Pseudospectral Reduction of Incompressible Two-Dimensional Turbulence

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2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega \doteq \hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\times\boldsymbol{u}$:

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega = \nu\nabla^2\omega + f.$$

- In Fourier space:

$$\frac{\partial\omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \int d\boldsymbol{p}\int d\boldsymbol{q}\,\frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega_{\boldsymbol{p}}^*\omega_{\boldsymbol{q}}^* + f_{\boldsymbol{k}},$$

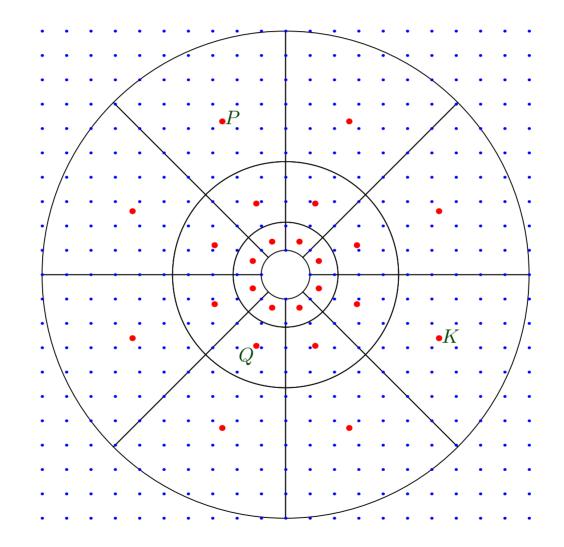where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ and $\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{p}\times\boldsymbol{q})\,\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$ is antisymmetric under permutation of any two indices.

- When $\nu = f_{\boldsymbol{k}} = 0$,

enstrophy $Z = \frac{1}{2} \sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ and energy $E = \frac{1}{2} \sum_{\boldsymbol{k}} \dfrac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ are conserved:

$$\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \qquad \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{p},$$

$$\frac{1}{k^2} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \qquad \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{q}.$$

Spectral Reduction

- Introduce a coarse-grained grid indexed by K:

Wavenumber Bin Geometry (8×3 bins)

- Define new variables

$$\Omega_{\boldsymbol{K}} = \langle \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{\Delta_{\boldsymbol{K}}} \int_{\Delta_{\boldsymbol{K}}} \omega_{\boldsymbol{k}} \, d\boldsymbol{k},$$

where $\Delta_{\boldsymbol{K}}$ is the area of bin $\boldsymbol{K}$.

- Evolution of $\Omega_{\boldsymbol{K}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* \right\rangle_{\boldsymbol{KPQ}},$$

where $\langle f \rangle_{\boldsymbol{KPQ}} = \dfrac{1}{\Delta_{\boldsymbol{K}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}}} \displaystyle\int_{\Delta_{\boldsymbol{K}}} d\boldsymbol{k} \int_{\Delta_{\boldsymbol{P}}} d\boldsymbol{p} \int_{\Delta_{\boldsymbol{Q}}} d\boldsymbol{q} \, f.$

- Approximate $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ by bin-averaged values $\Omega_{\boldsymbol{P}}$ and $\Omega_{\boldsymbol{Q}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*.$$

- Define the coarse-grained enstrophy Z and energy E:

$$E \doteq \frac{1}{2} \sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2} \Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2} \sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}.$$

- Enstrophy is still conserved by the nonlinearity since

$$\left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{P}.$$

- But energy conservation has been lost!

$$\frac{1}{K^2} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{NOT antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{Q}.$$

- Reinstate both desired symmetries with the modified coefficient

$$\frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2}.$$

Properties

- We call the forced-dissipative version of this approximation *spectral reduction* (SR):

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \, \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \, \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2} \, \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*.$$

- SR conserves both energy and enstrophy and reduces to the exact dynamics in the limit of small bin size.

- It has the same general structure and symmetries as the original equation and in this sense may be considered a *renormalization*.

- SR obeys a Liouville Theorem; in the inviscid limit, it yields statistical-mechanical (equipartition) solutions.

Moments

- Q. How accurate is spectral reduction?

- A. For large bins, the *instantaneous* dynamics of SR is inaccurate.

- However: the equations for the *time-averaged* (or ensemble-averaged) moments predicted by SR closely approximate those of the exact bin-averaged statistics.

