gamma^i} = 1$)

$$B_t^{\gamma^i} := \exp\left(\int_0^t \gamma_u^i\,du\right)$$

where γ^i is an arbitrary adapted and integrable process. Let $\gamma = (\gamma^1, \gamma^2, \dots, \gamma^d)$ and let $\mathbb{Q}^\gamma$ be a probability measure such that the process $\bar{S}^i = (B^{\gamma^i})^{-1} S^i$ is a $\mathbb{Q}^\gamma$–local martingale. This is equivalent to the property that the process

$$S_t^i - \int_0^t \gamma_u^i S_u^i\,du$$

is a $\mathbb{Q}^\gamma$–local martingale for $i = 1, 2, \dots, d$, meaning that $\mathbb{Q}^\gamma$ is an equivalent local martingale measure (ELMM) for the asset price S^i discounted with the process B^{γ^i}. Let φ be a self-financing trading strategy, in the sense of Definition 2.1, and let η be an arbitrary adapted and integrable process. We define

$$B_t^\eta := \exp\left(\int_0^t \eta_u\,du\right).$$

Lemma 2.2 *Let η be an arbitrary $\mathbb{F}$-adapted process and let the process $V^\eta(\varphi)$ be given by*

$$V_t^\eta(\varphi) := V_t^p(\varphi) + B_t^\eta \int_0^t \alpha_u F_u^f (B_u^\eta)^{-1}\,du + \sum_{i=1}^{d} B_t^\eta \int_0^t \beta_u^i H_u^i (B_u^\eta)^{-1}\,du - B_t^\eta \int_{(0,t]} (B_u^\eta)^{-1}\,dA_u^C. \tag{2.6}$$

If

$$\alpha_t = \eta_t - f_t, \quad \beta_t^i = h_t^i - \gamma_t^i, \tag{2.7}$$

then the process $\bar{V}^\eta := (B^\eta)^{-1} V^\eta$ is a local martingale under $\mathbb{Q}^\gamma$

Proof. Equation (2.6) implies that

$$dV_t^\eta(\varphi) = dV_t^p(\varphi) + \alpha_t F_t^f\,dt + \sum_{i=1}^{d} \beta_t^i H_t^i\,dt + (V_t^\eta - V_t^p)\eta_t\,dt - dA_t^C.$$

Since $V^p_t = \psi^f_t B^f_t = F^f_t$, we obtain

$$\begin{aligned}
dV^\eta_t - \eta_t V^\eta_t\,dt &= \psi^f_t\,dB^f_t - \sum_{i=1}^d \xi^i_t S^i_t (B^i_t)^{-1}\,dB^i_t + \sum_{i=1}^d \xi^i_t\,dS^i_t + \alpha_t F^f_t\,dt + \sum_{i=1}^d \beta^i_t H^i_t\,dt - \eta_t F^f_t\,dt \\
&= (\alpha_t + f_t - \eta_t)F^f_t\,dt + \sum_{i=1}^d (\beta^i_t - h^i_t + \gamma^i_t) H^i_t\,dt + \sum_{i=1}^d \xi^i_t\,(dS^i_t - \gamma^i_t S^i_t\,dt).
\end{aligned}$$

It is now clear that if α_t and β_t satisfy (2.7), then V^η is a local martingale under $\mathbb{Q}^\gamma$. □

The following result is a simple consequence of Lemma 2.2. We stress that the financial interpretation of processes η and γ^i for $i = 1, 2, \dots, d$, if any, is not relevant in the derivation of an abstract risk-neutral valuation formulae (2.8)–(2.9).

Proposition 2.1 *Assume that a contract (A, C) can be replicated by a trading strategy φ and the associated process $\bar{V}^\eta(\varphi)$ is a true martingale under $\mathbb{Q}^\gamma$. Then the ex-dividend price of (A, C) satisfies $\pi_t(A, C) = V_t(\varphi) = V^p_t(\varphi) - C_t$ where*

$$\begin{aligned}
V^p_t(\varphi) = B^\eta_t\,\mathbb{E}_{\mathbb{Q}^\gamma}\Big(&- \int_{(t,T]} (B^\eta_u)^{-1}\,dA^C_u + \int_t^T (\eta_u - f_u) F^f_u (B^\eta_u)^{-1}\,du \,\Big|\, \mathcal{F}_t\Big) \\
&+ B^\eta_t\,\mathbb{E}_{\mathbb{Q}^\gamma}\Big(\sum_{i=1}^d \int_t^T (h^i_u - \gamma^i_u) H^i_u (B^\eta_u)^{-1}\,du \,\Big|\, \mathcal{F}_t\Big)
\end{aligned} \tag{2.8}$$

