Funding, repo and credit in the Black-Scholes option pricing formula

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Reference

This lecture is based on:

D. Brigo, C. Buescu & M. Rutkowski (2015)
Funding, repo and credit in option pricing via dividends.
Working paper.

Goals

- Show how to replicate a zero-recovery defaultable bond in a financial market with an unsecured funding account and a market CDS on the bond's issuer.
- Examine the valuation of a vulnerable call option in the model with the underlying asset, funding account, repo contracts on the underlying and defaultable bond.
- Derive the pricing PDE and show that its solution is given by an extension of the classical Black-Scholes formula.
- Find the replicating strategy for the option in terms of the traded assets and funding arrangements.
- Analyze the dependence of the price and hedge on model parameters and the choice of a replicating strategy.

Replication of a defaultable bond

- We wish to replicate a zero-recovery defaultable bond in a financial market with an unsecured funding account with rate f called the treasury rate and a market CDS on the bond's issuer, which is traded at null price.
- The premium leg the CDS is assumed to pay a constant, continuous in time market spread κ and the protection leg pays one at the default of the bond and nothing otherwise. Recall that the market spread is computed by equating the value of the protection leg with the value of the premium leg.
- The price $D_t = D(t,T)$ of the zero-recovery defaultable bond maturing at T is given in terms of the process J, which jumps to one when default occurs and stays zero otherwise. Specifically, we have

$$D_t = \mathbb{1}_{\{J_t=0\}}\widetilde{D}_t = \mathbb{1}_{\{\tau>t\}}\widetilde{D}_t$$

where the yet unspecified process $\widetilde{D}$ represents the pre-default price of the bond.

Replication of a defaultable bond

- We will now provide intuitive replication arguments leading to the dynamics of the bond price.
- We assume here that there has been no default yet, but it may occur with a positive probability between the dates t and $t + dt$ for an arbitrarily small time increment dt.
- Let us consider the transactions an investor enters into at time $t < \tau \wedge T$:
  1. borrow $\widetilde{D}_t$ from the treasury and use it to buy one defaultable bond;
  2. buy a number $\widetilde{D}_t$ of CDS contracts on the same name.
- We have established a long position in the defaultable bond, and everything else forms the reverse of the replicating portfolio.
- Hence, formally, the replicating portfolio consists of the short position in the CDS and the long position in the treasury.
- We assume that the probability distribution of τ is continuous and its support includes $[0, T]$.

Valuation of a defaultable bond

- We now look at investor's portfolio at time $t + dt$:
  1. if there is a default ($J_{t+dt} = 1$) each of the $\widetilde{D}_t$ CDS contracts pays 1;
  2. if there is no default ($J_{t+dt} = 0$), he sells the bond for $\widetilde{D}_{t+dt}$;
  3. either way, he pays the premium leg $\kappa\, dt$ for each of the $\widetilde{D}_t$ CDS contracts and pays back the loan to the treasury, which amounts to $\widetilde{D}_t(1 + f\, dt)$.
- The overall gain over the time interval $(t, t + dt)$ is

$$\widetilde{D}_t \mathbb{1}_{\{J_{t+dt}=1\}} + \widetilde{D}_{t+dt} \mathbb{1}_{\{J_{t+dt}=0\}} - \kappa \widetilde{D}_t\, dt - \widetilde{D}_t(1 + f\, dt).$$

Equating this to zero to ensure replication and using the fact that the first indicator above is just dJ_t and that we assumed $J_t = 0$ (no default at time t), we obtain the dynamics for D

$$dD_t - D_t(\kappa + f)\, dt + D_{t-}\, dJ_t = 0$$

and thus, since $D_T = \mathbb{1}_{\{\tau > T\}}$, we have for all $t \in [0, T]$

$$D_t = \mathbb{1}_{\{\tau > t\}} e^{-(\kappa+f)(T-t)}.$$

Default time

- Denote by $\mathbb{F} = (\mathcal{F}_t)$ where $\mathcal{F}_t := \sigma(S_u, u \leq t)$ the natural filtration generated by the price process of a traded asset.
- Assume that the default time τ is a positive random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- The default time generates a filtration $\mathbb{H}$ where $\mathcal{H}_t := \sigma(\mathbb{1}_{\{\tau \leq u\}}, u \leq t)$, which is used to progressively enlarge $\mathbb{F}$ in order to obtain the full filtration $\mathbb{G} = (\mathcal{G}_t)$ where $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$.
- We work under the assumption that $F_t := \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t)$ is a continuous and strictly increasing function and $F_t < 1$ for all $t \in [0, T]$.
- We interpret τ as the default time of the counterparty, that is, the seller of the call option.
- The hedger is the buyer of the call option.

Vulnerable call option

- Let the maturity date T be fixed and let $X = \mathbb{1}_{\{\tau > T\}} g(S_T)$.
- We focus on the *vulnerable call option* with the payoff X at maturity time T given by
$$X = \mathbb{1}_{\{\tau > T\}} (S_T - K)^+.$$
- Hence τ is interpreted as the default time of the issuer of the call option.
- We wish to find the price P_t^h, $t \in [0, T]$, of this contract for the hedger, that is, for an investor who replicates a long position using financial instruments available in the market.
- We now consider a market model with the following traded assets:
  1. an unsecured funding account with the interest rate f;
  2. a stock (the underlying asset of the contract);
  3. a repurchase agreement on the stock with the repo rate h;
  4. a zero-recovery defaultable bond with the rate of return r^D issued by the counterparty.

Traded assets

- At time t, the price P_t^i of the ith asset is given by

$$P_t^1 = B_t^f, \quad P_t^2 = S_t, \quad P_t^3 = 0, \quad P_t^4 = D_t$$

  and the gains process since inception of the ith is denoted by G_t^i with $G_0^i = 0$.

- Model inputs: the treasury funding rate f, the repo rate h and the bond rate of return $r^D = \kappa + f$.

- We assume that under the real-world probability $\mathbb{P}$ the stock price is governed by

$$dS_t = \mu_t S_t \, dt + \sigma S_t \, dW_t$$

- Buying one repo contract amounts to selling the shares of stock against cash, under the agreement of repurchasing them back at the higher price so that $dG_t^3 = dS_t - hS_t \, dt$.

Gains of traded assets

- Under the standing assumption that the pre-default rate of return r^D on the counterparty's bond is constant, we obtain

$$D_t = \mathbb{1}_{\{\tau > t\}}\, e^{-r^D(T-t)} = (1 - J_t)e^{-r^D(T-t)}$$

  where $J_t := \mathbb{1}_{\{\tau \le t\}}$ models the jump to default of the counterparty.

- The gains G^4 have negative terms for outgoing cash flows corresponding to the drop in the bond value at the time of default.

- To summarize, the gains of primary assets are given by

$$\begin{aligned} dG_t^1 &= fB_t^f\, dt, \quad dG_t^2 = dS_t, \\ dG_t^3 &= dS_t - hS_t\, dt, \quad dG_t^4 = r^D D_t\, dt - D_{t-}\, dJ_t. \end{aligned}$$

- A *trading strategy* $\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4)$ gives the number of units of each primary asset purchased to build a portfolio.

Trading strategies

- Let $\beta \in [0,1]$ be a constant. A trading strategy φ is *admissible* if at any date t the investor can only use the repo market for a fraction β of the stock amount required and the rest has to be obtained in the stock market with funding from the treasury.
- The *wealth* at time $t \in [0,T]$ of an admissible strategy φ is denoted by $V_t(\varphi)$ and equals

$$V_t(\varphi) = \sum_{i=1}^{4} \varphi_t^i P_t^i$$

and the gains process associated with this strategy satisfies $G_0(\varphi) = 0$ and

$$dG_t(\varphi) := \sum_{i=1}^{4} \varphi_t^i \, dG_t^i + dA_t$$

- We then say that a strategy φ is *self-financing* if for all $t \in [0,T]$

$$V_t(\varphi) = V_0(\varphi) + G_t(\varphi).$$

Admissible strategies

- An admissible trading strategy φ *replicates* the long position in the call option if $V_T(\varphi) = 0$, that is, $V_{T-}(\varphi) = -(S_T - K)^+$.
- We define the time t price of a contract A as the wealth $V_t(\varphi)$ of the replicating strategy.
- The existence of the specific primary assets in our market ensures that any claim is *attainable*.
- In fact, the market under study is complete and no-arbitrage arguments show that the price of any contract is unique.
- At date t before default, the investor builds a replicating portfolio for a long position in the option knowing that the assumptions on τ imply that default may occur between t and $t + dt$ for an arbitrarily small dt.

Replication of the option

- To replicate the contract, the buyer of the option
  1. buys $\beta\xi_t$ repos, borrows $\beta\xi_t S_t$ from the treasury to buy and deliver $\beta\xi_t$ shares, and receives $\beta\xi_t S_t$ cash which is paid back to the treasury;
  2. borrows $(1-\beta)\xi_t S_t$ from the treasury and buys $(1-\beta)\xi_t$ shares;
  3. buys P^h_t/D_t units of the counterparty bond in order to match the value of this portfolio and the value of the contract.

- Hence we introduce the following admissible strategy θ replicating P^h

$$\theta_t := \left(-\frac{(1-\beta)\xi_t S_t}{B^f_t}, (1-\beta)\xi_t, \beta\xi_t, \frac{P^h_t}{D_t} \right).$$

- It is easy to check that $V_t(\theta) = P^h_t$ for all $t \in [0,T]$.

Replication of the option

- At time $t + dt$ the investor:
  1. receives $\beta\xi_t$ shares from repo and sells them for $\beta\xi_t S_{t+dt}$;
  2. borrows from treasury $\beta\xi_t S_t(1 + hdt)$ to close the repo;
  3. sells $(1 - \beta)\xi_t$ shares for $(1 - \beta)\xi_t S_{t+dt}$;
  4. sells the counterparty's bond for $P_t^h D_{t+dt}/D_t$;
  5. pays back to the treasury $(1 - \beta)\xi_t S_t(1 + fdt)$.

  From these transactions the change in the wealth of the replicating position is

$$
\begin{aligned}
V_{t+dt}(\theta) - V_t(\theta) &= \beta\xi_t S_{t+dt} - \beta\xi_t S_t(1 + hdt) + (1 - \beta)\xi_t S_{t+dt} + \frac{P_t^h}{D_t}\, dD_t \\
&\quad - (1 - \beta)\xi_t S_t(1 + fdt) \\
&= \beta\xi_t\, dS_t - \beta h\xi_t S_t\, dt + (1 - \beta)\xi_t\, dS_t + \frac{P_t^h}{D_t}\, dD_t - (1 - \beta) f\xi_t S_t\, dt \\
&= \xi_t\, dS_t - \big((1 - \beta) f + \beta h\big)\xi_t S_t\, dt + P_t^h(r^D dt - dJ_t).
\end{aligned}
$$

Formal derivation

- This can be derived formally by computing the gains process associated with the portfolio θ

$$dG_t(\theta) = -\frac{(1-\beta)\xi_t S_t}{B_t^f} f B_t^f \, dt + (1-\beta)\xi_t \, dS_t + \beta \xi_t (dS_t - hS_t \, dt)$$
$$+ \frac{P_t^h}{D_t}(r^D D_t \, dt - D_{t-} \, dJ_t)$$
$$= \xi_t \, dS_t - \big((1-\beta)f + \beta h\big)\xi_t S_t \, dt + P_{t-}^h (r^D dt - dJ_t)$$

where we used the equality $D_{t-} = D_t$, which holds before default.

- Note also that the wealth of θ at default equals zero, which is consistent with the option payoff at default. Hence we may set $\theta_t = (0,0,0,0)$ for $t > \tau$.
- Let us now focus on the pre-default pricing problem. Since $dV_t^\theta = dG_t^\theta$ (self-financing property) and $dP_t^h = dV_t^\theta$ (replication), we have

$$dP_t^h = \xi_t \, dS_t - \big((1-\beta)f + \beta h\big)\xi_t S_t \, dt + P_{t-}^h (r^D dt - dJ_t). \tag{0.1}$$

Pricing function

- To derive the pricing PDE, we assume that the hedger's price P_t^h can be expressed as
$$P_t^h = \mathbb{1}_{\{\tau > t\}} \widetilde{P}_t^h = \mathbb{1}_{\{\tau > t\}} v(t, S_t) = (1 - J_t) v(t, S_t)$$
for some function $v(t, s)$ of class $\mathcal{C}^{1,2}$.
- Then the Itô formula yields
$$dP_t^h = (1 - J_t)\, dv(t, S_t) + v(t, S_t)\, d(1 - J_t) = (1 - J_t)\, dv(t, S_t) - v(t, S_t)\, dJ_t$$
and
$$dP_t^h = (1 - J_t)\Big(v_t(t, S_t) + \frac{\sigma^2 S_t^2}{2} v_{ss}(t, S_t)\Big) dt + (1 - J_t) v_s(t, S_t)\, dS_t - v(t, S_t)\, dJ_t. \tag{0.2}$$

Pre-default pricing PDE

- By equating the dS_t, dt and the jump terms in (0.1) and (0.2), we obtain the following equalities where the variables (t, S_t) were suppressed

$$\xi_t = (1 - J_t)v_s, \quad P_t^h \, dJ_t = v \, dJ_t$$

and the pre-default pricing PDE for the function $v(t,s)$

$$v_t + \big((1-\beta)f + \beta h\big) s \, \frac{\partial v}{\partial s} + \frac{\sigma^2 s^2}{2} \frac{\partial v^2}{\partial s^2} - r^D v = 0$$

with terminal condition $v(T,s) = -(s-K)^+$.

- This is the Black-Scholes PDE when the underlying stock pays dividends.
- It suffices identify the risk-free rate with the return on the defaultable bond $r := r^D$ and the dividend yield with the bond spread over the effective funding rate: $q := r^D - f^\beta$ where by the *effective funding rate* we mean the weighted average $f^\beta := (1-\beta)f + \beta h$.