- E.g., time average the exact bin-averaged enstrophy equation:

$$\overline{\frac{\partial}{\partial t}\left\langle |\omega_{\boldsymbol{k}}|^2 \right\rangle_{\boldsymbol{K}}} + 2\,\text{Re}\left\langle \nu_{\boldsymbol{k}}\overline{|\omega_{\boldsymbol{k}}|^2} \right\rangle_{\boldsymbol{K}} = 2\,\text{Re}\sum_{\boldsymbol{P},\boldsymbol{Q}}\Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\overline{\omega_{\boldsymbol{k}}^*\omega_{\boldsymbol{p}}^*\omega_{\boldsymbol{q}}^*} \right\rangle_{\boldsymbol{KPQ}},$$

  where the bar means time average and $\langle\cdot\rangle_{\boldsymbol{K}}$ means bin average.

- Time-averaged quantities such as $\overline{|\omega_{\boldsymbol{k}}|^2}$ and $\overline{\omega_{\boldsymbol{k}}^*\omega_{\boldsymbol{p}}^*\omega_{\boldsymbol{q}}^*}$ are generally *smooth* functions of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ on the four-dimensional surface defined by the triad condition $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$.

- Mean Value Theorem for integrals: for some $\boldsymbol{\xi} \in \boldsymbol{K}$.

$$\overline{|\Omega_{\boldsymbol{K}}|^2} = \overline{|\omega_{\boldsymbol{\xi}}|^2} \approx \overline{|\omega_{\boldsymbol{k}}|^2} \qquad \forall \boldsymbol{k} \in \boldsymbol{K}.$$

- To good accuracy these statistical moments may therefore be evaluated at the characteristic wavenumbers $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$:

$$\overline{\frac{\partial}{\partial t} |\Omega_{\boldsymbol{K}}|^2} + 2 \operatorname{Re} \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \overline{|\Omega_{\boldsymbol{K}}|^2} = 2 \operatorname{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{kpq}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \overline{\Omega_{\boldsymbol{K}}^* \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*}.$$

- To the extent that the wavenumber magnitude q varies slowly over a bin:

$$\overline{\frac{\partial}{\partial t} |\Omega_{\boldsymbol{K}}|^2} + 2 \operatorname{Re} \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \overline{|\Omega_{\boldsymbol{K}}|^2} = 2 \operatorname{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{kpq} \rangle_{\boldsymbol{KPQ}}}{Q^2} \overline{\Omega_{\boldsymbol{K}}^* \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*}.$$

- But this is precisely the time-average of the SR equation!

Convergence

- The previous argument suggests that spectral reduction can indeed provide an accurate statistical description of turbulence, even when each bin contains many statistically independent modes.

- As the wavenumber partition is refined, one expects the solutions of the time-averaged SR equations to converge to the exact statistical solution.

- However, since spectral reduction smooths over the $\delta_{\bm{k}+\bm{p}+\bm{q},0}$ factor appearing in the nonlinear coefficient $\epsilon_{\bm{kpq}}$, the resulting equation is no longer a convolution. Thus, pseudospectral collocation is not directly applicable.

Noncanonical Hamiltonian Formulation

- Underlying *noncanonical* Hamiltonian formulation for inviscid 2D vorticity equation:

$$\dot{\omega}_{\boldsymbol{k}} = \int d\boldsymbol{q}\, J_{\boldsymbol{kq}}\frac{\delta H}{\delta \omega_{\boldsymbol{q}}},$$

where

$$H \doteq \frac{1}{2}\int d\boldsymbol{k}\,\frac{|\omega_{\boldsymbol{k}}|^2}{k^2},$$

$$J_{\boldsymbol{kq}} \doteq \int d\boldsymbol{p}\,\epsilon_{\boldsymbol{kpq}}\omega_{\boldsymbol{p}}^*.$$

- Leads to inviscid Navier–Stokes equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \int d\boldsymbol{p}\int d\boldsymbol{q}\,\frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega_{\boldsymbol{p}}^*\omega_{\boldsymbol{q}}^*.$$

Liouville Theorem

- Navier–Stokes:

$$J_{kq} \doteq \int dp\, \epsilon_{kpq}\omega_p^*$$

$$\Rightarrow \quad \int dk\, \frac{\delta\dot{\omega}_k}{\delta\omega_k} = \int dk \int dq\, \underbrace{\frac{\delta J_{kq}}{\delta\omega_k}}_{\epsilon_{k(-k)q}=0}\frac{\delta H}{\delta\omega_q} + J_{kq}\frac{\delta^2 H}{\delta\omega_k\delta\omega_q} = 0.$$