or, equivalently,

$$\begin{aligned}
V_t(\varphi) = B^\eta_t\,\mathbb{E}_{\mathbb{Q}^\gamma}\Big(&- \int_{(t,T]} (B^\eta_u)^{-1}\,dA_u + \int_t^T (\bar{c}_u - f_u) C_u (B^\eta_u)^{-1}\,du \,\Big|\, \mathcal{F}_t\Big) \\
&+ B^\eta_t\,\mathbb{E}_{\mathbb{Q}^\gamma}\Big(\int_t^T (\eta_u - f_u) V_u(\varphi) (B^\eta_u)^{-1}\,du + \sum_{i=1}^d \int_t^T (h^i_u - \gamma_u) H^i_u (B^\eta_u)^{-1}\,du \,\Big|\, \mathcal{F}_t\Big)
\end{aligned} \tag{2.9}$$

where we denote

$$\bar{c}_t := c^l_t \mathbb{1}_{\{C_t<0\}} + c^b_t \mathbb{1}_{\{C_t\geq 0\}}. \tag{2.10}$$

Proof. Equality (2.8) is an immediate consequence of the martingale property of $V^\eta(\varphi)$ and the equalities $V^p_T(\varphi) = 0$ and $V^p(\varphi) = F^f$. To derive (2.9) from (2.8), we note that

$$F^C_t = -\int_0^t \bar{c}_u C_u\,du$$

and we apply the integration by parts formula and the equality $C_T = 0$, to get

$$\begin{aligned}
\int_{(t,T]} (B^\eta_u)^{-1}\,dC_u &= C_T (B^\eta_T)^{-1} - C_t (B^\eta_t)^{-1} - \int_t^T C_u\,d(B^\eta_u)^{-1} \\
&= -C_t (B^\eta_t)^{-1} + \int_t^T \eta_u (B^\eta_u)^{-1} C_u\,du.
\end{aligned}$$

This completes the derivation of (2.9). □

As special cases, we obtain equations (1.5) and (1.6). To get (1.5), it suffices to set $\eta = f$ and $\gamma^i = h^i$ for $i = 1, 2, \dots, d$ so that

$$V^p_t(\varphi) = B^f_t\,\mathbb{E}_{\mathbb{Q}^h}\Big(- \int_{(t,T]} (B^f_u)^{-1}\,dA^C_u \,\Big|\, \mathcal{F}_t\Big).$$

Similarly, upon setting $\eta = \gamma^i = r$, we obtain the following formula

$$V_t^p(\varphi) = B_t^r \, \mathbb{E}_{\mathbb{Q}^r}\bigg(- \int_{(t,T]} (B_u^\eta)^{-1} \, dA_u^C + \int_t^T (r_u - f_u) F_u^f (B_u^r)^{-1} \, du \,\Big|\, \mathcal{F}_t \bigg) + B_t^r \, \mathbb{E}_{\mathbb{Q}^r}\bigg(\sum_{i=1}^d \int_t^T (h_u^i - r_u) H_u^i (B_u^r)^{-1} \, du \,\Big|\, \mathcal{F}_t \bigg),$$

which can be referred to as the risk-neutral valuation with adjusted cash flows.

2.2 Valuation in a Non-Linear Model with Funding Costs

A non-linear extension of the framework introduced in the preceding section is obtained when a single cash rate f is replaced by the lending and borrowing rates, denoted as f^l and f^b and, similarly, by introducing different repo rates for the long and short positions in stock, denoted as $h^{i,l}$ and $h^{i,b}$, respectively. Then we have the following generic decomposition of the value process $V^p(\varphi)$ of a trading strategy φ

$$V_t^p(\varphi) = \psi_t^l B_t^l + \psi_t^b B_t^{i,b} + \sum_{i=1}^d \big(\psi_t^{i,l} B_t^{i,l} + \psi_t^{i,b} B_t^{i,b} + \xi_t^i S_t^i \big) = F_t + \sum_{i=1}^d H_t^i = F_t + H_t$$

where in turn

$$F_t := \psi_t^l B_t^l + \psi_t^b B_t^b + \sum_{i=1}^d (\psi_t^{i,l} B_t^{i,l} + \psi_t^{i,b} B_t^{i,b}), \quad H_t^i := \xi_t^i S_t^i.$$

Moreover, we postulate that $\psi_t^l \geq 0, \psi_t^b \leq 0$ and $\psi_t^l \psi_t^b = 0$ for all t. Finally, we assume that $\psi_t^{i,l} \geq 0, \psi_t^{i,b} \leq 0, \psi_t^{i,l} \psi_t^{i,b} = 0$ and the following *repo funding condition* holds

$$\psi_t^{i,l} B_t^{i,l} + \psi_t^{i,b} B_t^{i,b} + \xi_t^i S_t^i = 0.$$

Consequently,

$$V^p(\varphi) = F_t^f := \psi_t^l B_t^l + \psi_t^b B_t^b.$$

As before, we set $V(\varphi) = V^p(\varphi) - C$ where C is the margin account.