Extended Black-Scholes formula

Proposition

The time t buyer's price of the vulnerable call option is given by the following extended version of the classic Black-Scholes formula

$$P_t^h = -\mathbb{1}_{\{\tau > t\}} \Big(S_t e^{-q(T-t)} N(d_1^q) - K e^{-r^D (T-t)} N(d_2^q) \Big)$$

with $q = r^D - f^\beta = r^D - (1-\beta) f - \beta h$ *and*

$$d_1^q = \frac{\log \frac{S_t}{K} + (r^D - q + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$$

and

$$d_2^q = d_1^q - \sigma \sqrt{T-t}.$$

Probabilistic derivation

- Note that the pricing formula can also be derived without resorting to the pricing PDE. From

$$dP_t^h = \xi_t \, dS_t - \big((1-\beta)f + \beta h\big)\xi_t S_t \, dt + P_{t-}^h (r^D \, dt - dJ_t)$$

we obtain the following equation for the pre-default price $\widetilde{P}^h$

$$d\widetilde{P}_t^h = \xi_t \, dS_t - f^\beta \xi_t S_t \, dt + r^D \widetilde{P}_t^h \, dt.$$

- Let $\mathbb{Q}^\beta$ be the probability measure equivalent to $\mathbb{P}$ and such that the drift of the risky asset S under $\mathbb{Q}^\beta$ is equal to the effective funding rate f^β.
- Then $\widetilde{P}^h$ is governed under $\mathbb{Q}^\beta$ by

$$d\widetilde{P}_t^h - r_t^D \widetilde{P}_t^h \, dt = \xi_t \sigma S_t \, dW_t^\beta$$

with $\widetilde{P}_T^h = -(S_T - K)^+$ where W^β is the Brownian motion under $\mathbb{Q}^\beta$.

Probabilistic derivation

- This leads to the following probabilistic representation for $\widetilde{P}_t^h$

$$\widetilde{P}_t^h = -e^{-r^D(T-t)}\,\mathbb{E}_{\mathbb{Q}^\beta}[(S_T - K)^+ \,|\, \mathcal{F}_t]$$

that is

$$\widetilde{P}_t^h = -e^{-(r^D - f^\beta)(T-t)}\,\mathbb{E}_{\mathbb{Q}^\beta}[e^{-f^\beta(T-t)}(S_T - K)^+ \,|\, \mathcal{F}_t].$$

- This in turn yield the pricing formula through either standard computations of conditional expectation (Feynman-Kac formula) or by simply noting that it is given by the Black-Scholes formula with the interest rate f^β and no dividends.
- Recall that the additional parameter $\beta \in [0,1]$ dictates the structure of the funding arrangements for the investor.

Funding sensitivities

- We obtain the following funding Greeks

$$\partial_f \widetilde{P}^h_t = \beta(T-t)\widetilde{P}^h_t + (1-\beta)(T-t)e^{(\beta(h-f)-\kappa)(T-t)}KN(d^q_2),$$
$$\partial_h \widetilde{P}^h_t = -\beta e^{(\beta(h-f)-\kappa)(T-t)}(T-t)S_t N(d^q_1) \leq 0,$$

where the last inequality is strict when $\beta > 0$.

- For $\beta = 1$ (pure repo funding), we get

$$\partial_f \widetilde{P}^h_t = \partial_{r^D} \widetilde{P}^h_t = (T-t)\widetilde{P}^h_t > 0,$$
$$\partial_h \widetilde{P}^h_t = -e^{-(h-f-\kappa)(T-t)}(T-t)S_t N(d^q_1) < 0,$$

which means that the pre-default call price increases in both the treasury rate f and the bond return r^D, but decreases in the repo rate h for the risky asset S.

Funding sensitivities

- For $\beta = 0$ (pure treasury funding), we get

$$\partial_f \widetilde{P}_t^h = -(T-t)e^{(f-r^D)(T-t)} K N(d_2^q) < 0,$$
$$\partial_h \widetilde{P}_t^h = 0,$$

  where $f - r^D = \kappa > 0$.

- In general, it is hard to determine the sign of the sensitivity $\partial_f \widetilde{P}_t^h$, although it is clear that it changes from a negative value for $\beta = 0$ to a positive value for $\beta = 1$.

- To give an interpretation of funding Greeks, we observe that the contract's payoff can be written as $X = D_T(S_T - K)^+$, so it can be seen as a hybrid contract which combines the call option on the stock with the counterparty bond.

Valuation and hedging of general contracts with funding costs and collateralization

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New Challenges

New challenges after the financial crisis

- The recent global financial crisis has led to major changes in the operations of financial markets. The defaultability of the counterparties became the central problem of financial management.
- The classical paradigm of discounting future cash flows using the risk-free rate is no longer accepted as a viable pricing rule, although some researchers advocate the use of the OIS (overnight index swap) rates as a proxy for the risk-free rate.
- In the presence of funding costs, counterparty credit risk, and collateral (margin account) the classical arbitrage pricing theory no longer applies.
- As a consequence, the analysis of the counterparty credit risk and price formation under differential funding costs and collateralisation rules are currently the most challenging problems in Mathematical Finance.

Mathematical challenges

- Non-uniqueness and non-linearity of one-sided prices: asymmetric pricing rules may fail to yield a mutually acceptable price for counterparties.
- Aggregation of pricing rules: non-linear effects of netting of exposures and/or margin accounts between two (or more) counterparties.
- Arbitrage opportunities: a general contract may introduce arbitrage opportunities to an arbitrage-free model with trading constraints.
- Multi-curve framework: effects of implied credit and liquidity risks premia.
- Benefit at default: marking to market for accounting purposes and bank's unrealised profits due to its own credit risk.
- Embedded options: valuation of a choice of a counterparty, funding mechanism and collateral (cash, currency, or risky assets).
- Asymmetric information: the game-theoretic approach to pricing of contracts with differential information.

Goals

- To describe trading strategies in the presence of funding costs (multiple yield curves), margin account (collateral) and default events.
- To propose suitable approaches to pricing of general European contracts within this novel framework.
- The focus will be put on one party (called ‘hedger’), but the same technique can be used to solve the valuation problem for the other party.
- The mark-to-market convention for collateral requires that both parties agree in respect of the fair value of the contract. Hence the actual problem is (at least) two-dimensional rather than one-dimensional.
- The issues of modeling of default times of counterparties, recoveries, close-out cash flows, and the benefit at default are not addressed in this lecture since they are covered by the commonly known methods of credit risk modeling.

Part I: Trading with Funding Costs

Market model: Linear case

- All processes introduced in what follows are implicitly assumed to be given on an underlying probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ where the filtration $\mathbb{G}$ models information available to traders.
- S^i is the *ex-dividend price* (or simply the *price*) of the ith risky security with the cumulative dividend stream after time 0 represented by the process A^i.
- B^i stands for the corresponding *funding account* representing the case of a secured funding of the ith asset.
- *Cash accounts* $B^{0,l}$ and $B^{0,b}$ are used for unsecured lending or borrowing of cash.
- We fix a finite trading horizon T.

Remark

We will sometimes assume that borrowing and lending rates coincide so that $B^{0,l} = B^{0,b} = B^0$.

Wealth of a trading strategy

- We consider a *trading strategy* $\phi = (\xi^1, \dots, \xi^d, \psi^0, \dots, \psi^d)$ in risky securities S^i, $i = 1, 2, \dots, d$, a cash account B^0 used for unsecured lending/borrowing, and accounts B^i, $i = 1, 2, \dots, d$, used for secured funding.
- Let A be càdlàg process of finite variation with $A_0 = 0$ and the initial price p.
- The wealth process depends also on the initial endowment x of the hedger.

Definition

A trading strategy (x, p, ϕ, A) with a cash stream process A is *self-financing* if the wealth process $V(x, p, \phi, A)$, which is given by the formula

$$V_t(x, p, \phi, A) := \sum_{i=1}^{d} \xi_t^i S_t^i + \sum_{j=0}^{d} \psi_t^j B_t^j,$$

satisfies for every $t \in [0, T]$

$$V_t(x, p, \phi, A) = x + p + \sum_{i=1}^{d} \int_0^t \xi_u^i \, d(S_u^i + A_u^i) + \sum_{j=0}^{d} \int_0^t \psi_u^j \, dB_u^j + A_t.$$

Cash stream process

Remark

- The *cash stream process* A (sometimes termed the *dividend process*) represents the cash flows associated with a contract A that are either paid or received by the hedger.
- In particular, the process A may include cash flows associated with transfer of collateral and the close-out cash flow at default.
- The dividend process A may depend on ϕ either explicitly or through the associated wealth process $V(\phi, A)$, so that it may happen that $A = A(\phi)$.
- The latter possibility explains why FVA, CVA and DVA computations are typically based on implicit valuation problems.
- For brevity, we will usually write $V(\phi)$ instead of $V(x, p, \phi, A)$.

Wealth decomposition

Remark

The wealth process admits the following decomposition

$$V_t(\phi) = V_0(\phi) + G_t^S(\phi) + G_t^F(\phi) + A_t$$

where

$$G_t^S(\phi) := \sum_{i=1}^{d} \int_0^t \xi_u^i \, (dS_u^i + dA_u^i)$$

represents gains/losses from trading in assets $S^1, \dots, S^d$ and

$$G_t^F(\phi) := \sum_{j=0}^{d} \int_0^t \psi_u^j \, dB_u^j$$

represents gains/losses from funding accounts.

Cumulative wealth

- Let

$$V_t^{\mathrm{cld}}(\phi) := V_t(\phi) + B_t^0 \int_0^t (B_u^0)^{-1}\, dA_u.$$

$$S_t^{i,\mathrm{cld}}(\phi) := S_t^i + B_t^i \int_0^t (B_u^i)^{-1}\, dA_u^i.$$

- We introduce the following notation

$$K_t^i := \int_0^t B_u^i\, d\widehat{S}_u^i + A_t^i = \int_0^t B_u^i\, d\widehat{S}_u^{i,\mathrm{cld}}$$

where we denote $\widehat{S}_t^i := S_t^i (B_t^i)^{-1}$ and $\widehat{S}_t^{i,\mathrm{cld}} := S_t^{i,\mathrm{cld}} (B_t^i)^{-1}$.

- We let

$$K_t^\phi := \int_0^t B_u^0\, d\widetilde{V}_u(\phi) + A_t = \int_0^t B_u^0\, d\widetilde{V}_u^{\mathrm{cld}}(\phi)$$

where we set $\widetilde{V}_t(\phi) := (B_t^0)^{-1} V_t(\phi)$ and $\widetilde{V}_t^{\mathrm{cld}}(\phi) := (B_t^0)^{-1} V_t^{\mathrm{cld}}(\phi)$.

Trading with funding costs

Remark

- The process K^i is equal to the wealth, discounted by the funding account B^i, of a self-financing strategy in risky security S^i and the associated funding account B^i in which B^i_t units of the cumulative-dividend price of the ith asset is held at time t.
- A similar interpretation can be given to the process K^ϕ.

Proposition

(i) For any self-financing strategy ϕ we have that for every $t \in [0,T]$

$$K^\phi_t = \sum_{i=1}^d \int_0^t \xi^i_u \, dK^i_u + \sum_{i=1}^d \int_0^t (\psi^i_u B^i_u + \xi^i_u S^i_u)(\widetilde{B}^i_u)^{-1} \, d\widetilde{B}^i_u \tag{2.1}$$

where we denote $\widetilde{B}^i_t = B^i_t / B^0_t$.

Trading with funding costs

Proposition (continued)

(ii) The equality

$$K_t^{\phi} = \sum_{i=1}^{d} \int_0^t \xi_u^i \, dK_u^i, \quad t \in [0,T], \tag{2.2}$$

holds if and only if

$$\sum_{i=1}^{d} \int_0^t (\psi_u^i B_u^i + \xi_u^i S_u^i)(\widetilde{B}_u^i)^{-1} \, d\widetilde{B}_u^i = 0, \quad t \in [0,T].$$

(iii) In particular, if for every $i = 1, 2, \ldots, d$ either $B_t^i = B_t^0$ for all $t \in [0,T]$ or

$$\psi_t^i B_t^i + \xi_t^i S_t^i = 0, \quad t \in [0,T], \tag{2.3}$$

then (2.2) holds.

Equivalent expressions

Corollary

Formula (2.1) is equivalent to the following expression

$$d\widetilde{V}_t^{cld}(\phi) = \sum_{i=1}^{d} \xi_t^i \, dK_t^i + \sum_{i=1}^{d} \zeta_t^i (B_t^i)^{-1} \, d\widetilde{B}_t^i$$

where $\zeta_t^i := \psi_t^i B_t^i + \xi_t^i S_t^i$. *More explicitly,*

$$dV_t(\phi) = \widetilde{V}_t(\phi) \, dB_t^0 + \sum_{i=1}^{d} \xi_t^i B_t^i \, d\widehat{S}_t^{i,cld} + \sum_{i=1}^{d} \zeta_t^i (\widetilde{B}_t^i)^{-1} \, d\widetilde{B}_t^i + dA_t.$$

Hence the funding costs of ϕ *satisfy*

$$G_t^F(\phi) = \int_0^t \widetilde{V}_u(\phi) \, dB_u^0 + \int_0^t \sum_{i=1}^{d} \zeta_u^i (\widetilde{B}_u^i)^{-1} \, d\widetilde{B}_u^i - \sum_{i=1}^{d} \int_0^t \xi_u^i \widehat{S}_u^i \, dB_u^i.$$

Fully secured funding of risky assets

Let us examine the case of a fully secured funding of risky assets (that is, the repo market)

- Note that condition (2.3) means that positive and negative positions in the ith risky security are maintained using the ith secured funding account only.
- In other words, this condition corresponds to the case of the fully secured funding of the ith asset by the corresponding repo rate.
- If condition (2.3) holds for every $i = 1, 2, \ldots, d$, then the wealth of a strategy ϕ satisfies

$$V_t(\phi) = \psi_t^0 B_t^0$$

  for every $t \in [0, T]$.
- Hence the net position of a trading strategy is funded from the unsecured account B^0.