- Spectral Reduction:

$$J_{KQ} \doteq \sum_P \Delta_P \left\langle\epsilon_{kpq}\right\rangle_{KPQ}\Omega_P^*$$

$$\Rightarrow \quad \sum_K \frac{\partial\dot{\Omega}_K}{\partial\Omega_K} = \sum_{K,Q} \underbrace{\frac{\partial J_{KQ}}{\partial\Omega_K}}_{\left\langle\epsilon_{kpq}\right\rangle_{K(-K)Q}=0}\frac{\partial H}{\partial\Omega_Q} + J_{KQ}\frac{\partial^2 H}{\partial\Omega_K\partial\Omega_Q} = 0.$$

Statistical Equipartition

- For *mixing* dynamics, the Liouville Theorem and the coarse-grained invariants

$$E \doteq \frac{1}{2} \sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2} \Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2} \sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \, \Delta_{\boldsymbol{K}},$$
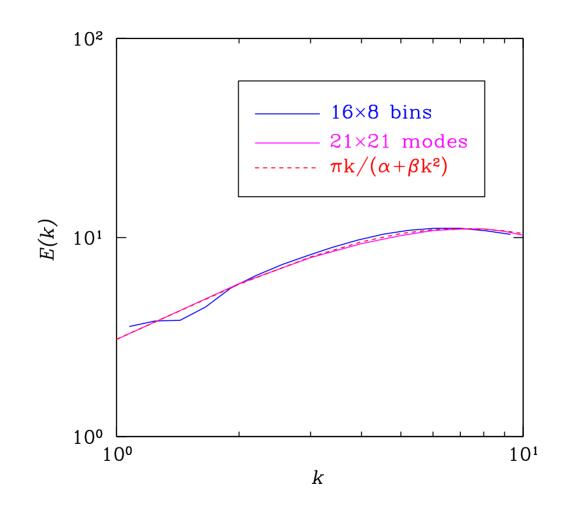
  lead to statistical equipartition of $(\alpha/K^2 + \beta) |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}$.

- This is the correct equipartition only for uniform bins.

- However, for nonuniform bins, a rescaling of time by $\Delta_{\boldsymbol{K}}$,

$$\boxed{\frac{1}{\Delta_{\boldsymbol{K}}} \frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*,}$$

  yields the correct inviscid equipartition: $\left\langle |\Omega_{\boldsymbol{K}}|^2 \right\rangle = \left(\frac{\alpha}{K^2} + \beta \right)^{-1}$.

- Unfortunately, the rescaled spectral reduction equations are hopelessly stiff [Bowman *et al.* 2001].

Relaxation of rescaled spectral reduction to equipartition.

Spectral Reduction on a Lattice

- Reluctantly, we accept the fact that each bin must contain the same number of modes.

- Imposing uniform bins has an important advantage: it affords a pseudospectral implementation of spectral reduction!

- Consider spectral reduction on a coarse-grained lattice, with $r \times r$ modes per rectangular bin.

- The bin-averaging operations become:

$$\langle f_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{r^2} \sum_{\boldsymbol{k} \in \boldsymbol{K}} f_{\boldsymbol{k}},$$

$$\langle f_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \doteq \frac{1}{r^6} \sum_{\boldsymbol{k} \in \boldsymbol{K}} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \sum_{\boldsymbol{q} \in \boldsymbol{Q}} f_{\boldsymbol{kpq}}.$$

- Uniform discrete spectral reduction:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = r^4 \sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^* + F_{\boldsymbol{K}} \xi(t).$$

- Let $\xi(t)$ be a unit Gaussian stochastic white-noise process and choose $F_{\boldsymbol{K}} = 2\epsilon_Z \dfrac{f_K}{\sqrt{\sum\limits_{\boldsymbol{K}} |f_K|^2}}$ to inject on average ϵ_Z units of enstrophy Novikov [1964].

Discrete Fast Fourier Transform

- Define the *N th primitive root of unity:*

$$\zeta_N = \exp\left(\frac{2\pi i}{N}\right).$$

- The fast Fourier transform (FFT) method exploits the properties that $\zeta_N^r = \zeta_{N/r}$ and $\zeta_N^N = 1$.

FFT of a Piecewise Constant Function

- Suppose $N = rM$ and $f_{rK+\ell} = F_K$ for $\ell = 0, 1, \ldots r - 1$ and $K = 0, 1, \ldots, M - 1$.