2.2.1 Dynamics of the Value Process of a Trading Strategy

We are now in a position to derive the non-linear dynamics of the value process and thus also obtain a generic non-linear pricing BSDE for the contract (A, C).

Lemma 2.3 *We have* $\psi_t^l = (B_t^l)^{-1} (V_t^p(\varphi))^+$, $\psi_t^b = -(B_t^b)^{-1} (V_t^p(\varphi))^-$ *and*

$$\psi_t^{i,l} = (B_t^{i,l})^{-1} (\xi_t^i S_t^i)^- = (B_t^{i,l})^{-1} (H_t^i)^-, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1} (\xi_t^i S_t^i)^+ = -(B_t^{i,b})^{-1} (H_t^i)^+.$$

Proof. Note that

$$\psi_t^l B_t^l + \psi_t^b B_t^b = V_t^p(\varphi), \quad \psi_t^{i,l} B_t^{i,l} + \psi_t^{i,b} B_t^{i,b} = -\xi_t^i S_t^i.$$

It thus suffices to use the postulated natural conditions $\psi_t^l \geq 0, \psi_t^b \leq 0, \psi_t^l \psi_t^b = 0$ and $\psi_t^{i,l} \geq 0, \psi_t^{i,b} \leq 0, \psi_t^{i,l} \psi_t^{i,b} = 0$. □

Using Lemma 2.3, we obtain the unique dynamics for the value process of a self-financing trading strategy φ inclusive of cash flows stream A^C. We postulate that

$$dV_t^p(\varphi) = \psi_t^l\, dB_t^l + \psi_t^b\, dB_t^b + \sum_{i=1}^d (\psi_t^{i,l}\, dB_t^{i,l} + \psi_t^{i,b}\, dB_t^{i,b} + \xi_t^i\, dS_t^i) + dA_t^C \tag{2.11}$$

and thus

$$\begin{aligned} dV_t^p(\varphi) = {} & (B_t^l)^{-1}(V_t^p(\varphi))^+\, dB_t^l - (B_t^b)^{-1}(V_t^p(\varphi))^-\, dB_t^b + \sum_{i=1}^d (B_t^{i,l})^{-1}(\xi_t^i S_t^i)^-\, dB_t^{i,l} \\ & - \sum_{i=1}^d (B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+\, dB_t^{i,b} + \sum_{i=1}^d \xi_t^i\, dS_t^i + dA_t^C. \end{aligned}$$

When all account processes are absolutely continuous, we obtain the following lemma.

Lemma 2.4 *The value process of a self-financing trading strategy φ satisfies*

$$dV_t^p(\varphi) = f_t^l(V_t^p(\varphi))^+\, dt - f_t^b(V_t^p(\varphi))^-\, dt + \sum_{i=1}^d \Big(\xi_t^i\, dS_t^i + h_t^{i,l}(H_t^i)^-\, dt - h_t^{i,b}(H_t^i)^+\, dt\Big) + dA_t^C. \tag{2.12}$$

where $H_t^i = \xi_t^i S_t^i$.

To formally 'linearize' the problem, we will now introduce the *effective rates* $\bar{f}$ and $\bar{h}^i$, which depend on the hedger's trading strategy.

Lemma 2.5 *The value process $V^p(\varphi)$ satisfies*

$$dV_t^p(\varphi) = \bar{f}_t F_t^f\, dt + \sum_{i=1}^d \xi_t^i \big(dS_t^i - \bar{h}_t^i S_t^i\, dt\big) + dA_t^C \tag{2.13}$$

where the effective rates $\bar{f}$ and $\bar{h}^i$ are given by

$$\bar{f}_t := f_t^l\, \mathbb{1}_{\{F_t^f \geq 0\}} + f_t^b\, \mathbb{1}_{\{F_t^f(\varphi) < 0\}} = f_t^l\, \mathbb{1}_{\{V_t(\varphi) + C_t \geq 0\}} + f_t^b\, \mathbb{1}_{\{V_t(\varphi) + C_t < 0\}} \tag{2.14}$$

and

$$\bar{h}_t^i := h_t^{i,l}\, \mathbb{1}_{\{H_t^i \leq 0\}} + h_t^{i,b}\, \mathbb{1}_{\{H_t^i(\varphi) > 0\}}. \tag{2.15}$$