Fully secured funding of risky assets

Corollary

Assume that (2.3) holds. Then

$$d\widetilde{V}_t^{cld}(\phi) = \sum_{i=1}^{d} \xi_t^i \, dK_t^i$$

or, more explicitly,

$$dV_t(\phi) = \widetilde{V}_t(\phi) \, dB_t^0 + \sum_{i=1}^{d} \xi_t^i B_t^i \, d\widehat{S}_t^{i,cld} + dA_t.$$

Hence the funding costs of ϕ satisfy

$$G_t^F(\phi) = \int_0^t \widetilde{V}_u(\phi) \, dB_u^0 - \sum_{i=1}^{d} \int_0^t \xi_u^i \widehat{S}_u^i \, dB_u^i.$$

Part II: Arbitrage Pricing

Netted wealth

- A generic definition of an arbitrage-free market model hinges on the concept of the *netted wealth* process $V^{\text{net}}(x, \phi, A)$.
- An explicit specification of the netted wealth process $V^{\text{net}}(x, \phi, A)$ depends on particular features of the model and contract at hand.
- Let $A = A^+ - A^-$ be the decomposition of A into increasing and decreasing components.
- When lending and borrowing rates differ, the simplest version of the netted wealth is given by (note that the netted wealth does not depend on p)

$$V_t^{\text{net}}(x, \phi, A) := V_t(x, 0, \phi, A) - B_t^{0,b} \int_0^t (B_u^{0,b})^{-1}\, dA_u^+ + B_t^{0,l} \int_0^t (B_u^{0,l})^{-1}\, dA_u^-$$

that is,

$$V_t^{\text{net}}(x, \phi, A) := V_t(x, 0, \phi, A) - V_t(0, 0, \widehat{\phi}, -A)$$

where $\widehat{\phi}$ means that all cash flows from $-A$ are invested in $B^{0,l}$ and $B^{0,b}$ without netting.

Arbitrage-free market model

Definition

We say that a market model is *arbitrage-free* whenever for any self-financing strategy ϕ and any process A representing the external cash flows if the discounted netted wealth process $\widetilde{V}^{\text{net}}(x, \phi, A)$ is bounded from below by a constant then

$$\mathbb{P}\big(V_T^{\text{net}}(x, \phi, A) < \widehat{V}_T(x)\big) > 0$$

where $\widehat{V}_T(x) := x^+ B_T^{0,l} - x^- B_T^{0,b}$.

Remark

For $B^{0,l} = B^{0,b} = B^0$, we obtain the equivalent condition

$$\mathbb{P}\big(\widetilde{V}_T^{\text{net}}(x, \phi, A) < x\big) > 0.$$

If we set $A = 0$ then the netted wealth process $V^{\text{net}}(x, \phi, 0)$ coincides with the wealth process $V(x, \phi)$ and thus our definition covers the classical definition of no-arbitrage since we may set $x = 0$.

Hedger's fair price

- Assume that the market model is arbitrage-free. The next step is to define the range of arbitrage prices of a contract with cash flows A.
- Let x be any initial wealth and let p stand for a price of a contract at time 0 for the hedger.
- A positive (resp. negative) value of p means that the hedger receives (resp. pays) the cash amount p at time 0.
- It is clear from the next definition that the price may depend on the initial wealth x and it is not unique, in general.

Definition

We say that p is a *hedger's fair price* for A whenever for any trading strategy ϕ with initial wealth $x+p$ and such the discounted wealth process $\widetilde{V}(x+p,\phi,A)$ is bounded from below by a constant the following condition holds

$$\mathbb{P}\big(V_T(x+p,\phi,A) < x^+B_T^{0,l} - x^-B_T^{0,b}\big) > 0.$$

First example: Basic market model

- We first consider the basic set-up with a single cash account B^0.
- The netted wealth equals

$$V_t^{\mathrm{net}}(x,\phi,A)=V_t(x,\phi,A)-B_t^0\int_0^t (B_u^0)^{-1}\,dA_u.$$

- We postulate, in addition, that self-financing trading strategies ϕ satisfy

$$\sum_{i=1}^{d}\int_0^t (\psi_u^i B_u^i+\xi_u^i S_u^i)(\widetilde{B}_u^i)^{-1}\,d\widetilde{B}_u^i=0,\quad t\in[0,T].$$

- For this basic model, we have the following result, which closely resembles classical results for models with a single funding account.

Proposition

Assume that there exists an equivalent probability measure $\widehat{\mathbb{P}}$ on $(\Omega,\mathcal{G}_T)$ such that the processes $\widehat{S}^{i,cld}$, $i=1,2,\dots,d$ are $(\widehat{\mathbb{P}},\mathbb{G})$-local martingales. Then the basic market model is arbitrage-free.

Proof

It suffices to observe that the discounted cumulative wealth $\widetilde{V}^{\text{net}}(\phi)$ satisfies

$$\begin{aligned}\widetilde{V}^{\text{net}}_t(\phi) &= \widetilde{V}^{\text{net}}_0(\phi) + \int_0^t (B^0_u)^{-1}\, dK^{\phi}_u \\ &= \widetilde{V}^{\text{net}}_0(\phi) + \sum_{i=1}^d \int_0^t (B^0_u)^{-1} \xi^i_u \, dK^i_u \\ &= \widetilde{V}^{\text{net}}_0(\phi) + \sum_{i=1}^d \int_0^t (B^0_u)^{-1} \xi^i_u B^i_u \, d\widehat{S}^{i,\text{cld}}_u .\end{aligned}$$

Hence the proposition follows from the standard argument, which runs as follows:

- since $\widetilde{V}^{\text{net}}(\phi)$ is local martingale, which is bounded from below by a constant, it is also a supermartingale under $\widehat{\mathbb{P}}$, which in turn means that arbitrage opportunities are precluded.

First example: Hedger's prices

- We consider the basic model and we assume that it is arbitrage-free.
- We aim to describe the set of hedger's prices of A.
- After simple computations, we obtain the following representation

$$\mathbb{P}\Big(p + \sum_{i=1}^{d} \int_0^T (B_u^0)^{-1} \xi_u^i B_u^i \, d\widehat{S}_u^{i,\mathrm{cld}} + \int_0^T (B_u^0)^{-1} \, dA_u < 0\Big) > 0.$$

- Note that here the range of arbitrage prices does not depend on the initial wealth x.
- Assume that $A_t = -X \mathbb{1}_{\{t=T\}}$ and $B^i = B^0$ for every $i = 1, \dots, d$. Then the last formula reduces to the classical case, namely,

$$\mathbb{P}\Big(p + \sum_{i=1}^{d} \int_0^T \xi_u^i \, d\widetilde{S}_u^{i,\mathrm{cld}} < B_T^{-1} X\Big) > 0.$$

Second example: Netting of short positions

- We now assume that $B^{i,l} = B^{0,l}$ for all $i = 1, 2, \ldots, d$ and we postulate that all positive cash amounts available, inclusive of proceeds from short-selling of risky assets, are invested in the unique account $B^{0,l}$.
- Long cash positions in risky assets S^i are assumed to be funded from accounts $B^{i,b}$. We thus deal here with the case of the partial netting across risky assets.
- Formally, we postulate that

$$V_t(\phi) = \psi_t^{0,l} B_t^{0,l} + \psi_t^{0,b} B_t^{0,b} + \sum_{i=1}^{d} (\xi_t^i S_t^i + \psi_t^{i,b} B_t^{i,b})$$

where, for every $i = 1, 2, \ldots, d$ and $t \in [0, T]$, the process $\psi_t^{i,b}$ satisfies

$$\psi_t^{i,b} = -(B_t^{i,b})^{-1} (\xi_t^i S_t^i)^+ \leq 0.$$

so that also

$$V_t(\phi) = \psi_t^{0,l} B_t^{0,l} + \psi_t^{0,b} B_t^{0,b} - \sum_{i=1}^{d} (\xi_t^i S_t^i)^-.$$

Second example: Self-financing condition

- Since $\psi_t^{0,l} \geq 0$ and $\psi_t^{0,b} \leq 0$, we obtain

$$\psi_t^{0,l} = (B_t^{0,l})^{-1} \Big(V_t(\phi) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^+$$

and

$$\psi_t^{0,b} = -(B_t^{0,b})^{-1} \Big(V_t(\phi) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^- .$$

- The self-financing condition takes the following form

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{d} \int_0^t \xi_u^i \, d(S_u^i + A_u^i) + \sum_{i=0}^{d} \int_0^t \psi_u^{i,b} \, dB_u^{i,b} + \int_0^t \psi_u^{0,l} \, dB_u^{0,l} + \int_0^t \psi_u^{0,b} \, dB_u^{0,b} + A_t .$$

Second example: Wealth dynamics

Proposition

The wealth dynamics are

$$dV_t(\phi) = \sum_{i=1}^{d} \xi_t^i \, (dS_t^i + dA_t^i) - \sum_{i=1}^{d} (\xi_t^i S_t^i)^+ (B_t^{i,b})^{-1} \, dB_t^{i,b} + dA_t$$

$$+ \Big(V_t(\phi) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^+ (B_t^{0,l})^{-1} \, dB_t^{0,l}$$

$$- \Big(V_t(\phi) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^- (B_t^{0,b})^{-1} \, dB_t^{0,b}.$$

If, in addition, all account processes $B^{i,l}$ and $B^{0,b}$ are absolutely continuous then

$$dV_t(\phi) = \sum_{i=1}^{k} \xi_t^i \big(dS_t^i + dA_t^i \big) - \sum_{i=1}^{d} r_t^{i,b} (\xi_t^i S_t^i)^+ \, dt + dA_t$$

$$+ r_t^{0,l} \Big(V_t(\phi) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^+ dt - r_t^{0,b} \Big(V_t(\phi) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^- dt$$

Second example: Arbitrage-free condition

We define the discounted wealth by setting

$$\widetilde{V}_t^l(x,\phi,A) := (B_t^{0,l})^{-1} V_t(x,\phi,A).$$

Proposition

Assume that $r_t^{0,l} \le r_t^{0,b}$ *and* $r_t^{0,l} \le r_t^{i,b}$ *for* $i = 1,2,\dots,d$. *Let us denote*

$$\widetilde{S}_t^{i,l,cld} = (B_t^{0,l})^{-1} S_t^i + \int_0^t (B_u^{0,l})^{-1} \, dA_u^i.$$

Suppose that there exists an equivalent probability measure $\widetilde{\mathbb{P}}$ *on* $(\Omega, \mathcal{G}_T)$ *such that the processes* $\widetilde{S}^{i,l,cld}$, $i = 1,2,\dots,d$ *are* $(\widetilde{\mathbb{P}}, \mathbb{G})$*-local martingales. Then the model with netting of short cash positions is arbitrage-free.*

Second example: Hedger's prices

- The set of hedger's prices p is now characterized by the following condition

$$\mathbb{P}\Bigg(x + p + \sum_{i=1}^{k} \int_0^T \xi_t^i \big(dS_t^i + dA_t^i\big) - \sum_{i=1}^{d} \int_0^T r_t^{i,b} (\xi_t^i S_t^i)^+ \, dt$$

$$+ A_T - A_0 + \int_0^T r_t^{0,l} \Big(V_t(\phi, A) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^+ \, dt$$

$$- \int_0^T r_t^{0,b} \Big(V_t(\phi, A) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^- \, dt < x^+ B_T^{0,b} - x^- B_T^{0,l} \Bigg) > 0.$$

- It is clear that the range of hedger's prices may depend on the initial wealth x, in general.

Part III: Replication of Contracts

Cash flows of a contract

- We denote by A the cumulative dividend paid by an OTC contract after its inception, as seen from the hedger's perspective.
- The cumulative dividend process accounts for all cash flows associated with a given security either received or paid after time 0 and before or at the contracts maturity date T, including the terminal payoff ΔA_T.

Example

If the the unique cash flow associated with the contract is the terminal payment occurring at time T, denoted as X, then the cumulative dividend process for this security takes form

$$A_t = X\mathbb{1}_{\{t=T\}}.$$

For the issuer of the European call option, there are no dividend payments and the terminal payoff equals $X = -(S_T - K)^+$, so that

$$A_t = -(S_T - K)^+\mathbb{1}_{\{t=T\}}.$$

Definition of replication

- In what follows, the price of an OTC contract will always be defined from the hedger's perspective with the initial endowment x

Definition

We say that a trading strategy (x, p, ϕ, A) *replicates* an OTC contract given by A if $V_T(x, p, \phi, A) = \widehat{V}_T(x)$.

Definition

If a contract can be replicated by a trading strategy (x, p, ϕ, A) then $V(x, p, \phi, A) - \widehat{V}(x)$ is called the *ex-dividend price* associated with ϕ and it is denoted by $S(x, \phi)$.

- It is not clear whether the uniqueness of the price $S(x, \phi)$ holds, in the sense that if (x, p, ϕ, A) and $(x, \widetilde{p}, \widetilde{\phi}, A)$ are two replicating strategies for a given contract then necessarily $V(x, p, \phi, A) = V(x, \widetilde{p}, \widetilde{\phi}, A)$.

First example: Replication in basic model

- Let $\widehat{\mathbb{P}}$ be any martingale measure for the model.
- It is assumed that random variables whose conditional expectations are evaluated are integrable.
- Recall that x is the hedger's initial endowment.
- Since the basic model is linear, we may and do set $x = 0$.