- For $J = 0, \ldots, M - 1$ and $s = 0, \ldots, r - 1$ the *backwards Fourier transform of the coarse-grained data* F_K is given by

$$\hat{f}_{sM+J} = \sum_{K=0}^{M-1} \sum_{\ell=0}^{r-1} \zeta_N^{(sM+J)(rK+\ell)} F_K = S_{J,s} \hat{F}_J,$$

where

$$S_{J,s} \doteq \sum_{\ell=0}^{r-1} \zeta_N^{J\ell} \zeta_r^{s\ell},$$

$$\hat{F}_J \doteq \sum_{K=0}^{M-1} \zeta_M^{JK} F_K.$$

- The *coarse-grained forwards Fourier transform* is given by:

$$F_K \doteq \frac{1}{Nr} \sum_{\ell=0}^{r-1} f_{rK+\ell} = \frac{1}{r^2 M} \sum_{\ell=0}^{r-1} \sum_{J=0}^{M-1} \sum_{s=0}^{r-1} \zeta_N^{-(rK+\ell)(sM+J)} \hat{f}_{sM+J}$$

$$= \frac{1}{r^2 M} \sum_{J=0}^{M-1} \zeta_M^{-KJ} \sum_{s=0}^{r-1} S_{J,s}^* \hat{f}_{sM+J}.$$

1D Coarse-Grained Convolution

- The coarse-grained convolution $\langle f * g \rangle_K$ of f and g can then be computed as

$$\langle f * g \rangle_K \doteq \frac{1}{r} \sum_{\ell=0}^{r-1} (f * g)_{rK+\ell} = \frac{1}{r^2 M} \sum_{J=0}^{M-1} \zeta_M^{-KJ} \sum_{s=0}^{r-1} S_{J,s}^* \hat{f}_{sM+J} \hat{g}_{sM+J}$$

$$= \frac{1}{r^2 M} \sum_{J=0}^{M-1} \zeta_M^{-KJ} W_J \hat{F}_J \hat{G}_J,$$

in terms of the spatial weight factors $W_J \doteq \sum_{s=0}^{r-1} |S_{J,s}|^2 S_{J,s}$.

- Similarly, the bin-averaged Fourier transform of F_K weighted by ℓ is given by

$$\hat{f}_{sM+J} = \sum_{K=0}^{M-1}\sum_{\ell=0}^{r-1} \zeta_N^{(sM+J)(rK+\ell)} \ell F_K = T_{J,s}\hat{F}_J,$$

where

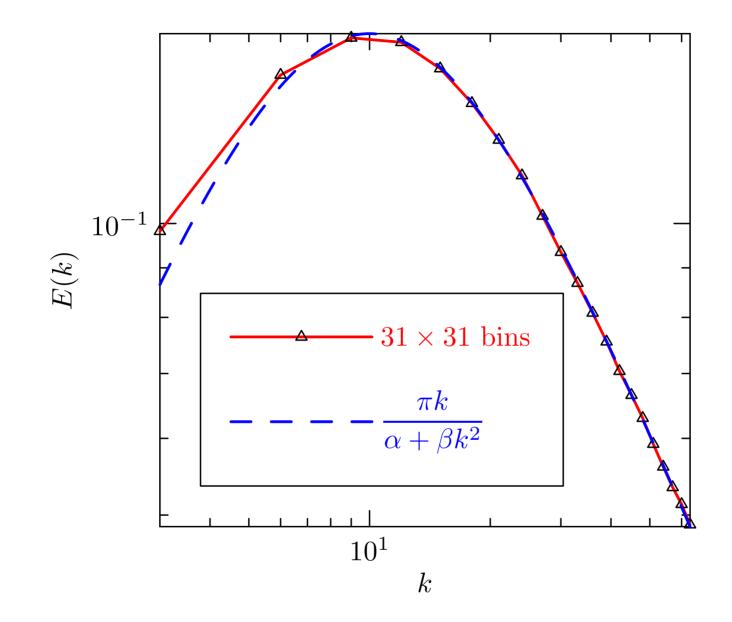$$T_{J,s} \doteq \sum_{\ell=0}^{r-1} \ell \zeta_N^{J\ell} \zeta_r^{s\ell}.$$

