Pseudospectral Reduction of Incompressible Two-Dimensional Turbulence

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2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega \doteq \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}$:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = \nu \nabla^2 \omega + f.$$

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- In Fourier space:

$$\frac{\partial\omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \int d\boldsymbol{p}\int d\boldsymbol{q}\,\frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}} + f_{\boldsymbol{k}},$$

where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ and $\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{q})\,\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$ is antisymmetric under permutation of any two indices.

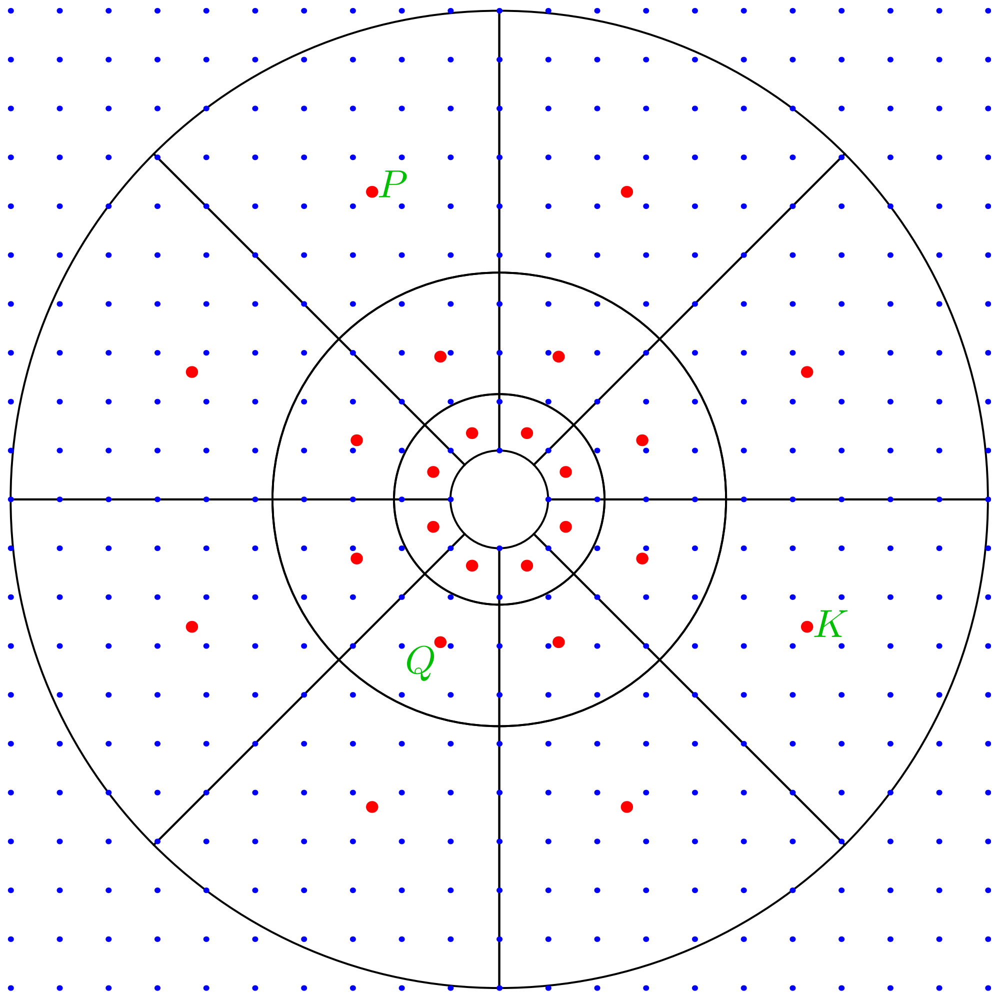
- When $\nu = f_{\boldsymbol{k}} = 0$,

enstrophy $Z = \frac{1}{2}\sum_{\boldsymbol{k}} \left|\omega_{\boldsymbol{k}}\right|^2$ and energy $E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{\left|\omega_{\boldsymbol{k}}\right|^2}{k^2}$ are conserved:

$$
\begin{aligned}
\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} &\quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{p}, \\
\frac{1}{k^2}\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} &\quad \text{antisymmetric in} \quad \boldsymbol{k} \leftrightarrow \boldsymbol{q}.
\end{aligned}
$$

Spectral Reduction

- Introduce a coarse-grained grid indexed by K:



Wavenumber Bin Geometry (8×3 bins)

- Define new variables

$$\Omega_{\boldsymbol{K}} = \langle \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{\Delta_{\boldsymbol{K}}} \int_{\Delta_{\boldsymbol{K}}} \omega_{\boldsymbol{k}} \, d\boldsymbol{k},$$

where $\Delta_{\boldsymbol{K}}$ is the area of bin $\boldsymbol{K}$.

