An Introduction to Backward Stochastic Differential Equations (BSDEs)

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- This introduction is based on Touzi [14], Bouchard [1] and Rutkowski [12] as well as papers given in the reference section.

- Throughout this introduction, we consider a d-dimensional Brownian motion W on a complete probability space $(\Omega, \mathcal{F}, P)$ over a finite horizon $[0, T]$ for a fixed $T > 0$.

- We denote by $(\mathcal{F}_t)_{0 \le t \le T}$ the augmented filtration generated by W.

1 Motivation

Consider $d = 1$ and a model of a financial market with

- one risk-free asset with interest rate r. We assume that r is bounded and predictable,

- one risky asset whose prices $(S_t)_{0 \le t \le T}$ are given by $S_0 > 0$ and

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t,$$

  where μ is bounded and predictable, and σ is positive, bounded away from zero and predictable,

- an investor, whose amount of money invested in S at time t is denoted by π_t, and his/her total wealth is denoted by Y_t,

- wealth process dynamics

$$\begin{aligned} \mathrm{d}Y_t &= \frac{\pi_t}{S_t} \,\mathrm{d}S_t + r_t(Y_t - \pi_t) \,\mathrm{d}t \\ &= (\pi_t \mu_t - \pi_t r_t + r_t Y_t) \,\mathrm{d}t + \pi_t \sigma_t \,\mathrm{d}W_t, \end{aligned}$$

  assuming that the strategy is self-financing,

- a European option with payoff ξ at time T, where ξ is a square-integrable, $\mathcal{F}_T$-measurable random variable.

Assuming that the investor wants to replicate the option payoff, we want to solve

$$\mathrm{d}Y_t = (\pi_t\mu_t - \pi_t r_t + r_t Y_t)\,\mathrm{d}t + \pi_t\sigma_t\,\mathrm{d}W_t, \qquad (1)$$
$$Y_T = \xi, \qquad (2)$$

which is a stochastic differential equation for Y with terminal condition $Y_T = \xi$.

In this case (see Section 4 below), we can solve explicitly the problem with Y given by

$$Y_t = E^Q\left[\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}\xi \middle| \mathcal{F}_t\right],$$

where the probability measure Q is defined by

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left(-\int_0^T \frac{\mu_s - r_s}{\sigma_s}\,\mathrm{d}W_s - \frac{1}{2}\int_0^T \frac{(\mu_s - r_s)^2}{\sigma_s^2}\,\mathrm{d}s\right).$$

We will see that the reason why we can solve explicitly this problem is that the term $\pi_t(\mu_t - r_t) + r_t Y_t$ in (1) is affine in π_t and Y_t.

However, the problem has no more an explicit solution if the interest rates for borrowing and lending are different, say, $\underline{r}_t$ for lending and $\overline{r}_t$ for borrowing. The wealth process dynamics then is

$$\begin{aligned}\mathrm{d}Y_t &= \Big(\pi_t\mu_t + \underline{r}_t(Y_t-\pi_t)^+ - \overline{r}_t(Y_t-\pi_t)^-\Big)\,\mathrm{d}t \\ &\quad + \pi_t\sigma_t\,\mathrm{d}W_t, \\ Y_T &= \xi,\end{aligned}$$

which has no more a dt-term affine in π_t and Y_t.

2 Definition of BSDEs

Definition: An n-dimensional BSDE is of the form

$$\begin{aligned}\mathrm{d}Y_t &= -f_t(Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t \text{ for } t\in[0,T], \\ Y_T &= \xi,\end{aligned} \tag{3}$$

where **given** are

- the *generator* (also called *driver*) f, which is a mapping

  $$f : [0,T]\times\Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\to\mathbb{R}^n,$$

  which satisfied appropriate measurability conditions, namely, for every fixed $(y,z)\in\mathbb{R}^n\times\mathbb{R}^{n\times d}$, the process $(f_t(y,z))_{0\le t\le T}$ is predictable;

- the *terminal condition* ξ, which is an $\mathcal{F}_T$-measurable, square-integrable random variable with values in $\mathbb{R}^n$.

A **solution** to (3) consists of (Y, Z) for

- an $\mathbb{R}^n$-valued, adapted process $(Y_t)_{0 \le t \le T}$
- an $\mathbb{R}^{n \times d}$-valued, predictable process $(Z_t)_{0 \le t \le T}$

satisfying (3).