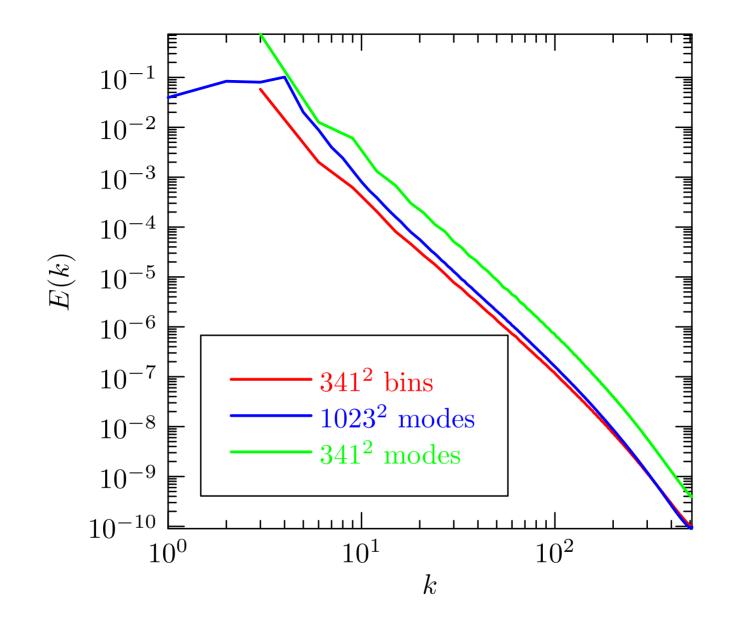- Let $W'_J \doteq \sum_{s=0}^{r-1} |S_{J,s}|^2 \, T_{J,s}.$

Pseudospectral reduction

- In terms of $F^0 \doteq K_x \Omega_{\boldsymbol{K}}$, $F^1 \doteq K_y \Omega_{\boldsymbol{K}}$, $F^2 \doteq \Omega_{\boldsymbol{K}}$, $G^0 \doteq K_x K^{-2} \Omega_{\boldsymbol{K}}$, $G^1 \doteq K_y K^{-2} \Omega_{\boldsymbol{K}}$, and $G^2 \doteq K^{-2} \Omega_{\boldsymbol{K}}$:

$$
\begin{aligned}
&\sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \left\langle \delta_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{k}} (p_x q_y - p_y q_x) \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}} \Omega_{\boldsymbol{Q}} \\
&= \frac{1}{r^2} \sum_{\boldsymbol{\ell}} \left([(rK_x + \ell_x)\Omega_{\boldsymbol{K}}] * [(rK_y + \ell_y)K^{-2}\Omega_{\boldsymbol{K}}] \right)_{r\boldsymbol{K}+\boldsymbol{\ell}} \\
&\quad\quad -\frac{1}{r^2} \sum_{\boldsymbol{\ell}} \left([(rK_y + \ell_y)\Omega_{\boldsymbol{K}}] * [(rK_x + \ell_x)K^{-2}\Omega_{\boldsymbol{K}}] \right)_{r\boldsymbol{K}+\boldsymbol{\ell}} \\
&= \frac{1}{r^4 M^2} \sum_{\boldsymbol{J}} \zeta_M^{-\boldsymbol{K}\cdot\boldsymbol{J}} \left[r^2 W_{J_x} W_{J_y} (\hat{F}^0_{\boldsymbol{J}} \hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}} \hat{G}^0_{\boldsymbol{J}}) \right. \\
&\quad\quad \left. + r W'_{J_x} W_{J_y} (\hat{F}^2_{\boldsymbol{J}} \hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}} \hat{G}^2_{\boldsymbol{J}}) + r W_{J_x} W'_{J_y} (\hat{F}^0_{\boldsymbol{J}} \hat{G}^2_{\boldsymbol{J}} - \hat{F}^2_{\boldsymbol{J}} \hat{G}^0_{\boldsymbol{J}}) \right].
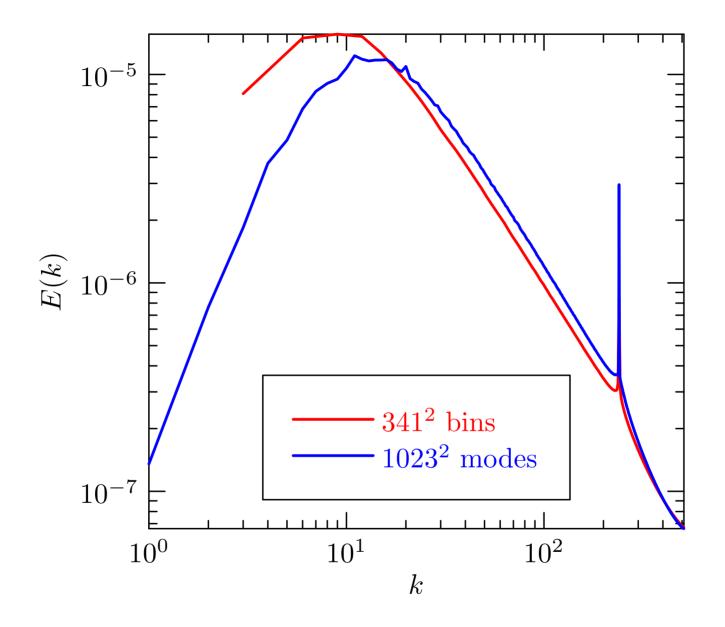\end{aligned}
$$

- Computational complexity is $\mathcal{O}(N \log N)$, with a coefficient $7/5 = 1.4$ times greater that for pseudospectral collocation.

