Turbulence, Scientific Computing, and Visualization

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Turbulence

> *Big whirls have little whirls that feed on their velocity, and little whirls have littler whirls and so on to viscosity...*

- In 1941, Kolmogorov conjectured that the energy spectrum of 3D turbulence exhibits a self-similar power-law scaling characterized by a uniform *cascade* of energy to molecular (viscous) scales:

$$E(k) = C\epsilon^{2/3}k^{-5/3}.$$

- Here k the Fourier wavenumber.
- Kolmogorov suggested that C is a universal constant.

2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega \doteq \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{u}$ of an incompressible ($\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$) fluid:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega = \nu\nabla^2\omega + f.$$

- In Fourier space:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}} + f_{\boldsymbol{k}},$$

where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ and $\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{q})\,\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$ is antisymmetric under permutation of any two indices.

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q} \, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}} + f_{\boldsymbol{k}},$$

- When $\nu = f_{\boldsymbol{k}} = 0$,

  enstrophy $Z = \frac{1}{2} \int |\omega_{\boldsymbol{k}}|^2 \, d\boldsymbol{k}$ and energy $E = \frac{1}{2} \int \frac{|\omega_{\boldsymbol{k}}|^2}{k^2} d\boldsymbol{k}$ are conserved:

$$\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{p},$$

$$\frac{1}{k^2} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{q}.$$

Casimir Invariants

- Inviscid unforced two dimensional turbulence has uncountably many other Casimir invariants.

- Any continuously differentiable function of the (scalar) vorticity is conserved by the nonlinearity:

$$\frac{d}{dt}\int f(\omega)\,d\boldsymbol{x} = \int f'(\omega)\frac{\partial\omega}{\partial t}\,d\boldsymbol{x} = -\int f'(\omega)\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega\,d\boldsymbol{x}$$
$$= -\int \boldsymbol{u}\cdot\boldsymbol{\nabla} f(\omega)\,d\boldsymbol{x} = \int f(\omega)\boldsymbol{\nabla}\cdot\boldsymbol{u}\,d\boldsymbol{x} = 0.$$

- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit cascades?

Fast Variably Restricted Dealiased Convolutions.

- Develop practical algorithm for computing many *partial* Fourier transforms at once:

$$u_{\boldsymbol{j}} \doteq \sum_{|\boldsymbol{k}|<c(\boldsymbol{j})} \zeta_N^{\boldsymbol{k}\cdot\boldsymbol{j}} U_{\boldsymbol{k}}$$

  where $\zeta_N = e^{2\pi i/N}$ is the Nth primitive root of unity.

- Here $c(\boldsymbol{j})$ is a spatially-dependent constraint on the summation limits.

- The fast Fourier transform (FFT) method exploits the properties that $\zeta_N^r = \zeta_{N/r}$ and $\zeta_N^N = 1$.