Remark: We can write (3) equivalently as

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)\, \mathrm{d}s - \int_t^T Z_s\, \mathrm{d}W_s. \tag{4}$$

3 BSDEs with zero generator

In the case of $f \equiv 0$, (4) reduces to

$$Y_t = \xi - \int_t^T Z_s\, \mathrm{d}W_s. \tag{5}$$

To find a solution (Y, Z), recall the martingale representation theorem, which says that every $\mathcal{F}_T$-measurable, square-integrable ξ can be written as

$$\xi = E[\xi] + \int_0^T \beta_s \, \mathrm{d}W_s$$

for a unique predictable, square-integrable process β. By setting $Y_t = E[\xi|\mathcal{F}_t]$ and $Z = \beta$, we obtain a solution to (5), which is unique in the class of square-integrable solutions.

4 BSDEs with affine generator

We now consider the case $n = 1$ and

$$f_t(y, z) = a_t + b_t y + c_t \cdot z,$$

where

- a is an $\mathbb{R}$-valued, predictable process such that $E\Big[\int_0^T |a_t| \, \mathrm{d}t\Big] < \infty$,
- b is an $\mathbb{R}$-valued, bounded, predictable process,
- c is an $\mathbb{R}^d$-valued, bounded, predictable process.

In this case, we can reduce the corresponding BSDE to a problem of a BSDE with zero generator.

1. To eliminate the term $c_t \cdot z$ in the generator, we apply Girsanov's theorem. Recall that under the measure Q given by

   $$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left(\int_0^T c_t \, \mathrm{d}W_t - \frac{1}{2} \int_0^T |c_t|^2 \, \mathrm{d}t \right),$$

   the process $(B_t)_{0 \le t \le T}$ defined by

   $$B_t = W_t - \int_0^t c_s \, \mathrm{d}s, \quad t \in [0, T]$$

   is a Brownian motion.

   We can rewrite

   $$\mathrm{d}Y_t = -(a_t + b_t Y_t + c_t \cdot Z_t) \, \mathrm{d}t + Z_t \, \mathrm{d}W_t$$

   as

   $$\mathrm{d}Y_t = -(a_t + b_t Y_t) \, \mathrm{d}t + Z_t \, \mathrm{d}B_t$$

   and consider this BSDE under Q.

2. To eliminate the term $b_t y$ in the generator, we use the transformation

$$\overline{Y}_t = Y_t \mathrm{e}^{\int_0^t b_s \,\mathrm{d}s}$$

so that by Itô's formula

$$\begin{aligned} \mathrm{d}\overline{Y}_t &= \mathrm{e}^{\int_0^t b_s \,\mathrm{d}s} \mathrm{d}Y_t + b_t Y_t \mathrm{e}^{\int_0^t b_s \,\mathrm{d}s}\,\mathrm{d}t \\ &= -a_t \mathrm{e}^{\int_0^t b_s \,\mathrm{d}s}\,\mathrm{d}t + Z_t \mathrm{e}^{\int_0^t b_s \,\mathrm{d}s}\,\mathrm{d}B_t, \qquad (6) \end{aligned}$$

with $\overline{Y}_T = \overline{\xi} := \mathrm{e}^{\int_0^T b_s \,\mathrm{d}s} \xi$,

3. To eliminate the remaining term in the generator, we write (6) as

$$\mathrm{d} \underbrace{\left(\overline{Y}_t + \int_0^t a_u \mathrm{e}^{\int_0^u b_s \,\mathrm{d}s}\,\mathrm{d}u \right)}_{=:\underline{Y}_t} = \underbrace{Z_t \mathrm{e}^{\int_0^t b_s \,\mathrm{d}s}}_{=:\underline{Z}_t}\,\mathrm{d}B_t.$$

with $\underline{Y}_T = \underline{\xi} = \mathrm{e}^{\int_0^T b_s \,\mathrm{d}s} \xi + \int_0^T a_u \mathrm{e}^{\int_0^u b_s \,\mathrm{d}s}\,\mathrm{d}u$,

4. By Section 3, we know that

$$\begin{aligned} \underline{Y}_t &= E^Q[\underline{\xi} | \mathcal{F}_t] \\ &= E^Q\left[\mathrm{e}^{\int_0^T b_s \,\mathrm{d}s} \xi + \int_0^T a_u \mathrm{e}^{\int_0^u b_s \,\mathrm{d}s}\,\mathrm{d}u \middle| \mathcal{F}_t \right], \end{aligned}$$

hence

$$\overline{Y}_t = E^Q\Big[\mathrm{e}^{\int_0^T b_s\,\mathrm{d}s}\xi + \int_t^T a_u \mathrm{e}^{\int_0^u b_s\,\mathrm{d}s}\,\mathrm{d}u\Big|\mathcal{F}_t\Big],$$
$$Y_t = E^Q\Big[\mathrm{e}^{\int_t^T b_s\,\mathrm{d}s}\xi + \int_t^T a_u \mathrm{e}^{-\int_u^t b_s\,\mathrm{d}s}\,\mathrm{d}u\Big|\mathcal{F}_t\Big].$$