Inviscid equipartition of a 31×31 pseudospectrally reduced simulation with radix $r = 3$.

Direct cascade.
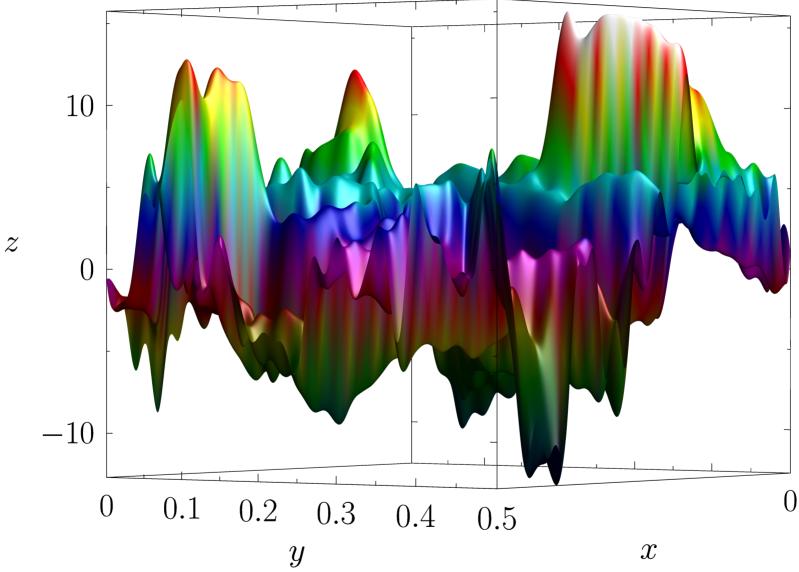
Inverse cascade.

Conclusions

- Spectral reduction affords a dramatic reduction in the number of degrees of freedom that must be explicitly evolved in turbulence simulations.

- One can evolve a turbulent system for thousands of eddy turnover times to obtain energy spectra smooth enough to compare with theory.

- Recognizing that spectral reduction yields correct inviscid equipartition spectra only with uniform binning and restricting our attention to this case only, an efficient FFT-based implementation, which we call pseudospectral reduction, is proposed.

- Even with uniform binning, the resulting energy spectrum is much closer to the predictions of the full dynamics than, say, the spectrum obtained by simply using a smaller spatial domain (larger mode spacing).

- We have recently generalized our efficient `FFTW++` [Bowman & Roberts 2011] library to support implicitly dealiased 2D coarse-grained Hermitian convolutions:

  `http://fftwpp.sourceforge.net`

- Spectral reduction could be used to develop a reliable dynamic subgrid model: Malcolm Roberts' Ph.D. thesis (expected 2011) explores ways to couple a pseudospectrally reduced subgrid model to a large-eddy simulation.
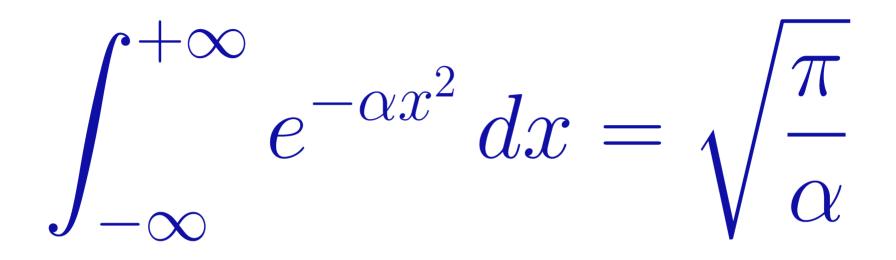
Vorticity Surface Plot

Asymptote: 2D & 3D Vector Graphics Language

Andy Hammerlindl, John C. Bowman, Tom Prince

`http://asymptote.sf.net`

(freely available under the GNU public license)

Asymptote Lifts TeX to 3D

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$$

`http://asymptote.sf.net`

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