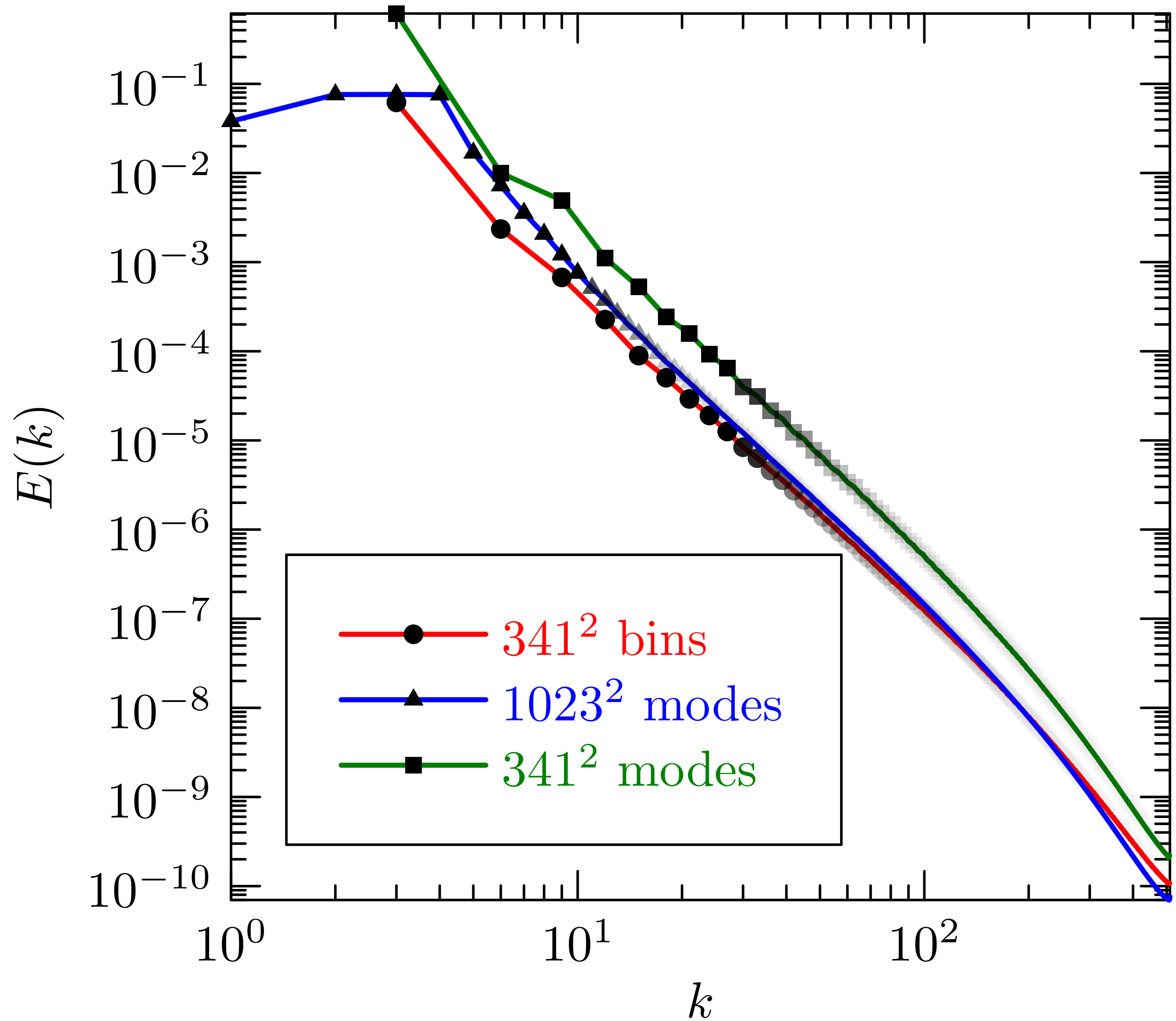
- Goal: obtain a 'fast' computational scaling, following Ying & Fomel [2009] but with a smaller overall coefficient.