Proposition

Assume that a contract A can be replicated by a trading strategy (ϕ, A). Then its ex-dividend price process $S(\phi)$ associated with ϕ equals

$$S_t(\phi) = -B_t^0 \, \mathbb{E}_{\widehat{\mathbb{P}}}\bigg(\int_{(t,T]} (B_u^0)^{-1} \, dA_u \bigg), \quad t \in [0,T].$$

Second example: Wealth dynamics

- We now proceed to the second model and we assume that it is arbitrage-free. In particular, $r_t^{0,l} \leq r_t^{0,b}$ and $r_t^{0,l} \leq r_t^{i,b}$ for $i = 1, 2, \ldots, d$.
- We postulate the existence of a probability measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{G}_T)$ such that the processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i = 1, 2, \ldots, d$ are $(\widetilde{\mathbb{P}}, \mathbb{G})$-local martingales where

$$\widetilde{S}_t^{i,l,\mathrm{cld}} = (B_t^{0,l})^{-1} S_t^i + \int_0^t (B_u^{0,l})^{-1} \, dA_u^i.$$

- Recall that the wealth process now satisfies

$$\begin{aligned} dV_t(\phi) &= \sum_{i=1}^{k} \xi_t^i \big(dS_t^i + dA_t^i \big) - \sum_{i=1}^{d} r_t^{i,b} (\xi_t^i S_t^i)^+ \, dt + dA_t \\ &\quad + r_t^{0,l} \Big(V_t(\phi) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^+ dt - r_t^{0,b} \Big(V_t(\phi) + \sum_{i=1}^{d} (\xi_t^i S_t^i)^- \Big)^- dt. \end{aligned}$$

Second example: Replication

- Let us consider a contract with the dividend process A. We ask the following question: how to find the least expensive way of replication?
- We thus search for a strategy (p, ϕ) satisfying $V_T(x, p, \phi, A) = \widehat{V}_T(x)$ with the minimal cost p.
- For brevity, we represent the dynamics of $V(\phi, A)$ as follows

$$dV_t(\phi, A) = \sum_{i=1}^{k} \xi_t^i \big(dS_t^i - r_t^{0,l} S_t^i \, dt + dA_t^i \big) + f(t, \xi_t, S_t, V(\phi, A) \, dt + dA_t.$$

- The discounted wealth $\widetilde{V}^{0,l}(\phi, A) := (B^{0,l})^{-1} V^{0,l}(\phi, A)$ satisfies

$$\begin{aligned} d\widetilde{V}_t^{0,l}(\phi, A) &= \sum_{i=1}^{k} \xi_t^i \, d\widetilde{S}_t^{i,l,\mathrm{cld}} - r_t^{0,l} \widetilde{V}_t^{0,l}(\phi, A) \, dt \\ &\quad + (B_t^{0,l})^{-1} \, \widetilde{f}(t, \xi_t, S_t, \widetilde{V}^{0,l}(\phi, A)) \, dt + (B_t^{0,l})^{-1} \, dA_t. \end{aligned}$$

Second example: Pricing BSDE

- For $x = 0$, the pricing problem can now be intuitively represented as the problem of finding the strategy ξ which minimizes the following expectation

$$S_0(\phi) = -\widetilde{\mathbb{E}}\bigg(\int_0^T (B_u^{0,l})^{-1} \left(\widetilde{f}(u, \xi_u, S_u, \widetilde{V}_u^{0,l})\, du + dA_u \right) \bigg).$$

- More precisely, we search for a minimal solution (Y, ξ) to the BSDE

$$dY_t = \sum_{i=1}^{k} \xi_t^i \, d\widetilde{S}_t^{i,l,\mathrm{cld}} + (B_t^{0,l})^{-1} \, \widetilde{f}(t, \xi_t, S_t, Y_t)\, dt + (B_t^{0,l})^{-1} \, dA_t$$

with the terminal value $Y_T = 0$.

- We also address the issue of finding the least expensive way of super-hedging by postulating that $Y_T \geq 0$ rather than $Y_T = 0$. The strict comparison theorem for solutions to BSDES is a convenient tool for this purpose.
- The case of an arbitrary x can be dealt with in a similar way.

Part IV: Collateralized Contracts

Margin account

- Hedger enters a contract with contractual cash flows A and he either receives or posts cash collateral given by some stochastic process C.
- Let

$$C_t = C_t^+ \mathbb{1}_{\{C_t \geq 0\}} - C_t^- \mathbb{1}_{\{C_t < 0\}} = C_t^+ - C_t^-$$

  be the decomposition of C into its positive and negative parts.
- By convention, C^+ stands for the amount of collateral received, whereas $-C_t^-$ represents the amount of collateral paid.
- Recall that the mechanism of posting or receiving collateral is referred to as the *margin account*.
- We consider a *collateralized trading strategy* (ϕ, A, C) where

$$\phi = \left(\xi^1, \dots, \xi^d, \psi^0, \dots, \psi^d, \eta^l, \eta^b\right)$$

  in risky assets S^i, the unsecured account B^0, the funding accounts B^i and the collateral accounts $B^{c,l}$ and $B^{c,b}$.

Self-financing strategies

Definition

A collateralized trading strategy (x, ϕ, A, C) is *self-financing* whenever the portfolio's value $V^p(x, \phi, A, C)$, which is given by the equality

$$V_t^p(x, \phi, A, C) = \sum_{i=1}^{d} \xi_t^i S_t^i + \sum_{j=0}^{d} \psi_t^j B_t^j$$

satisfies, for every $t \in [0, T]$

$$V_t^p(x, \phi, A, C) = x + p + \sum_{i=1}^{d} \int_0^t \xi_u^i \, d(S_u^i + A_u^i) + \sum_{j=0}^{d} \int_0^t \psi_u^j \, dB_u^j + A_t$$
$$+ \int_0^t \eta_u^l \, dB_u^{c,l} + \int_0^t \eta_u^b \, dB_u^{c,b} + g(C_t).$$

Wealth dynamics

- In practice, the collateral amounts may either be held in segregated accounts or be available for rehypothecation by the collateral taker.
- The quantity $g(C_t)$ depends on the adopted collateral convention.
  - In the case of rehypothecation, we set $g(C_t) = C_t$.
  - In the case of segregation, we set $g(C_t) = -C_t^-$.
- In both cases, the remuneration of the margin account leads to the equality

$$F_t^C := \int_0^t C_u^- (B_u^{c,l})^{-1} \, dB_u^{c,l} - \int_0^t C_u^+ (B_u^{c,b})^{-1} \, dB_u^{c,b}.$$

- The wealth process $V(\phi)$ satisfies $V(\phi) = V^p(\phi) - g(C_t)$.

First example with segregation of collateral

Proposition

Let the remuneration of collateral F^C *be given by*

$$F_t^C = \int_0^t C_u^- (B_u^{c,l})^{-1}\, dB_u^{c,l} - \int_0^t C_u^+ (B_u^{c,b})^{-1}\, dB_u^{c,b}.$$

Then the dynamics of $\widetilde{V}^p(\phi, A) = (B_t^0)^{-1} V^p(\phi, A)$ *are*

$$d\widetilde{V}_t^p(\phi, A) = \sum_{i=1}^d \xi_t^i B_t^i \, d\widehat{S}_t^{i,cld} + \sum_{i=1}^d \zeta_t^i (\widetilde{B}_t^i)^{-1}\, d\widetilde{B}_t^i + dF_t^C - dC_t^- + dA_t.$$

If $\zeta_t^i = \psi_t^i B_t^i + \xi_t^i S_t^i = 0$ *for all* $t \in [0, T]$*, then*

$$d\widetilde{V}_t^p(\phi, A) = \sum_{i=1}^d \xi_t^i B_t^i \, d\widehat{S}_t^{i,cld} + dF_t^C - dC_t^- + dA_t.$$

Default events

Let $\tau := \tau_h \wedge \tau_c$ be the moment of the first default. On the event $\{\tau \leq T\}$, we define the random variable Υ as

$$\Upsilon = Q_\tau + \Delta A_\tau - C_\tau, \tag{5.1}$$

where

- Q is the Credit Support Annex (CSA) closeout valuation of the contract A,
- $\Delta A_\tau = A_\tau - A_{\tau-}$ is the jump of A at τ equal to a promised dividend at τ,
- C_τ is the value of the collateral process C at time τ.

In the financial interpretation, Υ^+ is the amount the counterparty owes to the hedger at time τ, whereas Υ^- is the amount the hedger owes to the counterparty at time τ.

It accounts for the legal value Q_τ of the contract, plus the bullet dividend ΔA_τ to be received/paid at time τ, less the collateral amount C_τ since it is already held by either the hedger (if $C_\tau > 0$) or the counterparty (if $C_\tau < 0$).

Closeout payoff $\mathfrak{K}$

- The closeout payoff occurs if at least one of the parties defaults before or at the maturity of the contract.
- We define the closeout payoff from the perspective of the hedger.
- The random variables R_c and R_h taking values in $[0,1]$ represent the recovery rates of the counterparty and the hedger, respectively.

Definition

The *CSA closeout payoff* $\mathfrak{K}$ is defined as

$$\mathfrak{K} := C_\tau + \mathbb{1}_{\{\tau^c < \tau^h\}}(R_c \Upsilon^+ - \Upsilon^-) + \mathbb{1}_{\{\tau^h < \tau^c\}}(\Upsilon^+ - R_h \Upsilon^-) + \mathbb{1}_{\{\tau^h = \tau^c\}}(R_c \Upsilon^+ - R_h \Upsilon^-).$$

The *counterparty risky cumulative cash flows* process $\widehat{A}$ is given by

$$\widehat{A}_t := \mathbb{1}_{\{t<\tau\}} A_t + \mathbb{1}_{\{t \geq \tau\}} (A_{\tau-} + \mathfrak{K}), \quad t \in [0,T].$$

Comments on the closeout payoff $\mathfrak{K}$

- The term C_τ is due to the fact that legal title to the collateral amount comes into force only at the time of the first default.
- The following three terms correspond to the CSA convention that, in principle, the nominal cash flow at the first default from the perspective of the hedger is given as $Q_\tau + \Delta A_\tau$.
- Let us consider, for instance, the event $\{\tau_c < \tau_h\}$. If $\Upsilon^+ > 0$, then we obtain

  $$\mathfrak{K} = C_\tau + R_c(Q_\tau + \Delta A_\tau - C_\tau) \leq Q_\tau + \Delta A_\tau,$$

  where the equality holds whenever $R_c = 1$. If $\Upsilon^- > 0$, then we get

  $$\mathfrak{K} = C_\tau - (-Q_\tau - \Delta A_\tau + C_\tau) = Q_\tau + \Delta A_\tau.$$

  Finally, if $\Upsilon = 0$, then $\mathfrak{K} = C_\tau = Q_\tau + \Delta A_\tau$.
- Similar analysis can be done on the remaining two events in the closeout formula.

A BSDE approach to fair bilateral pricing under funding costs and collateralization

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Outline

Classical Model

The classical approach to hedging the position in such a contract hinges on creation of a self-financing trading strategy, say $\phi = (\xi^1, \ldots, \xi^d, \psi^0)$, with the corresponding wealth process

$$V_t(\phi) = \sum_{i=1}^{d} \xi_t^i S_t^i + \psi_t^0 B_t^0.$$

- All trades are fully funded using a single money market account B_0.
- Both parties have access to the same traded risky assets, money account, and market information.
- The discounted cash flows are symmetric, i.e., the discounted cash flows, as seen from one party, were the negative of the discounted cash flows as seen from the other party.
- Hence the hedging and pricing exercise is symmetric in an analogous way.

Realistic Model

Classical models need to be extended since:

- contracts now tend to be collateralized;
- parties may need to account for different funding rates;
- counterparty and systemic risks need to be accounted for;
- netting of portfolio positions becomes an important issue.

Consequently,

- hedging portfolio will now refer to **multiple funding accounts**.
- the discounted cash flows (and thus also prices) will typically be **asymmetric** relative to the parties in the contract, since their funding sources are no longer assumed to be identical.

Previous lecture

- Provided a sound theoretical underpinning for some results presented in above papers by developing a unified framework for the nonlinear approach to hedging and pricing of OTC financial contracts.
- In particular, the impact that various funding bases and margin covenants exert on the values and hedging strategies for OTC contracts was examined.
- Provided a blueprint for derivation of dynamics of the wealth process corresponding to self-financing trading strategy and to examine such dynamics under various trading covenants,
- Introduced the relevant concepts of arbitrage and no-arbitrage valuation. It was shown that the problem of no-arbitrage under funding costs can be dealt with using a specific form of a martingale measure.
- Examined fair pricing under funding costs and collateralization through a nonlinear BSDE.

In this lecture

Under a generic non-linear market model, which includes several risky assets, multiple funding accounts and margin accounts

- We examine the pricing and hedging of contract both from the perspective of the hedger and the counterparty with arbitrary initial endowments.
- We derive inequalities for unilateral prices and we study the range of fair bilateral prices.
- We also examine the positive homogeneity and monotonicity of unilateral prices with respect to the initial endowments.
- our study hinges on results for BSDEs driven by continuous martingales.
- We derive the pricing PDEs for path-independent contingent claims of a European style in a Markovian framework.

Outline

Asset prices and funding accounts

- $T > 0$ finite trading horizon date for our model. $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ a filtered probability space where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and completeness.
- A – a *bilateral financial contract*. The process A is of finite variation and it represents the *cumulative cash flows* of a given contract from time 0 till its maturity date T.
- B^l (resp., B^b) – the unsecured *lending* (resp., *borrowing*) *cash account*.
- S^i – the *ex-dividend price* of the ith risky asset with the *cumulative dividend stream* A^i for $i = 1, 2, \dots, d$.
- $B^{i,b}$ – the *funding account* for long (cash) positions in the ith risky asset.
- $B^{c,l}$ (resp., $B^{c,b}$) – the *collateral remuneration* processes specifying the interest received (resp., paid) on the cash collateral pledged (resp., received) by the hedger.
- C – the *cash collateral*, a $\mathbb{G}$-adapted process.

Contract

Definition

A *bilateral financial contract* is an arbitrary càdlàg process A of finite variation, which represents the *cumulative cash flows* of a given contract from time 0 till its maturity date T.

- A models all cash flows, which are either paid out from the wealth or added to the wealth, as seen from the *hedger*. Hence the process $-A$ plays an analogous role for the *counterparty*.
- By convention, A includes the initial cash flow $p = A_0$ of a contract at its inception date $t_0 = 0$.
- For instance, if a contract has the initial price p and stipulates that the hedger will receive the cash flows $\bar{A}_1, \dots, \bar{A}_k$ at times $t_1, \dots, t_k \in (0, T]$, then we set $A_0 = p_0$ so that

$$A_t = p\mathbb{1}_{[0,T]}(t) + \sum_{l=1}^{k} \bar{A}_l \, \mathbb{1}_{[t_l,T]}(t).$$

Collateral with rehypothecation

- $C_t^+ := C_t \mathbb{1}_{\{C_t \geq 0\}}$ is the cash collateral received at time t by the hedger, whereas $C_t^- := -C_t \mathbb{1}_{\{C_t < 0\}}$ represents the cash collateral provided at time t by the hedger.
- We postulate that $C_T = 0$ to ensure that the collateral is returned in full to the pledging party when a contract matures, provided that the default event has not occurred prior to or at time T.
- *Rehypothecation* allows to reuse the collateral pledged by counterparties as collateral, as opposed to *segregation* where the reuse of collateral is prohibited.
- We work under a stylized convention of full rehypothecation, meaning that the cash collateral received by the hedger is used for trading without any restrictions. If the hedger is a collateral provider, then segregation or rehypothecation is obviously immaterial for the dynamics of the value process of his portfolios.