5 BSDEs with Lipschitz-continuous generator

Theorem 1 (Pardou and Peng [10]) *Assume that* $E[|\xi|^2] < \infty$, $E\Big[\int_0^T |f_t(0,0)|^2\,dt\Big] < \infty$ *and*

$$|f_t(y,z) - f_t(y',z')| \leq C(|y-y'| + |z-z'|) \quad (7)$$

for a constant C *and all* $y, y' \in \mathbb{R}^n, z, z' \in \mathbb{R}^{n\times d}$. *Then the BSDE* (3) *has a unique square-integrable solution.*

Remarks:

- The proof is based on fixed point argument, working in a combined space for (Y, Z) with norm
$$\sqrt{E\Big[\int_0^T \mathrm{e}^{\alpha t}(|Y_t|^2 + |Z_t|^2)\,\mathrm{d}t\Big]}$$
for a suitably chosen constant α. A sequence $(Y^{(n)}, Z^{(n)})$ can be defined by
$$Y_t^{(n)} = \xi + \int_t^T f_s\Big(Y_s^{(n-1)}, Z_s^{(n-1)}\Big)\,\mathrm{d}s - \int_t^T Z_s^{(n)}\,\mathrm{d}W_s$$
using the martingale representation for the random variable $\xi + \int_0^T f_s\Big(Y_s^{(n-1)}, Z_s^{(n-1)}\Big)\,\mathrm{d}s$.

- Theorem 1 holds for general n (multidimensional Y), but in applications from mathematical finance, the Lipschitz continuity for the generator is often too restrictive.

Having established existence and uniqueness of BSDE solutions, we can compare them for different terminal conditions and generators.

Theorem 2 (Peng [11] and El Karoui et al. [4]) *Assume $n = 1$. Consider two square-integrable solutions (Y^i, Z^i) to BSDEs with terminal conditions ξ^i and generator f^i such that $E\Big[\int_0^T |f_t^i(0,0)|^2\, dt\Big] < \infty$, $E[|\xi^i|^2] < \infty$ and (7) is satisfied for f^i for $i = 1, 2$. Assume further that*

$$\xi^1 \geq \xi^2, \quad f_t^1(Y_t^2, Z_t^2) \geq f_t^2(Y_t^2, Z_t^2).$$

Then $Y_t^1 \geq Y_t^2$ for all $t \in [0, T]$.

While the Brownian motion W can be multidimensional in Theorem 2, we need that Y is one-dimensional. Under additional conditions, comparison results for multidimensional Y are available, see for example, Hu and Peng [7] or Cohen et al. [2], but they only hold under additional conditions on the generators. This is one of the reasons why the existence and uniqueness results can be extended to the quadratic generators only in the case of $n = 1$; see next section.

6 BSDEs with quadratic generator

Let us start with an example of a quadratic BSDE for $n = 1$ and $d = 1$ by considering

$$\mathrm{d}Y_t = -\frac{1}{2}|Z_t|^2\,\mathrm{d}t + Z_t\,\mathrm{d}W_t.$$

Recall that the stochastic exponential

$$X_t = \exp\left(\int_0^t Z_s\,\mathrm{d}W_s - \int_0^t \frac{1}{2}|Z_s|^2\,\mathrm{d}s\right)$$

is a martingale if $\int Z_s\,\mathrm{d}W_s$ satisfies sufficient integrability conditions, for example, if it is a BMO (bounded mean oscillation) martingale or if the Novikov condition

$$E\left[\exp\left(\int_0^T \frac{1}{2}|Z_s|^2\,\mathrm{d}s\right)\right] < \infty$$

is satisfied. Note that

$$\mathrm{d}\Big(\ln(X_t)\Big) = -\frac{1}{2}|Z_t|^2\,\mathrm{d}t + Z_t\,\mathrm{d}W_t.$$