Proof. From (2.12), we obtain

$$\begin{aligned} dV_t^p(\varphi) &= f_t^l(V_t^p(\varphi))^+\, dt - f_t^b(V_t^p(\varphi))^-\, dt + \sum_{i=1}^d \Big(\xi_t^i\, dS_t^i + h_t^{i,l}(H_t^i)^-\, dt - h_t^{i,b}(H_t^i)^+\, dt\Big) + dA_t^C \\ &= \bar{f}_t F_t^f(\varphi)\, dt + \sum_{i=1}^d \xi_t^i \big(dS_t^i - \bar{h}_t^i S_t^i\, dt\big) + dA_t^C \end{aligned} \tag{2.16}$$

where $\bar{f}$ and $\bar{h}^i$ are given by (2.14) and (2.15), respectively. □

2.2.2 Nonlinear BSDE

Under the standing assumption that the initial endowment of the hedger is null, equation (2.16) leads to the following nonlinear BSDE for the portfolio's value and the hedging strategy

$$dY_t = \Big(\bar{f}_t Y_t - \sum_{i=1}^d \bar{h}_t^i Z_t^i S_t^i\Big)\, dt + \sum_{i=1}^d Z_t^i\, dS_t^i + dA_t^C \tag{2.17}$$

with the terminal condition $Y_T = 0$ where from (2.14)–(2.15)

$$\bar{f}_t := f_t^l\, \mathbb{1}_{\{Y_t \geq 0\}} + f_t^b\, \mathbb{1}_{\{Y_t < 0\}}$$

and

$$\bar{h}_t^i := h_t^{i,l}\, \mathbb{1}_{\{Z_t^i S_t^i \leq 0\}} + h_t^{i,b}\, \mathbb{1}_{\{Z_t^i S_t^i > 0\}}.$$

We henceforth assume that BSDE (2.17) has a unique solution in a suitable space of stochastic processes.

2.2.3 Abstract Risk-Neutral Valuation Formulae

Recall that

$$B_t^\eta := \exp\left(\int_0^t \eta_u\, du\right)$$

and

$$B_t^{\gamma^i} := \exp\left(\int_0^t \gamma_u^i\, du\right)$$

where η and γ^i for $i = 1, 2, \dots, d$ are arbitrary adapted and integrable processes. Let $\mathbb{Q}^\gamma$ be a probability measure such that the processes $(B^{\gamma^i})^{-1} S^i$, $i = 1, 2, \dots, d$ are $\mathbb{Q}^\gamma$-local martingales. Then Lemmas 2.2 and 2.5 yield the following result, which extends Proposition 2.1 and agrees with the pricing formula postulated in [3]. For the validity of this result, one needs to impose some mild integrability assumptions.

Proposition 2.2 *Assume that a collateralized contract (A, C) can be replicated by a trading strategy φ. Then its ex-dividend price at time t equals $V_t(\varphi) = V^p(\varphi) - C_t$ where*

$$\begin{aligned} V_t^p(\varphi) &= B_t^\eta\, \mathbb{E}_{\mathbb{Q}^\gamma}\bigg(- \int_{(t,T]} (B_u^\eta)^{-1}\, dA_u^C + \int_t^T (\eta_u - \bar{f}_u) F_u^f (B_u^\eta)^{-1}\, du \,\Big|\, \mathcal{F}_t\bigg) \\ &\quad + B_t^\eta\, \mathbb{E}_{\mathbb{Q}^\gamma}\bigg(\sum_{i=1}^d \int_t^T (\bar{h}_u^i - \gamma_u^i) H_u^i (B_u^\eta)^{-1}\, du \,\Big|\, \mathcal{F}_t\bigg) \end{aligned}$$

or, equivalently,

$$\begin{aligned} V_t(\varphi) &= B_t^\eta\, \mathbb{E}_{\mathbb{Q}^\gamma}\bigg(- \int_{(t,T]} (B_u^\eta)^{-1}\, dA_u + \int_t^T (\bar{c}_u - \bar{f}_u) C_u (B_u^\eta)^{-1}\, du \,\Big|\, \mathcal{F}_t\bigg) \\ &\quad + B_t^\eta\, \mathbb{E}_{\mathbb{Q}^\gamma}\bigg(\int_t^T (\eta_u - \bar{f}_u) V_u(\varphi) (B_u^\eta)^{-1}\, du + \sum_{i=1}^d \int_t^T (\bar{h}_u^i - \gamma_u^i) H_u^i (B_u^\eta)^{-1}\, du \,\Big|\, \mathcal{F}_t\bigg) \end{aligned} \tag{2.18}$$

where

$$\bar{c}_t := c_t^l \mathbb{1}_{\{C_t < 0\}} + c_t^b \mathbb{1}_{\{C_t \geq 0\}}.$$

Proof. The first asserted formula follows from (2.16). The second is easy consequences of the first one. □

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