Collateral account

The collateral accounts $B^{c,l}$ and $B^{c,b}$ play the following roles.

- If the hedger provides (resp., receives) cash collateral with the amount C^- (resp., C^+), then he receives (resp., pays) interest on this amount, as specified by the process $B^{c,l}$ (resp., $B^{c,b}$).
- Hence the counterparties are exposed to different conditions, unless $B^{c,l}$ coincides with $B^{c,b}$.
- For the sake of simplicity, the hedger and counterparty are implicitly assumed to be *default-free* before the maturity date T of a contract at hand. In the presence of defaults, we would need to specify the close-out payoff and to deal with the pricing BSDE up to a random time horizon.
- It is worth stressing that none of the processes $C, B^{c,l}$ and $B^{c,b}$ is assumed to be a traded asset, they should rather be seen as market frictions.

Market model with partial netting

In the model with partial netting, we postulate that:

- Short positions in risky assets $S^1, S^2, \ldots, S^d$ are aggregated and the proceeds from short-selling are used for trading.
- Long positions in risky assets S^i are assumed to be funded from their respective funding accounts $B^{i,b}$, which can be interpreted as secured loans in the repo market.
- All positive and negative cash flows from a contract (A, C) and a trading strategy ϕ, inclusive of the proceeds from short-selling of risky assets, are reinvested in traded assets.
- This convention corresponds to the case of a synthetic short-selling of an asset through the repo market, as opposed to the classical short-selling of a borrowed asset where the broker keeps cash in his account.

Hedger's portfolio

- Hedger's portfolio is composed of initial endowment x, a process

$$\phi = \left(\xi^1, \dots, \xi^d, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta^l, \eta^b\right)$$

and a long position in a contract (A, C).

- The process ϕ represent positions in the risky assets S^i, the funding accounts $B^{i,b}$ for risky assets, the unsecured lending cash account B^l, the unsecured borrowing cash account B^b, and the collateral remuneration accounts $B^{c,l}$ and $B^{c,b}$.
- We postulate that:
(i) $\psi^l_t \geq 0,\ \psi^b_t \leq 0$ and $\psi^l_t \psi^b_t = 0$ for all $t \in [0, T]$,
(ii) for every $i = 1, 2, \dots, d$ and all $t \in [0, T]$, $\psi^{i,b}_t = -(B^{i,b}_t)^{-1}(\xi^i_t S^i_t)^+$,
(iii) $\eta^l_t = (B^{c,l}_t)^{-1} C^-_t$ and $\eta^b_t = -(B^{c,b}_t)^{-1} C^+_t$ for all $t \in [0, T]$.

Self-financing trading strategy under rehypothecation

Definition

A hedger's trading strategy (x, ϕ, A, C) is *self-financing* whenever the *portfolio's value* $V^p(x, \phi, A, C)$ given by

$$V_t^p(x, \phi, A, C) := \sum_{i=1}^{d} \xi_t^i S_t^i + \sum_{i=1}^{d} \psi_t^{i,b} B_t^{i,b} + \psi_t^l B_t^l + \psi_t^b B_t^b \qquad (2.1)$$

satisfies, for every $t \in [0, T]$,

$$V_t^p(x, \phi, A, C) = x + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \, d(S_u^i + A_u^i) + \sum_{i=1}^{d} \int_0^t \psi_u^{i,b} \, dB_u^{i,b} + \int_0^t \psi_u^l \, dB_u^l + \int_0^t \psi_u^b \, dB_u^b + \int_0^t \eta_u^l \, dB_u^{c,l} + \int_0^t \eta_u^b \, dB_u^{c,b} + C_t + A_t.$$

Self-financing trading strategy under rehypothecation

- The integrals $\int_0^t \eta_u^l \, dB_u^{c,l}$ and $\int_0^t \eta_u^b \, dB_u^{c,b}$ represent the accrued interest generated by the margin account.
- $B^{c,l}$ and $B^{c,b}$ do not appear in (2.1) since they are not traded assets.
- For any self-financing trading strategy (x, ϕ, A, C), the hedger's *wealth* is given by the equality

$$V(x, \phi, A, C) = V^p(x, \phi, A, C) - C.$$

- Formally, the self-financing property of the hedger's strategy can be defined either in terms of the dynamics of the portfolio's value process $V^p(x, \phi, A, C)$ or, equivalently, in term of the dynamics of the hedger's wealth $V(x, \phi, A, C)$.
- We prefer to focus on the process $V^p(x, \phi, A, C)$ to emphasize the fact that the self-financing property is primarily concerned with specifying the manner in which the hedger's portfolio of traded assets can be continuously rebalanced by the hedger

Wealth dynamics of self-financing trading strategy

Lemma

The dynamics of a trading strategy (x, ϕ, A, C) *are uniquely determined by the initial endowment* x *and processes* ξ, A *and* C *through the following equation*

$$
\begin{aligned}
dV_t^p(x,\phi,A,C) = & \sum_{i=1}^d \xi_t^i \, (dS_t^i + dA_t^i) - \sum_{i=1}^d (\xi_t^i S_t^i)^+ (B_t^{i,b})^{-1} \, dB_t^{i,b} + dA_t^C \\
& + \Big(V_t^p(x,\phi,A,C) + \sum_{i=1}^d (\xi_t^i S_t^i)^- \Big)^+ (B_t^l)^{-1} \, dB_t^l \\
& - \Big(V_t^p(x,\phi,A,C) + \sum_{i=1}^d (\xi_t^i S_t^i)^- \Big)^- (B_t^b)^{-1} \, dB_t^b
\end{aligned}
$$

where $A^C := A + C + F^C$ *and*

$$
F_t^C := \int_0^t C_u^- (B_u^{c,l})^{-1} \, dB_u^{c,l} - \int_0^t C_u^+ (B_u^{c,b})^{-1} \, dB_u^{c,b}.
$$

Wealth dynamics of a self-financing trading strategy

To compute the so-called *generator* for the associated BSDEs we assume that:

- $dB_t^l = r_t^l B_t^l \, dt, \;\; dB_t^b = r_t^b B_t^b \, dt, \;\; dB_t^{i,b} = r_t^{i,b} B_t^{i,b} \, dt$, for $\mathbb{G}$-adapted processes r^l, r^b and $r^{i,b}$ such that $0 \le r^l \le r^b$ and $r^l \le r^{i,b}$ for $i = 1, 2, \ldots, d$,
- $dB_t^{c,b} = r_t^{c,b} B_t^{c,b} \, dt, \;\; dB_t^{c,l} = r_t^{c,l} B_t^{c,l} \, dt$, for some $\mathbb{G}$-adapted processes $r^{c,b}$ and $r^{c,l}$ satisfying $r^{c,l} \le r^{c,b}$.

We sometimes work under the following assumption, which restores the symmetry between the two parties in regard to remuneration of collateral.
Assumption (C) The processes $B^{c,l}$ and $B^{c,b}$ satisfy $B^{c,l} = B^{c,b} = B^c$ where $dB_t^c = r_t^c B_t^c \, dt$ so that, for all $t \in [0, T]$,

$$F_t^C = - \int_0^t C_u (B_u^c)^{-1} \, dB_u^c = - \int_0^t r_u^c C_u \, du = -F_t^{-C}$$

and thus the equality $(-A)^{-C} := -A - C + F^{-C} = -A^C$ holds.

Notation for discounted processes

The discounted cumulative prices of risky assets

$$\widetilde{S}_t^{i,l,\mathrm{cld}} := (B_t^l)^{-1} S_t^i + \int_{(0,t]} (B_u^l)^{-1} \, dA_u^i$$

and

$$S_t^{i,b,\mathrm{cld}} := (B_t^b)^{-1} S_t^i + \int_{(0,t]} (B_u^b)^{-1} \, dA_u^i.$$

Also let

$$A_t^{C,l} := \int_{(0,t]} (B_u^l)^{-1} \, dA_u^C$$

and

$$A_t^{C,b} := \int_{(0,t]} (B_u^b)^{-1} \, dA_u^C.$$

Wealth dynamics of self-financing trading strategy

Lemma

The discounted wealth $Y^l := \widetilde{V}^{p,l}(x,\phi,A,C) = (B^l)^{-1}V^p(x,\phi,A,C)$ *satisfies*

$$dY_t^l = \sum_{i=1}^d \xi_t^i \, d\widetilde{S}_t^{i,l,cld} + \widetilde{f}_l(t, Y_t^l, \xi_t)\, dt + dA_t^{C,l}$$

where the mapping $\widetilde{f}_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *is given by*

$$\widetilde{f}_l(t,y,z) := (B_t^l)^{-1} f_l(t, B_t^l y, z) - r_t^l y$$

and $f_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *satisfies* $f_l(t,y,z) :=$

$$\sum_{i=1}^d r_t^l z^i S_t^i - \sum_{i=1}^d r_t^{i,b}(z^i S_t^i)^+ + r_t^l\Big(y + \sum_{i=1}^d (z^i S_t^i)^-\Big)^+ - r_t^b\Big(y + \sum_{i=1}^d (z^i S_t^i)^-\Big)^-.$$

Wealth dynamics of self-financing trading strategy

For a party with a negative initial endowment, we have the following lemma.

Lemma

The discounted wealth $Y^b := \widetilde{V}^{p,b}(x,\phi,A,C) = (B^b)^{-1}V^p(x,\phi,A,C)$ *satisfies*

$$dY_t^b = \sum_{i=1}^d \xi_t^i \, d\widetilde{S}_t^{i,b,cld} + \widetilde{f}_b(t, Y_t^b, \xi_t)\, dt + dA_t^{C,b}$$

where the mapping $\widetilde{f}_b : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *is given by*

$$\widetilde{f}_b(t,y,z) := (B_t^b)^{-1} f_b(t, B_t^b y, z) - r_t^b y$$

and $f_b : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *satisfies* $f_b(t,y,z) :=$

$$\sum_{i=1}^d r_t^b z^i S_t^i - \sum_{i=1}^d r_t^{i,b}(z^i S_t^i)^+ + r_t^l \Big(y + \sum_{i=1}^d (z^i S_t^i)^- \Big)^+ - r_t^b \Big(y + \sum_{i=1}^d (z^i S_t^i)^- \Big)^- .$$

Outline

Netted wealth and arbitrage opportunities

We denote by

$$\widehat{V}_t(x) = xB_t^l \mathbb{1}_{\{x \geq 0\}} + xB_t^b \mathbb{1}_{\{x < 0\}}$$

the wealth of the hedger if he decides not to engage in any contract and simply invest his initial endowment x in the cash account.

Definition

An *arbitrage opportunity* with respect to a contract (A, C) for the hedger with an initial endowment x is a pair $(\widehat{x}, \widehat{\phi}, A, C)$ and $(\widetilde{x}, \widetilde{\phi}, -A, -C)$ of trading strategies such that $x = \widehat{x} + \widetilde{x}$ and

$$\mathbb{P}(V_T^{\mathrm{net}} \geq \widehat{V}_T(x)) = 1 \quad \text{and} \quad \mathbb{P}(V_T^{\mathrm{net}} > \widehat{V}_T(x)) > 0$$

where the *netted wealth* $V^{\mathrm{net}} = V^{\mathrm{net}}(\widehat{x}, \widetilde{x}, \widehat{\phi}, \widetilde{\phi}, A, C)$ is given by

$$V^{\mathrm{net}}(\widehat{x}, \widetilde{x}, \widehat{\phi}, \widetilde{\phi}, A, C) := V(\widehat{x}, \widehat{\phi}, A, C) + V(\widetilde{x}, \widetilde{\phi}, -A, -C).$$

Admissibility and no-arbitrage

Definition

A pair $(\widehat{x}, \widehat{\phi}, A, C)$ and $(\widetilde{x}, \widetilde{\phi}, -A, -C)$ of self-financing trading strategies is *admissible for the hedger* if the discounted netted wealth

$$\begin{aligned}\widehat{V}^{\text{net}}(\widehat{x}, \widetilde{x}, \widehat{\phi}, \widetilde{\phi}, A, C) := {} & (B^l)^{-1} V^{\text{net}}(\widehat{x}, \widetilde{x}, \widehat{\phi}, \widetilde{\phi}, A, C) \mathbb{1}_{\{x \geq 0\}} \\ & + (B^b)^{-1} V^{\text{net}}(\widehat{x}, \widetilde{x}, \widehat{\phi}, \widetilde{\phi}, A, C) \mathbb{1}_{\{x < 0\}}\end{aligned}$$

is bounded from below by a constant.

Proposition

If there exists a probability measure $\widetilde{\mathbb{P}}^l$, which is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, such that the processes $\widetilde{S}^{i,l,cld}$, $i = 1, 2, \dots, d$ are $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-local martingales. Then no extended arbitrage opportunity in regard to any contract (A, C) exists for the hedger and the counterparty with a non-negative initial endowment.

Replication of a contract

When dealing with the hedger with a negative initial endowment, it is suitable to assume:

- There exists a probability measure $\widetilde{\mathbb{P}}^b$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, such that the processes $\widetilde{S}^{i,b,\mathrm{cld}}$, $i = 1, 2, \dots, d$ are $(\widetilde{\mathbb{P}}^b, \mathbb{G})$-local martingales.
- $r^b \leq r^{i,b}$ holds for every i. It is too strong from the practical point of view and thus we will focus on the case of nonnegative endowments.
- For a fixed t, let us denote $A^t_u = A_u - A_t$ for $u \in [t, T]$.