Therefore, we use an exponential transformation

$$\begin{aligned}\mathrm{d}\Big(\mathrm{e}^{Y_t}\Big) &= \mathrm{e}^{Y_t}\mathrm{d}Y_t + \frac{1}{2}\mathrm{e}^{Y_t}\mathrm{d}\langle Y\rangle_t \\ &= -\mathrm{e}^{Y_t}\frac{1}{2}|Z_t|^2\,\mathrm{d}t + \mathrm{e}^{Y_t}Z_t\,\mathrm{d}W_t + \frac{1}{2}\mathrm{e}^{Y_t}|Z_t|^2\,\mathrm{d}t \\ &= \mathrm{e}^{Y_t}Z_t\,\mathrm{d}W_t,\end{aligned}$$

which yields

$$\mathrm{e}^{\xi} - \mathrm{e}^{Y_t} = \int_t^T \mathrm{d}\Big(\mathrm{e}^{Y_s}\Big) = \int_t^T \mathrm{e}^{Y_s}Z_s\,\mathrm{d}W_s,$$

so that

$$Y_t = \ln\Big(E\Big[\mathrm{e}^{\xi}\Big|\mathcal{F}_t\Big]\Big),$$

assuming that $\int \mathrm{e}^{Y_s}Z_s\,\mathrm{d}W_s$ is a true martingale (and not merely a local martingale).

The example shows that BSDEs with quadratic generators are still meaningful, but square integrability of the terminal condition may not be enough because e^{ξ} appears in the solution of the above example.

Theorem 3 (Simplified version from Kobylanski [9]) *Assume $n = 1$, that ξ is bounded and*

$$\begin{aligned} |f_t(y,z)| &\le c + c|z|^2, \\ \left|\frac{\partial f_t}{\partial z}(y,z)\right| &\le c + c|z| \\ \frac{\partial f_t}{\partial y}(y,z) &\le c + c|z|^2 \end{aligned}$$

for a constant c and all $y \in \mathbb{R}, z \in \mathbb{R}^d$. Then the BSDE (3) *has a unique solution (Y, Z) with bounded Y and square-integrable Z.*

Remarks:

- A comparison theorem similarly to Theorem 2 holds for BSDEs with quadratic generators and $n = 1$.
- Theorem 3 cannot be generalized to BSDEs with generators of super-quadratic growth in z. Counterexamples for such a case can be found in Delbaen et al. [3].

Example (from Frei and dos Reis [6]):
We take $d = 1$ (dimension of W) and consider the two-dimensional ($n = 2$) BSDE

$$\mathrm{d}Y_t^1 = Z_t^1 \,\mathrm{d}W_t, \quad Y_T^1 = \xi, \tag{8}$$

$$\mathrm{d}Y_t^2 = -\Big(|Z_t^1|^2 + \frac{1}{2}|Z_t^2|^2\Big)\,\mathrm{d}t + Z_t^2\,\mathrm{d}W_t, \quad Y_T^2 = 0, \tag{9}$$

for a given bounded random variable ξ.

For some bounded ξ, the BSDE (8), (9) has no square-integrable solution. Main ideas to show this:

- assume that a square-integrable solution (Y, Z) exists;
- (8) determines Y_1 and Z_1 uniquely;
- using this Z_1 in (9) gives

$$E\bigg[\exp\bigg(\int_0^T |Z_t^1|^2\,\mathrm{d}t\bigg)\bigg]$$
$$= \exp\Big(Y_0^2\Big) E\bigg[\exp\bigg(\int_0^T Z_t^2\,\mathrm{d}W_t - \frac{1}{2}\int_0^T |Z_t^2|^2\,\mathrm{d}t\bigg)\bigg]$$
$$\le \exp\Big(Y_0^2\Big);$$

- it is possible to construct a process β such that $\int \beta_t \, \mathrm{d}W_t$ is bounded but
$$E\left[\exp\left(\int_0^T |\beta_t|^2 \, \mathrm{d}t\right)\right] = \infty,$$
then set $\xi = \int_0^T \beta_t \, \mathrm{d}W_t$.

Comments:

- The questions about existence of solutions to multidimensional, quadratic BSDEs are related to interested economic questions about the existence of a Nash equilibrium in a model of a financial market where different investors take the relative performance compared to each other into account (see [6]).

- The study of multidimensional, quadratic BSDEs is a very recent research topic. There are existence results available for particular cases, for example, if

– the terminal condition is small enough (see Tevzadze [13]),

– the time horizon $[0, T]$ is short (see Frei [5]),

– the generator has a particularly quadratic diagonal structure (see Hu and Tang [8]),

– under some Markovian assumptions on the generator and terminal condition (see Xing and Žitković [15]).

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