Numerical Study of Kolmogorov Self-Similarity

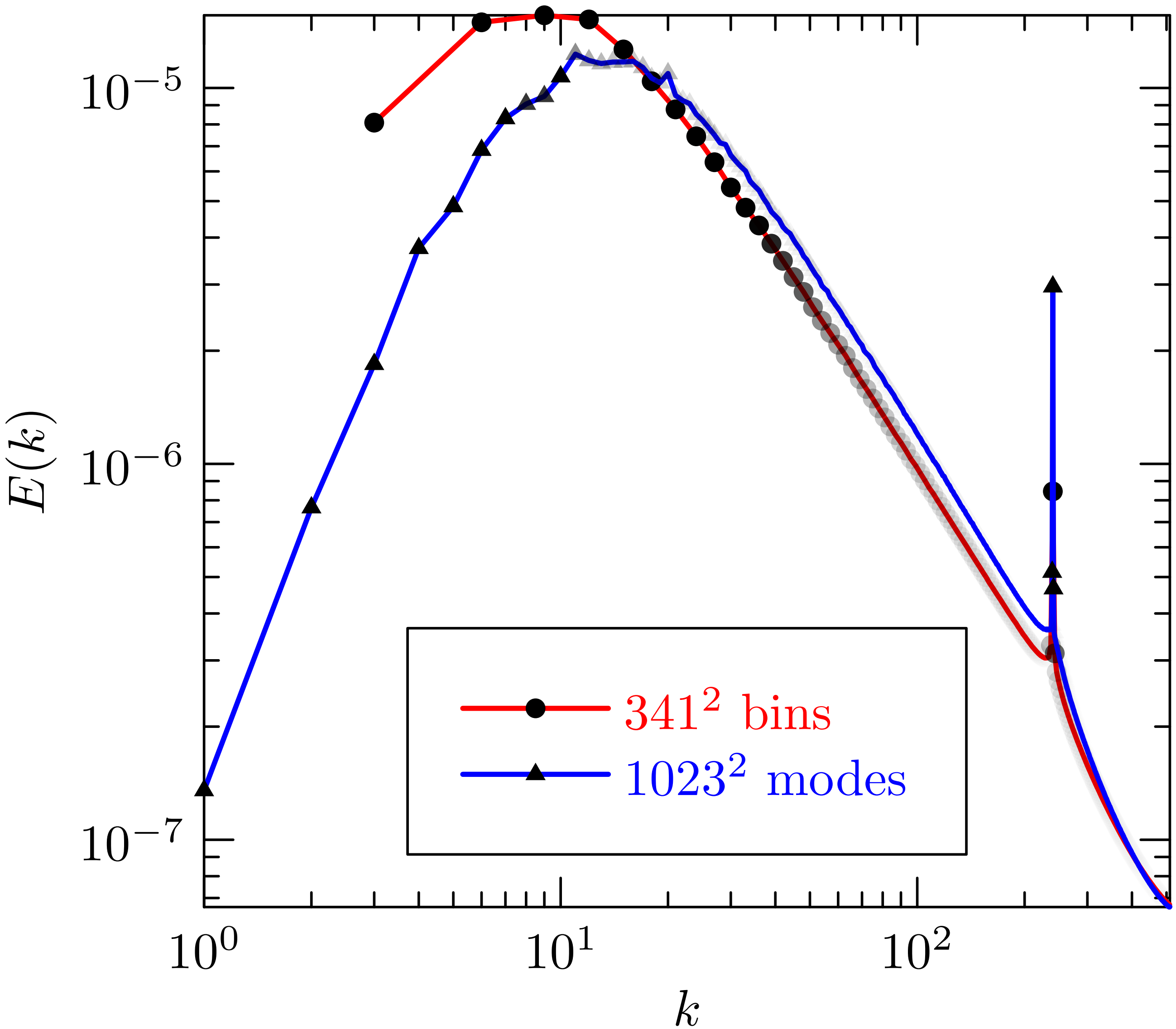
- Variably restricted dealiased convolutions can be used to compute detailed inertial-range flux profiles and for the first time verify a key underpinning assumption of Kolmogorov's famous power-law conjecture for turbulence.

Pseudospectral Reduction

- Spectral reduction is a technique for dramatically reducing the number of Fourier modes that must be retained in simulations of turbulent phenomena [Bowman *et al.* 1999].
- It can be used to develop a reliable dynamic subgrid model for large eddy simulations.
- Malcolm Roberts' Ph.D. thesis (2011) explores ways of doing this.
- In 2D, an efficient pseudospectral (FFT-based) formulation of spectral reduction was recently developed [Bowman & Roberts 2012].
- Spectral reduction has already been formulated in 3D; there even exists a two-field formulation that conserves both energy and helicity.
- It should be straightforward to develop pseudospectral versions of these reduced 3D models.



Direct cascade.



Inverse cascade.

Multispectral Reduction

- Variably restricted convolutions can be used to join multiple uniform spectrally reduced grids interacting via Kolmogorov self-similarity.

- Analogous to the geometric multigrid method for elliptic equations, except the equations are hyperbolic and the refinement is done in Fourier space.

High-Order Adaptive Exponential Integrators

- Following the excellent Ph.D. thesis of Berland [2006], Lie group techniques can be used to develop an efficient embedded Runge-Kutta $(5,4)$ exponential pair to improve on the low-order $(3,2)$ exponential adaptive integrator previously developed for turbulent shell models [Bowman *et al.* 2006].

Implicitly Dealiased Convolutions

- Develop useful implicitly dealiased convolutions not yet included in our convolution library `FFTW++`.