Definition

For $t \in [0, T]$, a trading strategy $(\widehat{V}_t(x) + p^{A,C}_t, \phi, A^t, C)$ where $p^{A,C}_t$ is a $\mathcal{G}_t$-measurable random variable, is said to *replicate a collateralized contract* (A, C) on $[t, T]$ whenever

$$V_T(\widehat{V}_t(x) + p^{A,C}_t, \phi, A^t, C) = \widehat{V}_T(x).$$

Ex-dividend prices

Denote initial endowment of hedger (resp., the counterparty) by x_1 (resp., x_2).

Definition

Any $\mathcal{G}_t$-measurable random variable for which a replicating strategy for (A, C) over $[t, T]$ exists is called a *hedger's ex-dividend price* at time t for (A, C) and denoted by $P_t^h(x_1, A, C)$, so that for some strategy ϕ replicating (A, C)

$$V_T(\widehat{V}_t(x_1) + P_t^h(x_1, A, C), \phi, A^t, C) = \widehat{V}_T(x_1).$$

Similarly, for an arbitrary level x_2 of the counterparty's initial endowment and any trading strategy ϕ replicating $(-A, -C)$, the *counterparty's ex-dividend price* $P_t^c(x_2, -A, -C)$ at time t for the contract $(-A, -C)$ is implicitly given by

$$V_T(\widehat{V}_t(x_2) - P_t^c(x_2, -A, -C), \phi, -A^t, -C) = \widehat{V}_T(x_2).$$

Range of fair bilateral prices

Since we place ourselves in a non-linear framework, a natural asymmetry arises between the counterparties, the price discrepancy may occur, that is, it may happen that

$$P_t^h(x_1, A, C) \neq P_t^c(x_2, -A, -C).$$

To be more specific, the following situations may arise:

- **(H.1)** $0 \le P_t^c(x_2, -A, -C) < P_t^h(x_1, A, C)$,
- **(H.2)** $P_t^c(x_2, -A, -C) \le 0 < P_t^h(x_1, A, C)$,
- **(H.3)** $P_t^c(x_2, -A, -C) < P_t^h(x_1, A, C) \le 0$,

and, symmetrically,

- **(C.1)** $0 \le P_t^h(x_1, A, C) < P_t^c(x_2, -A, -C)$,
- **(C.2)** $P_t^h(x_1, A, C) \le 0 < P_t^c(x_2, -A, -C)$,
- **(C.3)** $P_t^h(x_1, A, C) < P_t^c(x_2, -A, -C) \le 0$.

Range of fair bilateral prices

Definition

A real number $p^{A,C} = A_0$ is a *hedger's fair price* for (A, C) at time 0 if for any trading strategy (x, ϕ, A, C), such that the discounted wealth

$$\widehat{V}(x, \phi, A, C) := (B^l)^{-1} V(x, \phi, A, C) \mathbb{1}_{\{x \geq 0\}} + (B^b)^{-1} V(x, \phi, A, C) \mathbb{1}_{\{x < 0\}}$$

is bounded from below, we have $\mathbb{P}\big(V_T(x, \phi, A, C) = \widehat{V}_T(x)\big) = 1$ or $\mathbb{P}\big(V_T(x, \phi, A, C) < \widehat{V}_T(x)\big) > 0.$

For an arbitrage-free model for both parties, in all cases (H.1)–(H.3), any $\mathcal{G}_t$-measurable random variable P_t^f satisfying

$$P_t^f \in \big[P_t^c(x_2, -A, -C), P_t^h(x_1, A, C)\big]$$

is a *fair price* for both parties, in the sense that a bilateral transaction executed at P_t^f will not generate an arbitrage opportunity for either of them.

Range of fair bilateral prices

Definition

The interval $\mathcal{R}^f_t(x_1, x_2) := \big[P^c_t(x_2, -A, -C), P^h_t(x_1, A, C)\big]$ is called the *range of fair bilateral prices* at time t of a contract (A, C) between the hedger and the counterparty.

The analysis for the cases (C.1)–(C.3) can be done analogously, the financial interpretation and conclusions are quite different. For any $\mathcal{G}_t$-measurable random variable P^p_t satisfying

$$P^p_t \in \big[P^h_t(x_1, A, C), P^c_t(x_2, -A, -C)\big]$$

can be interpreted as a bilaterally acceptable price. Note that, unless $P^h_t(x_1, A, C) = P^c_t(x_2, -A, -C)$, the price P^p_t is not a fair bilateral price, since an arbitrage opportunity may arise for at least one party involved when a contract (A, C) is traded between them at the price P^p_t.

Outline

Hedger's price

Proposition

Under suitable assumptions, for $x_1 \geq 0$ and any contract (A, C) admissible under $\widetilde{\mathbb{P}}^l$, hedger's ex-dividend price satisfies, for every $t \in [0, T)$,

$$P_t^h(x_1, A, C) = B_t^l(Y_t^{h,l,x_1} - x_1) - C_t$$

where the pair $(Y^{h,l,x_1}, Z^{h,l,x_1})$ is the unique solution to the BSDE

$$\begin{cases} dY_t^{h,l,x_1} = Z_t^{h,l,x_1,*}\, d\widetilde{S}_t^{l,cld} + \widetilde{f}_l\big(t, Y_t^{h,l,x_1}, Z_t^{h,l,x_1}\big)\, dt + dA_t^{C,l}, \\ Y_T^{h,l,x_1} = x_1, \end{cases}$$

with the generator $\widetilde{f}_l$.

Hedger's replicating strategy

Proposition

The unique replicating strategy for the hedger equals

$$\phi = \left(\xi^1, \dots, \xi^d, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta^b, \eta^l\right)$$

where for every $t \in [0, T]$ *and* $i = 1, 2, \dots, d$

$$\xi^i_t = Z^{h,l,x_1,i}_t, \quad \psi^{i,b}_t = -(B^{i,b}_t)^{-1}(\xi^i_t S^i_t)^+,$$

$$\eta^l_t = (B^{c,l}_t)^{-1} C^-_t, \quad \eta^b_t = -(B^{c,b}_t)^{-1} C^+_t,$$

$$\psi^l_t = (B^l_t)^{-1}\Big(B^l_t Y^{h,l,x_1}_t + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^+,$$

$$\psi^b_t = -(B^b_t)^{-1}\Big(B^l_t Y^{h,l,x_1}_t + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^-.$$

Counterparty's price

Proposition

Under suitable assumptions, for $x_2 \geq 0$ *and any contract* (A, C) *admissible under* $\widetilde{\mathbb{P}}^l$*, counterparty's ex-dividend price is, for* $t \in [0, T)$,

$$P_t^c(x_2, -A, -C) = -B_t^l(Y_t^{c,l,x_2} - x_2) - C_t$$

where the pair $(Y^{c,l,x_2}, Z^{c,l,x_2})$ *is the unique solution to the BSDE*

$$\begin{cases} dY_t^{c,l,x_2} = Z_t^{c,l,x_2,*}\, d\widetilde{S}_t^{l,cld} + \widetilde{f}_l\big(t, Y_t^{c,l,x_2}, Z_t^{c,l,x_2}\big)\, dt + d(-A)_t^{-C,l}, \\ Y_T^{c,l,x_2} = x_2, \end{cases}$$

where

$$(-A)_t^{-C,l} := \int_{(0,t]} (B_u^l)^{-1}\, d(-A)_u^{-C} = -A_t^{C,l}$$

Counterparty's replicating strategy

Proposition

The unique replicating strategy for the counterparty is

$$\phi = \left(\xi^1, \dots, \xi^d, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta^b, \eta^l\right)$$

where for every $t \in [0, T]$ *and* $i = 1, 2, \dots, d$

$$\xi_t = Z_t^{c,l,x_2}, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+,$$

$$\eta_t^l = (B_t^{c,l})^{-1} C_t^+, \quad \eta_t^b = -(B_t^{c,b})^{-1} C_t^-$$

$$\psi_t^l = (B_t^l)^{-1}\left(B_t^l Y_t^{c,l,x_2} + \sum_{i=1}^d (\xi_t^i S_t^i)^-\right)^+,$$

$$\psi_t^b = -(B_t^b)^{-1}\left(B_t^l Y_t^{c,l,x_2} + \sum_{i=1}^d (\xi_t^i S_t^i)^-\right)^-.$$

Outline

Properties of Prices

Range of fair bilateral prices. If $x_1 \geq 0$ and $x_2 \geq 0$, then for any contract (A, C) admissible under $\widetilde{\mathbb{P}}^l$ we have, for all $t \in [0, T]$,

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \widetilde{\mathbb{P}}^l - \text{a.s.},$$

so that the range of fair bilateral prices $\mathcal{R}_t^f(x_1, x_2)$ is non-empty almost surely.

Monotonicity w.r.t. the initial endowment. Let a contract (A, C) be admissible under $\widetilde{\mathbb{P}}^l$. Then the hedger's price satisfies: if $\bar{x} \geq x \geq 0$, then for all $t \in [0, T]$

$$P_t^h(\bar{x}, A, C) \leq P_t^h(x, A, C),$$

and the counterparty's price satisfies: if $\bar{x} \geq x \geq 0$, then for all $t \in [0, T]$

$$P_t^c(\bar{x}, -A, -C) \geq P_t^c(x, -A, -C).$$

Properties of Prices

Asymptotic properties of unilateral prices. For any contract (A, C) admissible under $\widetilde{\mathbb{P}}^l$ and any date $t \in [0, T]$, there exist $\mathbb{G}$-adapted processes $P_t^{h,A,C,+}$ and $P_t^{c,-A,-C,+}$, s.t.

$$P_t^{h,A,C,+},\, P_t^{c,-A,-C,+} \in [P_t^c(0,-A,-C), P_t^h(0,A,C)] = \mathcal{R}_t^f(0,0)$$

$$\lim_{x \to +\infty} P_t^c(x,-A,-C) = P_t^{c,-A,-C,+} \leq P_t^{h,A,C,+} = \lim_{x \to +\infty} P_t^h(x,A,C).$$

Price independence of initial endowment. Let a contract (A, C) be admissible under $\widetilde{\mathbb{P}}^l$ such that $A^C - A_0^C$ is decreasing. The price $P_t^h(x_1, A, C)$ is independent of $x_1 \geq 0$, so that $P_t^h(x_1, A, C) = P_t^h(0, A, C)$ for all $x_1 \geq 0$ and $t \in [0, T]$.

Positive homogeneity of the hedger's price. The hedger's price is positively homogeneous, in the sense that the equality $P_t^h(\lambda x_1, \lambda A, \lambda C) = \lambda P_t^h(x_1, A, C)$ is valid for all $\lambda \in \mathbb{R}_+$.

Outline

Setting

- For simplicity, we assume there is only one risky asset $S = S^1$ and

$$dS_u = \mu(u, S_u)\, du + \sigma(u, S_u)\, dW_u, \quad S_t = s \in \mathcal{O},$$

- The dividend process for the asset S is given by

$$A_u^1 = \int_t^u \kappa(v, S_v)\, dv.$$

- We examine the valuation and hedging of an uncollateralized European contingent claim starting from a fixed time $t \in [0, T]$:

$$A_t - A_0 = -H(S_T)\mathbb{1}_{[T,T]}(t)$$

and we set $C = 0$.

Hedger's PDE

Proposition

Let $v^h(t,s) \in C^{1,2}([0,T] \times \mathcal{O})$ such that $v^h(t,s)$ and $\frac{\partial v^h}{\partial s}(t,s)$ have a polynomial growth in s, be the solution to the quasi-linear PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t,s) + \frac{1}{2}\sigma^2(t,s)\frac{\partial^2 v}{\partial s^2}(t,s) = \kappa(t,s)\frac{\partial v}{\partial s}(t,s) \\ \qquad - x_1 r_t^l B_t^l - r_t^{1,b}\big(s\frac{\partial v}{\partial s}(t,s)\big)^+ \\ \qquad + r_t^l\Big(v(t,s) + x_1 B_t^l + \big(s\frac{\partial v}{\partial s}(t,s)\big)^-\Big)^+ \\ \qquad - r_t^b\Big(v(t,s) + x_1 B_t^l + \big(s\frac{\partial v}{\partial s}(t,s)\big)^-\Big)^-, \\ v(T,s) = H(s), \quad s \in \mathcal{O}. \end{cases}$$

Hedger's replicating strategy

Proposition

The hedger's price of the European contingent claim $H(S_T)$ *is given by* $v^h(t, S_t)$ *and the unique replicating strategy* $\phi = \big(\xi, \psi^l, \psi^b, \psi^{1,b}\big)$ *for the hedger is given by*

$$\xi_u = \frac{\partial v}{\partial s}(u, S_u),$$

$$\psi_t^{1,b} = -(B_u^{1,b})^{-1}\Big(S_u \frac{\partial v}{\partial s}(u, S_u)\Big)^+,$$

$$\psi_u^l = (B_u^l)^{-1}\Big(v(u, S_u) + x_1 B_u^l + \big(S_u \frac{\partial v}{\partial s}(u, S_u)\big)^-\Big)^+,$$

$$\psi_u^b = -(B_u^b)^{-1}\Big(v(u, S_u) + x_1 B_u^l + \big(S_u \frac{\partial v}{\partial s}(u, S_u)\big)^-\Big)^-,$$

where $v = v^h$.