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- Evolution of $\Omega_{\boldsymbol{K}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}} \right\rangle_{\boldsymbol{KPQ}},$$

where $\langle f \rangle_{\boldsymbol{KPQ}} = \dfrac{1}{\Delta_{\boldsymbol{K}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}}} \displaystyle\int_{\Delta_{\boldsymbol{K}}} d\boldsymbol{k} \int_{\Delta_{\boldsymbol{P}}} d\boldsymbol{p} \int_{\Delta_{\boldsymbol{Q}}} d\boldsymbol{q} \, f.$

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- Approximate $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ by bin-averaged values $\Omega_{\boldsymbol{P}}$ and $\Omega_{\boldsymbol{Q}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*.$$

- Define the coarse-grained enstrophy Z and energy E:

$$Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \left|\Omega_{\boldsymbol{K}}\right|^2 \Delta_{\boldsymbol{K}}, \qquad E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{\left|\Omega_{\boldsymbol{K}}\right|^2}{K^2}\Delta_{\boldsymbol{K}}.$$

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- Reinstate both desired symmetries with the modified coefficient

$$\frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2}.$$

Properties

- We call the forced-dissipative version of this approximation *spectral reduction* (SR):

$$\boxed{\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}.}$$

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- However: since the $\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},0}$ factor in the nonlinear coefficient $\epsilon_{\boldsymbol{kpq}}$ has been smoothed over, spectral reduction is no longer a convolution: pseudospectral collocation does not apply.

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- E.g., time average the exact bin-averaged enstrophy equation:

$$\overline{\frac{\partial}{\partial t}\left\langle |\omega_{\boldsymbol{k}}|^2 \right\rangle_{\boldsymbol{K}}} + 2\,\mathrm{Re}\left\langle \nu_{\boldsymbol{k}} \overline{|\omega_{\boldsymbol{k}}|^2} \right\rangle_{\boldsymbol{K}} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \overline{\omega^*_{\boldsymbol{k}} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}} \right\rangle_{\boldsymbol{KPQ}},$$

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where the bar means time average and $\langle\cdot\rangle_{\boldsymbol{K}}$ means bin average.

- Time-averaged quantities such as $\overline{|\omega_{\boldsymbol{k}}|^2}$ and $\overline{\omega^*_{\boldsymbol{k}}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}}}$ are generally *smooth* functions of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ on the four-dimensional surface defined by the triad condition $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$.

- Mean Value Theorem for integrals: for some $\boldsymbol{\xi} \in \boldsymbol{K}$.

$$\overline{|\Omega_{\boldsymbol{K}}|^2} = \overline{|\omega_{\boldsymbol{\xi}}|^2} \approx \overline{|\omega_{\boldsymbol{k}}|^2} \qquad \forall \boldsymbol{k} \in \boldsymbol{K}.$$

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- To good accuracy these statistical moments may therefore be evaluated at the characteristic wavenumbers $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$:

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- But this is precisely the time-average of the SR equation!

Noncanonical Hamiltonian Formulation

- Underlying *noncanonical* Hamiltonian formulation for inviscid 2D vorticity equation:

$$\dot{\omega}_{\boldsymbol{k}} = \int d\boldsymbol{q}\, J_{\boldsymbol{k}\boldsymbol{q}} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}},$$

where

$$H \doteq \frac{1}{2} \int d\boldsymbol{k}\, \frac{|\omega_{\boldsymbol{k}}|^2}{k^2},$$

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- Leads to inviscid Navier–Stokes equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}.$$