  `http://fftwpp.sourceforge.net`

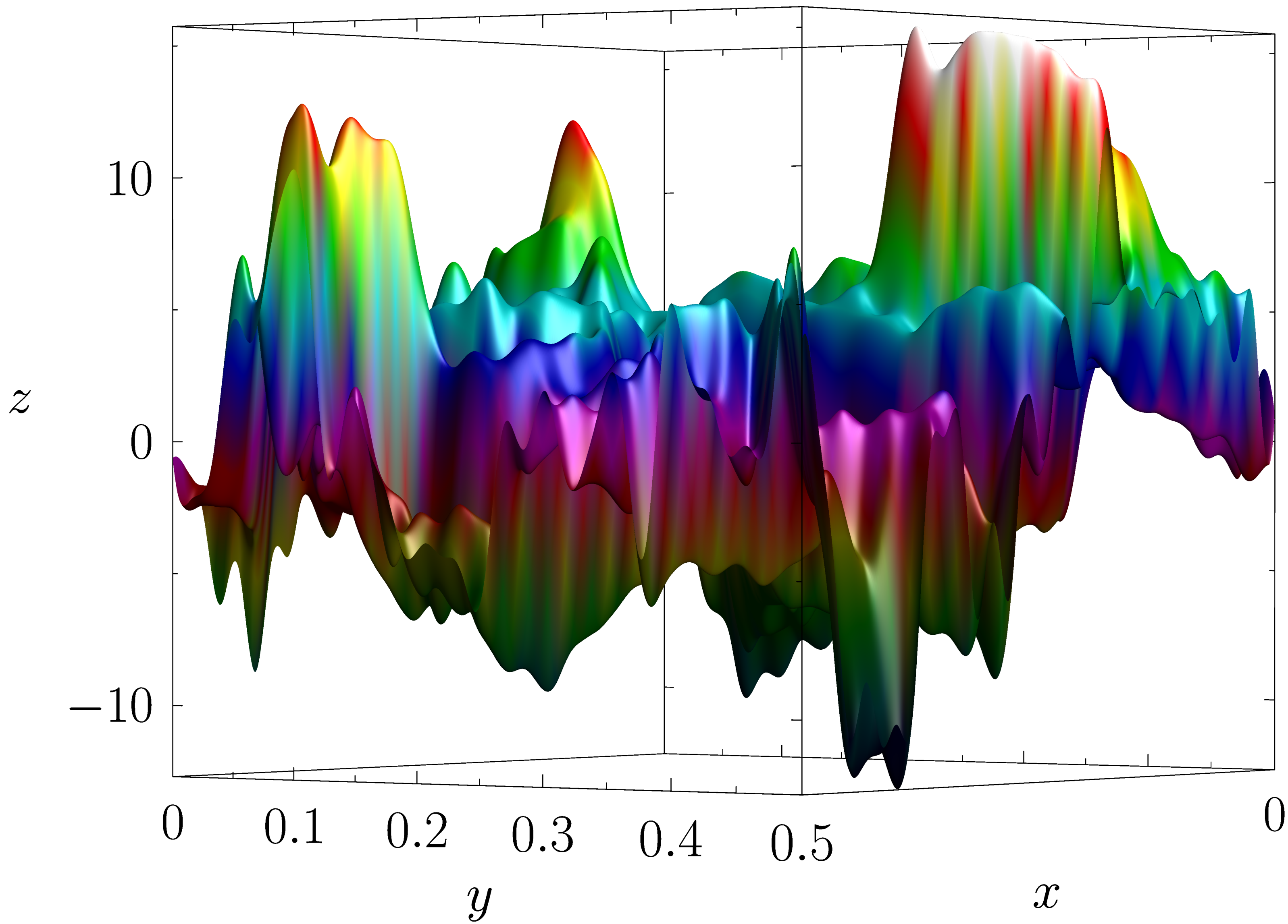
- The multithreaded algorithms also need to be parallelized on distributed memory architectures, minimizing communication costs.

Clement W. Bowman Mathematical Turbulence Scholarship

- Value: \$2,000

- Eligibility: Awarded annually to an incoming or continuing student registered full-time in a graduate degree or post-doctoral program in the Department of Mathematical and Statistical Sciences who is studying the mathematical analysis of turbulence. The recipient will be chosen on the basis of academic performance (minimum grade point average of 3.5) and/or publication record. If no suitable candidate is found at the time of selection, the scholarship will not be awarded in that year.

Vorticity Surface Plot



Asymptote Lifts TeX to 3D

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$$

http://asymptote.sf.net

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