Counterparty's PDE

Proposition

Let $v^c(t,s) \in C^{1,2}([0,T] \times \mathcal{O})$ such that $v^c(t,s)$ and $\frac{\partial v^c}{\partial s}(t,s)$ have a polynomial growth in s be the solution to the quasi-linear PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t,s) + \frac{1}{2}\sigma^2(t,s)\frac{\partial^2 v}{\partial s^2}(t,s) = \kappa(t,s)\frac{\partial v}{\partial s}(t,s) \\ \qquad + x_2 r_t^l B_t^l + r_t^{1,b}\big(- s\frac{\partial v}{\partial s}(t,s)\big)^+ \\ \qquad - r_t^l\Big(- v(t,s) + x_2 B_t^l + \big(- s\frac{\partial v}{\partial s}(t,s)\big)^-\Big)^+ \\ \qquad + r_t^b\Big(- v(t,s) + x_2 B_t^l + \big(- s\frac{\partial v}{\partial s}(t,s)\big)^-\Big)^-, \\ v(T,s) = H(s), \quad s \in \mathcal{O}. \end{cases}$$

Counterparty's replicating strategy

Proposition

The counterparty's price of the European contingent claim $H(S_T)$ *is given by* $v^c(t, S_t)$ *and the unique replicating strategy* $\phi = \big(\xi, \psi^l, \psi^b, \psi^{1,b}\big)$ *for the counterparty is given by*

$$
\begin{aligned}
\xi_u &= -\frac{\partial v}{\partial s}(u, S_u), \\
\psi_u^{1,b} &= -(B_u^{1,b})^{-1}\Big(- S_u \frac{\partial v}{\partial s}(u, S_u)\Big)^+, \\
\psi_u^l &= (B_u^l)^{-1}\Big(- v(u, S_u) + x_2 B_u^l + \big(- S_u \frac{\partial v}{\partial s}(u, S_u)\big)^- \Big)^+, \\
\psi_u^b &= -(B_u^b)^{-1}\Big(- v(u, S_u) + x_2 B_u^l + \big(- S_u \frac{\partial v}{\partial s}(u, S_u)\big)^- \Big)^-,
\end{aligned}
$$

where $v = v^c$.

Fair valuation and hedging of contracts with endogenous collateralization

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1. Trading with Funding Costs and Collateral
2. Arbitrage-Free Property
3. Replication and Fair Bilateral Prices
4. Endogenous Collateral
5. Hedger's Collateral
6. Two-Sided Collateral
7. Model with a partial netting

New challenges

- The financial crisis of 2007-2009 has led to major changes in the operations of financial markets.
- The defaultability of the counterparties became the central problem of financial management.
- The classic paradigm of discounting future cash flows using the risk-free rate is no longer accepted as a viable pricing rule (multiple yield curves).
- In the presence of funding costs, counterparty credit risk, and collateral (margin account) the classic arbitrage pricing theory no longer applies.
- As a consequence, the analysis of the counterparty credit risk and price formation for collateralized contracts under differential funding costs are currently the most challenging problems in Mathematical Finance.
- A non-linear and asymmetric pricing and hedging paradigm is emerging.

Goals

- To describe trading strategies in the presence of funding costs (multiple yield curves) and margin account (collateral).
- To propose suitable approaches to pricing of financial contracts within this novel framework.
- We mainly focus on one party (called the *hedger*), but the same technique can be used to solve the problem for the *counterparty*.
- The mark-to-market convention for collateral requires that both parties agree in respect of the fair bilateral value of the contract. Hence the actual problem is two-dimensional, rather than one-dimensional.
- The latter issue is especially important in the case of the so-called *endogenous collateral* where we deal with a two-dimensional fully-coupled backward stochastic differential equation (BSDE).

Extended Bergman's (1995) model

- The semimartingale S^i is the price of the ith risky security.
- Cash accounts B^l and B^b for unsecured *lending* and *borrowing* of cash.
- The *collateral* accounts $B^{c,l}$ and $B^{c,b}$ are strictly positive and continuous processes of finite variation.
- A *contract* is a process A representing the *cumulative cash flows*.
- The *collateral process* C with $C_T = 0$ can be represented as

$$C_t = C_t \mathbb{1}_{\{C_t \geq 0\}} + C_t \mathbb{1}_{\{C_t < 0\}} = C_t^+ - C_t^-$$

  where C_t^+ is the cash collateral received at time t by the hedger and C_t^- represents the cash collateral posted by him.
- The process $V(x, \varphi, A, C)$ represents the hedger's wealth.
- The process $V^p(x, \varphi, A, C) = V(x, \varphi, A, C) + C_t$ is the portfolio's value.
- The initial endowment is denoted by x (or rather x_1 and x_2).

Self-financing trading strategy

- For a portfolio $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \eta^b, \eta^l)$, the hedger's wealth process equals

$$V_t(x, \varphi, A, C) = \sum_{i=1}^{d} \xi_t^i S_t^i + \psi_t^l B_t^l + \psi_t^b B_t^b + \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l}$$

where $\eta_t^b = -(B_t^{c,b})^{-1} C_t^+$ and $\eta_t^l = (B_t^{c,l})^{-1} C_t^-$.

- A trading strategy (x, φ, A, C) is *self-financing* when the value process

$$V_t^p(x, \varphi, A, C) := \sum_{i=1}^{d} \xi_t^i S_t^i + \psi_t^l B_t^l + \psi_t^b B_t^b$$

satisfies

$$V_t^p(x, \varphi, A, C) = x + \sum_{i=1}^{d} \int_0^t \xi_u^i \, dS_u^i + \int_0^t \psi_u^l \, dB_u^l + \int_0^t \psi_u^b \, dB_u^b + A_t$$
$$+ \int_0^t \eta_u^b \, dB_u^{c,b} + \int_0^t \eta_u^l \, dB_u^{c,l} + C_t.$$

Funding costs

- We have

$$\psi_t^l = (B_t^l)^{-1}\Big(V_t^p(x,\varphi,A,C) - \sum_{i=1}^d \xi_t^i S_t^i\Big)^+$$

and

$$\psi_t^b = -(B_t^b)^{-1}\Big(V_t^p(x,\varphi,A,C) - \sum_{i=1}^d \xi_t^i S_t^i\Big)^-.$$

- Let $dB_t^l = r_t^l B_t^l\,dt$ and $dB_t^b = r_t^b B_t^b\,dt$ for some processes $0 \le r^l \le r^b$.
- Let $B^{c,l} = B^{c,b} = B^c$ where $dB_t^c = r_t^c B_t^c\,dt$ for some process r^c.
- We define the process F^C

$$F_t^C := \int_0^t \eta_u^b\,dB_u^{c,b} + \int_0^t \eta_u^l\,dB_u^{c,l} = -\int_0^t r_u^c C_u\,du$$

and we denote $A^C := A + C + F^C$.

Dynamics of discounted portfolio's value

Proposition

Let $\widetilde{S}^{i,l}_t := (B^l_t)^{-1} S^i_t$. *The process* $Y^l := (B^l)^{-1} V^p(x, \varphi, A, C)$ *satisfies*

$$dY^l_t = \sum_{i=1}^{d} Z^{l,i}_t \, d\widetilde{S}^{i,l}_t + G_l(t, Y^l_t, Z^l_t)\, dt + (B^l_t)^{-1}\, dA^C_t$$

where $Z^{l,i} = \xi^i$, $i = 1, 2, \ldots, d$ *and the mapping* G_l *equals*

$$G_l(t,y,z) = \sum_{i=1}^{d} r^l_t (B^l_t)^{-1} z^i S^i_t + (B^l_t)^{-1} \left(r^l_t \Big(yB^l_t - \sum_{i=1}^{d} z^i S^i_t \Big)^+ - r^b_t \Big(yB^l_t - \sum_{i=1}^{d} z^i S^i_t \Big)^- \right) - r^l_t y.$$

Let $\widetilde{S}^{i,b}_t := (B^b_t)^{-1} S^i_t$. *The process* $Y^b := (B^b)^{-1} V^p(x, \varphi, A, C)$ *satisfies*

$$dY^b_t = \sum_{i=1}^{d} Z^{b,i}_t \, d\widetilde{S}^{i,b}_t + G_b(t, Y^b_t, Z^b_t)\, dt + (B^b_t)^{-1}\, dA^C_t$$

where $Z^{b,i} = \xi^i$, $i = 1, 2, \ldots, d$ *and the mapping* G_b *equals*

$$G_b(t,y,z) = \sum_{i=1}^{d} r^b_t (B^b_t)^{-1} z^i S^i_t + (B^b_t)^{-1} \left(r^l_t \Big(yB^b_t - \sum_{i=1}^{d} z^i S^i_t \Big)^+ - r^b_t \Big(yB^b_t - \sum_{i=1}^{d} z^i S^i_t \Big)^- \right) - r^b_t y.$$

Definition of netted wealth

The concept of the netted wealth is the gateway to study arbitrage issues in our non-linear and asymmetric approach.

Definition

The *netted wealth* $V^{\mathrm{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C)$ is given by

$$V^{\mathrm{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C) := V(y_1, \varphi, A, C) + V(y_2, \widetilde{\varphi}, -A, -C)$$

where $x = y_1 + y_2$ and φ, $\widetilde{\varphi}$ are self-financing trading strategies.

Note that $V_0^{\mathrm{net}}(x, \varphi, A, C) = x$ for any contract (A, C) and any strategy φ.

Definition

A self-financing trading strategy $(y_1, y_2, \varphi, \widetilde{\varphi}, A, C)$ is *admissible* if the discounted netted wealth process $\widetilde{V}^{l,\mathrm{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C) := V^{\mathrm{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C)/B^l$ is bounded from below.

Arbitrage opportunity

Definition

An admissible strategy (x, φ, A, C) is an *arbitrage opportunity for the hedger* with respect to (A, C) whenever

$$\mathbb{P}(V_T^{\text{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C) \geq V_T^0(x)) = 1$$

and

$$\mathbb{P}(V_T^{\text{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C) > V_T^0(x)) > 0$$

where

$$V_t^0(x) := x^+ B_t^l - x^- B_t^b$$

for all $t \in [0, T]$. A model is *arbitrage-free* for the hedger if there is no arbitrage opportunity in regard to any contract (A, C).

Martingale measure and ex-dividend prices

Assumption

There exists a probability measure $\widetilde{\mathbb{P}}^l$ equivalent to $\mathbb{P}$ such that the processes $\widetilde{S}^{i,l}$, $i = 1, 2, \dots, d$ are $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-local martingales

Proposition

If a martingale measure $\widetilde{\mathbb{P}}^l$ exists and $x_1 \geq 0$, $x_2 \geq 0$ then the model is arbitrage-free for the hedger and for the counterparty.

Definition

Any $\mathcal{G}_t$-measurable random variable for which a replicating strategy for (A, C) over $[t, T]$ exists is called the *hedger's ex-dividend price* at time t for a contract (A, C) and it is denoted by $P_t^h(x_1, A, C)$. Hence for some self-financing strategy φ

$$V_T(V_t^0(x_1) + P_t^h(x_1, A, C), \varphi, A - A_t, C) = V_T^0(x_1).$$

Fair bilateral prices

Definition

For an arbitrary level x_2 of the counterparty's initial endowment and a strategy $\widetilde{\varphi}$ replicating $(-A, -C)$, the *counterparty's ex-dividend price* $P_t^c(x_2, -A, -C)$ at time t for a contract $(-A, -C)$ is implicitly given by the equality

$$V_T(V_t^0(x_2) - P_t^c(x_2, -A, -C), \widetilde{\varphi}, -A + A_t, -C) = V_T^0(x_2).$$

By a *fair bilateral price*, we mean the price level at which no arbitrage opportunity arises for either party. Hence the following definition.

Definition

The $\mathcal{G}_t$-measurable interval

$$\mathcal{R}_t^f(x_1, x_2) := \left[P_t^c(x_2, -A, -C), P_t^h(x_1, A, C)\right]$$

is called the *range of fair bilateral prices* at time t for the contract (A, C).

Bilaterally profitable prices

Definition

Assume that the inequality $P_t^h(x_1, A, C) \neq P_t^c(x_2, -A, -C)$ holds. Then the $\mathcal{G}_t$-measurable interval $\mathcal{R}_t^p(x_1, x_2) := \left[P_t^h(x_1, A, C), P_t^c(x_2, -A, -C)\right]$ is called the *range of bilaterally profitable prices* at time t of an OTC contract (A, C).

Three concepts of arbitrage:

- **(A.1)** The classic definition of an arbitrage opportunity that may arise by trading in primary assets, as in the classic FTAP.
- **(A.2)** An arbitrage opportunity associated with a long hedged position in some contract combined with a short hedged position in the same contract. The contract's price is considered to be exogenously given, but is arbitrary.
- **(A.3)** An arbitrage opportunity related to the fact that the hedger and the counterparty may require different premia to implement their respective (super-)replicating strategies. Here an arbitrage opportunity is simultaneously available to both parties at a *negotiated* OTC price.

Endogenous collateral

- We wish to find out whether the range of fair bilateral prices is non-empty, at least for some classes of contracts (A, C).
- In general, the process C may depend on both the hedger's value $V^h := V(x_1, \varphi, A, C)$ and the counterparty's value $V^c := V(x_2, \widetilde{\varphi}, -A, -C)$.
- It is given as follows

$$C_t = q\big(V_t^0(x_1) - V_t^h, V_t^c - V_t^0(x_2)\big) = q(-P_t^h, -P_t^c)$$

where $q : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous function with $q(0,0) = 0$.

- The *convex collateralization* is given by $q(y_1, y_2) = \alpha y_1 + (1-\alpha) y_2$ for some $\alpha \in [0,1]$, so that

$$C_t = \alpha(V_t^0(x) - V_t^h) + (1-\alpha)(V_t^c - V_t^0(x)) = -(\alpha P_t^h + (1-\alpha) P_t^c).$$

- One can also introduce the so-called *haircuts*.

Model assumptions

Assumption

We postulate that:
(i) there exists a probability measure $\widetilde{\mathbb{P}}^l$ equivalent to $\mathbb{P}$ such that $\widetilde{S}^l$ is a continuous, square-integrable, $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-martingale and has the predictable representation property with respect to the filtration $\mathbb{G}$ under $\widetilde{\mathbb{P}}^l$,
(ii) there exists an $\mathbb{R}^{d\times d}$-valued, $\mathbb{G}$-adapted process m^l such that

$$\langle \widetilde{S}^l \rangle_t = \int_0^t m^l_u (m^l_u)^* \, du$$

where the process $m^l(m^l)^*$ is invertible and satisfies $m^l(m^l)^* = \mathbb{S}\sigma\sigma^*\mathbb{S}$ where σ is a d-dimensional square matrix of $\mathbb{G}$-adapted processes satisfying the *ellipticity condition*: there exists a constant $\Lambda > 0$

$$\sum_{i,j=1}^{d} (\sigma_t \sigma_t^*)_{ij} \, a_i a_j \geq \Lambda |a|^2 = \Lambda a^* a, \quad \forall\, a \in \mathbb{R}^d, \, t \in [0, T].$$

The case of hedger's collateral

- Assume first that C depends only on the hedger's value

$$C_t = q(V_t^0(x_1) - V_t^h) = q(-P_t^h)$$

for some Lipschitz continuous function $q : \mathbb{R} \to \mathbb{R}$ such that $q(0) = 0$.