Liouville Theorem

- Navier–Stokes:

$$J_{\boldsymbol{kq}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{kpq}} \omega^*_{\boldsymbol{p}}$$

$$\Rightarrow \qquad \int d\boldsymbol{k}\, \frac{\delta \dot{\omega}_{\boldsymbol{k}}}{\delta \omega_{\boldsymbol{k}}} = \int d\boldsymbol{k} \int d\boldsymbol{q}\ \underbrace{\frac{\delta J_{\boldsymbol{kq}}}{\delta \omega_{\boldsymbol{k}}}}_{\epsilon_{\boldsymbol{k}(-\boldsymbol{k})\boldsymbol{q}}=0} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}} + J_{\boldsymbol{kq}} \frac{\delta^2 H}{\delta \omega_{\boldsymbol{k}} \delta \omega_{\boldsymbol{q}}} = 0.$$

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- Spectral Reduction:

$$J_{\boldsymbol{KQ}} \doteq \sum_{\boldsymbol{P}} \Delta_{\boldsymbol{P}} \left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}}$$

$$\Rightarrow \qquad \sum_{\boldsymbol{K}} \frac{\partial \dot{\Omega}_{\boldsymbol{K}}}{\partial \Omega_{\boldsymbol{K}}} = \sum_{\boldsymbol{K},\boldsymbol{Q}} \underbrace{\frac{\partial J_{\boldsymbol{KQ}}}{\partial \Omega_{\boldsymbol{K}}}}_{\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{K}(-\boldsymbol{K})\boldsymbol{Q}}=0} \frac{\partial H}{\partial \Omega_{\boldsymbol{Q}}} + J_{\boldsymbol{KQ}} \frac{\partial^2 H}{\partial \Omega_{\boldsymbol{K}} \partial \Omega_{\boldsymbol{Q}}} = 0.$$

Statistical Equipartition

- For *mixing* dynamics, the Liouville Theorem and the coarse-grained invariants

$$E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2}\Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}},$$

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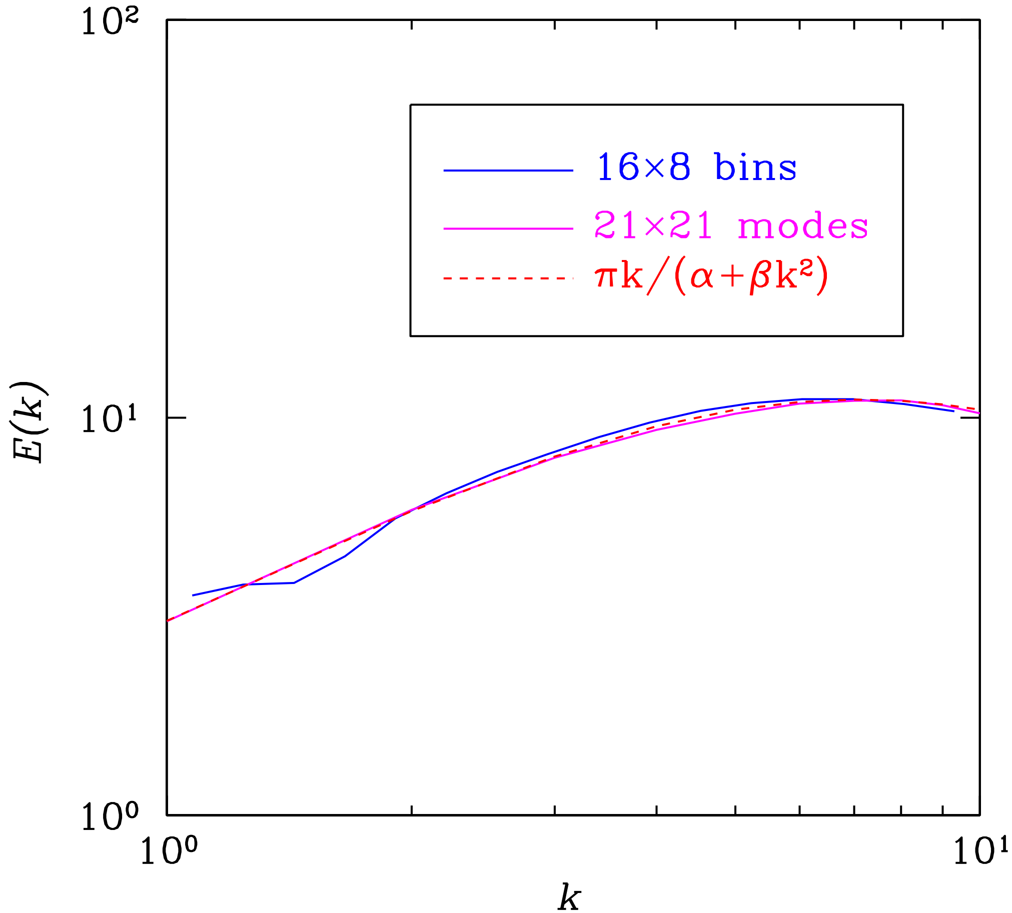
lead to statistical equipartition of $(\alpha/K^2+\beta)\,|\Omega_{\boldsymbol{K}}|^2\,\Delta_{\boldsymbol{K}}$.