- The price P^c solves the BSDE, which depends on the solution P^h. Hence the pricing/hedging BSDEs for the pair (P^h, P^c) are partially coupled.

Proposition

If $x_1 \geq 0$, $x_2 \geq 0$, then for any contract (A, C) we have for every $t \in [0, T]$

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}$$

so that the range of fair bilateral prices $\mathcal{R}_t^f(x_1, x_2)$ is non-empty, $\widetilde{\mathbb{P}}^l - \text{a.s.}$

The range may be empty, in general, if the initial endowments have opposite signs, that is, when $x_1 > 0$ and $x_2 < 0$.

Partially coupled pricing BSDEs

Proposition

Let $x_1 \geq 0$ and $x_2 \geq 0$. The hedger's price equals $P^h := P^h(x_1, A, C) = Y^1$ where (Y^1, Z^1) is the unique solution to the BSDE

$$\begin{cases} dY_t^1 = Z_t^{1,*}\, d\widetilde{S}_t^l + f_l\big(t, x_1, Y_t^1, Z_t^1\big)\, dt + dA_t, \\ Y_T^1 = 0, \end{cases}$$

where

$$\begin{aligned} f_l(t, x_1, y, z) &= r_t^l (B_t^l)^{-1} z^* S_t - (B_t^l)^{-1} \sum_{i=1}^{d} r_t^{i,b} (z^i S_t^i)^+ - x_1 B_t^l r_t^l - r_t^c q(-y) \\ &\quad + r_t^l \Big(y + q(-y) + x_1 B_t^l + (B_t^l)^{-1} \sum_{i=1}^{d} (z^i S_t^i)^- \Big)^+ \\ &\quad - r_t^b \Big(y + q(-y) + x_1 B_t^l + (B_t^l)^{-1} \sum_{i=1}^{d} (z^i S_t^i)^- \Big)^- . \end{aligned}$$

Partially coupled pricing BSDEs

Proposition

The counterparty's price equals $P^c := P^c(x_2, -A, -C) = Y^2$ *where* (Y^2, Z^2) *is the unique solution to the BSDE*

$$\begin{cases} dY_t^2 = Z_t^{2,*}\, d\widetilde{S}_t^l + g_l\big(t, x_2, Y_t^2, Z_t^2, Y_t^1\big)\, dt + dA_t, \\ Y_T^2 = 0, \end{cases}$$

where

$$\begin{aligned} g_l(t, x_2, y, z, Y_t^1) = {} & r_t^l (B_t^l)^{-1} z^* S_t + (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (-z^i S_t^i)^+ + x_2 B_t^l r_t^l - r_t^c q(-Y_t^1) \\ & - r_t^l \Big(-y - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (-z^i S_t^i)^- \Big)^+ \\ & + r_t^b \Big(-y - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (-z^i S_t^i)^- \Big)^- . \end{aligned}$$

Fully coupled pricing BSDEs

- We now consider the case where

$$C_t = q\big(V_t^0(x_1) - V_t^h, V_t^c - V_t^0(x_2)\big) = q(-P_t^h, -P_t^c).$$

- Then the BSDEs for the hedger's and counterparty's prices are fully coupled.

Proposition

Assume that $x_1 \geq 0$ and $x_2 \geq 0$. Then the hedger's and counterparty's prices satisfy $(P^h, P^c)^* = (Y^1, Y^2) = Y$ where (Y, Z) solves the following two-dimensional, fully-coupled BSDE

$$\begin{cases} dY_t = Z_t^* \, d\widetilde{S}_t^l + g\big(t, Y_t, Z_t\big)\, dt + d\overline{A}_t, \\ Y_T = 0, \end{cases}$$

where $g = (g^1, g^2)^*$, $\overline{A} = (A, A)^*$ and ...

Fully coupled pricing BSDEs

Proposition

for all $y = (y_1, y_2)^* \in \mathbb{R}^2$ *and* $z = (z_1, z_2) \in \mathbb{R}^{d \times 2}$,

$$\begin{aligned} g^1(t,y,z) &= r_t^l (B_t^l)^{-1} z_1^* S_t - x_1 B_t^l r_t^l - r_t^c q(-y_1, y_2) \\ &\quad + r_t^l \Big(y_1 + q(-y_1,-y_2) + x_1 B_t^l - (B_t^l)^{-1} z_1^* S_t \Big)^+ \\ &\quad - r_t^b \Big(y_1 + q(-y_1,-y_2) + x_1 B_t^l - (B_t^l)^{-1} z_1^* S_t \Big)^- \end{aligned}$$

and

$$\begin{aligned} g^2(t,y,z) &= r_t^l (B_t^l)^{-1} z_2^* S_t + x_2 B_t^l r_t^l - r_t^c q(-y_1, y_2) \\ &\quad - r_t^l \Big(-y_2 - q(-y_1,-y_2) + x_2 B_t^l + (B_t^l)^{-1} z_2^* S_t \Big)^+ \\ &\quad + r_t^b \Big(-y_2 - q(-y_1,-y_2) + x_2 B_t^l + (B_t^l)^{-1} z_2^* S_t \Big)^- . \end{aligned}$$

Backward stochastic viability property (BSVP)

- Fix $T > 0$ and consider the n-dimensional BSDE

$$Y_t = \eta + \int_t^T h(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s.$$

- The following definition was introduced by Buckdahn, Quincampoix and Rascanu (2000) for a non-empty, closed, convex set of $K \subset \mathbb{R}^n$.

Definition

We say that BSDE has the *backward stochastic viability property* (BSVP) in K if: for any $U \in [0, T]$ and any square-integrable $\eta \in K$ the unique solution (Y, Z) to

$$Y_t = \eta + \int_t^U h(s, Y_s, Z_s)\, ds - \int_t^U Z_s\, dW_s$$

satisfies $Y_t \in K$ for all $t \in [0, U]$, $\mathbb{P}$-a.s.

Multi-dimensional viability theorem

- Let $\Pi_K(y)$ be the projection of a point $y \in \mathbb{R}^n$ onto K.
- Let $d_K(y)$ be the distance between y and K.
- The following result is due to Buckdahn, Quincampoix and Rascanu (2000).

Theorem

Let the generator h of BSDE satisfy the Lipschitz condition and some additional assumptions. Then BSDE has the BSVP in K if and only if for any $t \in [0,T]$, $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ such that $d_K^2(\cdot)$ is twice differentiable at y we have

$$4\langle y - \Pi_K(y), h(t, \Pi_K(y), z)\rangle \leq \langle D^2 d_K^2(y) z, z\rangle + M d_K^2(y)$$

where $M > 0$ is a constant independent of (t, y, z).

Comparison theorem for two-dimensional BSDE

Theorem

Consider the two-dimensional BSDE

$$Y_t = \eta + \int_t^T h(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s.$$

The following statements are equivalent:

(i) for any $U \in [0, T]$ *and* $\eta^1, \eta^2 \in L^2(\Omega, \mathcal{F}_U, \mathbb{P})$ *such that* $\eta^1 \geq \eta^2$*, the unique solution* (Y, Z) *to the BSDE on* $[0, U]$ *satisfies* $Y_t^1 \geq Y_t^2$ *for all* $t \in [0, U]$*,*

(ii) there exists a constant M *such that for all* $y, z \in \mathbb{R}^2$

$$\begin{aligned} &-4y_1^-[h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)] \\ &\leq M|y_1^-|^2 + 2|z_1|^2 \mathbb{1}_{\{y_1 < 0\}}. \end{aligned}$$

Diffusion-type market model

- The risky asset S is governed by the SDE

$$dS_t = \mu(t, S_t)\, dt + \sigma(t, S_t)\, dW_t$$

where W is a one-dimensional Brownian motion.
- The filtration $\mathbb{G}$ is assumed to be generated by the Brownian motion W.
- The coefficients μ and σ are such that the SDE has a unique strong solution.
- The dividend process equals $A^1_t = \int_0^t \kappa(u, S_u)\, du$.
- We denote

$$a_t := (\sigma(t, S_t))^{-1} \big(\mu(t, S_t) + \kappa(t, S_t) - r^l_t S_t\big).$$

Assumption

We postulate that the processes a, $(\sigma(\cdot, S))^{-1}$ and all interest rates are continuous and the processes a and $(\sigma(\cdot, S))^{-1} S$ are bounded.

Fair prices of European claims

- For a European claim, we have

$$A_t - A_0 = H_T \mathbb{1}_{[T,T]}(t).$$

- Using the comparison theorem for a fully-coupled two-dimensional BSDE, we obtain the following result.

Proposition

Let $x_1 \geq 0$, $x_2 \geq 0$. For any European claim (H_T, C) where $H_T \in L^2(\Omega, \mathcal{F}_T, \widetilde{\mathbb{P}}^l)$ we have for every $t \in [0, T]$

$$P^c_t(x_2, -H_T, -C) \leq P^h_t(x_1, H_T, C), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}$$

so that the range of fair bilateral prices $\mathcal{R}^f_t(x_1, x_2)$ is non-empty.

A similar result holds for any contract (A, C) when $H_{t_i} \in L^2(\Omega, \mathcal{F}_{t_i}, \widetilde{\mathbb{P}}^l)$ and

$$A_t - A_0 = \sum_{i=1}^{l} H_{t_i} \mathbb{1}_{[t_i,T]}(t).$$

Model with a partial netting

- Let $B^{i,b}$ be the *borrowing funding account* for the ith risky asset.
- We consider a trading strategy

$$\varphi = (\xi^1, \dots, \xi^d, \varphi^l, \varphi^b, \varphi^{1,b}, \varphi^{2,b}, \dots, \varphi^{d,b}, \eta).$$

- The hedger's trading strategy (x, φ, A, C) is *self-financing* whenever the process $V^p(x, \varphi, A, C)$, which is given by

$$V_t^p(x, \varphi, A, C) = \psi_t^l B_t^l + \psi_t^b B_t^b + \sum_{i=1}^{d} \big(\xi_t^i S_t^i + \psi_t^{i,b} B_t^{i,b}\big),$$

satisfies $\psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+$ and

$$V_t^p(x, \varphi, A, C) = x + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \, d(S_u^i + A_u^i) + \sum_{j=1}^{d} \int_0^t \psi_u^{i,b} \, dB_u^{i,b} + \int_0^t \psi_u^l \, dB_u^l$$
$$+ \int_0^t \psi_u^b \, dB_u^b + \int_0^t \eta_u \, dB_u^c + A_t + C_t.$$

Model with a partial netting

- Let

$$A_t^C = A_t + C_t - \int_0^t C_u (B_u^c)^{-1} \, dB_u^c.$$

- For a self-financing trading strategy φ the processes $Z^{l,i} = \xi^i$, $i = 1, 2, \dots, d$ and $Y^l := (B^l)^{-1} V^p(x, \varphi, A, C)$ satisfy

$$dY_t^l = \sum_{i=1}^d Z_t^{l,i} \, d\widetilde{S}_t^{i,l,\text{cld}} + G_l(t, Y_t^l, Z_t^l) \, dt + dA_t^{C,l}$$

where the generator G_l equals, for all $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$G_l(t, y, z) = (B_t^l)^{-1} \textstyle\sum_{i=1}^d r_t^l z^i S_t^i - (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ - r_t^l y$$
$$+ (B_t^l)^{-1} \Big(r_t^l \Big(y B_t^l + \textstyle\sum_{i=1}^d (z^i S_t^i)^- \Big)^+ - r_t^b \Big(y B_t^l + \sum_{i=1}^d (z^i S_t^i)^- \Big)^- \Big).$$

Case of a hedger's collateral

Proposition

If $x_1 \geq 0$, $x_2 \geq 0$ *then for any contract* (A, C) *and every* $t \in [0, T]$

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C).$$

Proposition

Assume that:

(i) the process $A - A_0$ *is decreasing,*

(ii) $C_t = q(V_t^0(x_1) - V_t^h)$ *where* q *satisfies* $y + q(-y) \geq 0$ *for all* $y \geq 0$.

If $x_1 \geq 0$, $x_2 \leq 0$ *then the inequality* $P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C)$ *holds for every* $t \in [0, T]$. *Moreover, the price* $P_t^h(x_1, A, C)$ *is independent of* $x_1 \geq 0$.

- Condition (ii) holds, for instance, when $q(y) = (1 + \alpha_1)y^+ - (1 + \alpha_2)y^-$ for some haircut processes α_1, α_2 such that $\alpha_2 \leq 0$.

Model with an uncertain money market rate

- Let a $\mathbb{G}$-adapted interest rate process r satisfy

$$r_t \in [r^l_t, r^b_t] \quad \text{for every } t \in [0, T].$$

- Consider the market model with the single money market rate r.
- Then the hedger and the counterparty have the same price P^r independent of their initial endowments.

Proposition

(i) For any contract (A, C), the unique no-arbitrage price in the market model with the money market rate r satisfies $P^r_t \leq P^h_t(0, A, C)$ for all $t \in [0, T]$.
(ii) If the function q satisfies for all $y_1 \geq y_2$ and $t \in [0, T]$

$$(r_t - r^c_t)(q(y_1) - q(y_2)) \leq 0$$

then also $P^c_t(0, -A, -C) \leq P^r_t$ and thus $P^r_t \in [P^c_t(0, -A, -C), P^h_t(0, A, C)]$.

Concluding remarks

- Note that only contracts of European style were covered by these lectures.
- American and game options are even more challenging – please refer to the lectures by Professor Agnés Sulem.
- The counterparty risk may also be included in the present framework, but new existence and comparison theorems for BSDEs are required to deal with jumps at default.
- For a BSDE approach to mean-variance hedging of CVA, see papers by Crépey (2015) and the monograph by Crépey and Bielecki (2014).
- An interesting concept of *partial replication* (aka warehousing of risk) was introduced by Burgard and Kjaer (2013).

Thank you!