- This is the correct equipartition only for uniform bins.
- However, for nonuniform bins, a rescaling of time by $\Delta_{\boldsymbol{K}}$,

$$\boxed{\frac{1}{\Delta_{\boldsymbol{K}}}\frac{\partial\Omega_{\boldsymbol{K}}}{\partial t} + \langle\nu_{\boldsymbol{k}}\rangle_{\boldsymbol{K}}\,\Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\,\frac{\langle\epsilon_{\boldsymbol{kpq}}\rangle_{\boldsymbol{KPQ}}}{Q^2}\,\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}},}$$

yields the correct inviscid equipartition: $\left\langle|\Omega_{\boldsymbol{K}}|^2\right\rangle = \left(\frac{\alpha}{K^2}+\beta\right)^{-1}$.

- Unfortunately, the rescaled spectral reduction equations are hopelessly stiff [Bowman *et al.* 2001].



Relaxation of rescaled spectral reduction to equipartition.

Spectral Reduction on a Lattice

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- Imposing uniform bins has an important advantage: it affords a pseudospectral implementation of spectral reduction!
- Consider spectral reduction on a coarse-grained lattice, with $r \times r$ modes per rectangular bin.

- The bin-averaging operations become:

$$\langle f_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{r^2} \sum_{\boldsymbol{k} \in \boldsymbol{K}} f_{\boldsymbol{k}},$$

$$\langle f_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \doteq \frac{1}{r^6} \sum_{\boldsymbol{k} \in \boldsymbol{K}} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \sum_{\boldsymbol{q} \in \boldsymbol{Q}} f_{\boldsymbol{kpq}}.$$

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- Uniform discrete spectral reduction:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = r^4 \sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}} + F_{\boldsymbol{K}} \xi(t).$$

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- Let $\xi(t)$ be a unit Gaussian stochastic white-noise process and choose $F_{\boldsymbol{K}} = 2\epsilon_Z \dfrac{f_K}{\sqrt{\sum_{\boldsymbol{K}} |f_K|^2}}$ to inject on average ϵ_Z units of enstrophy Novikov [1964].

Discrete Fast Fourier Transform

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- The fast Fourier transform (FFT) method exploits the properties that $\zeta_N^r = \zeta_{N/r}$ and $\zeta_N^N = 1$.

FFT of a Piecewise Constant Function

- Suppose $N = rM$ and $f_{rK+\ell} = F_K$ for $\ell = 0, 1, \ldots r - 1$ and $K = 0, 1, \ldots, M - 1$.

FFT of a Piecewise Constant Function

- Suppose $N = rM$ and $f_{rK+\ell} = F_K$ for $\ell = 0, 1, \dots r-1$ and $K = 0, 1, \dots, M-1$.

- For $J = 0, \dots, M-1$ and $s = 0, \dots, r-1$ the *backwards Fourier transform of the coarse-grained data* F_K is given by

$$\hat{f}_{sM+J} = \sum_{K=0}^{M-1} \sum_{\ell=0}^{r-1} \zeta_N^{(sM+J)(rK+\ell)} F_K = S_{J,s} \hat{F}_J,$$

where

$$S_{J,s} \doteq \sum_{\ell=0}^{r-1} \zeta_N^{J\ell} \zeta_r^{s\ell},$$

$$\hat{F}_J \doteq \sum_{K=0}^{M-1} \zeta_M^{JK} F_K.$$

- The *coarse-grained forwards Fourier transform* is given by:

$$F_K \doteq \frac{1}{Nr}\sum_{\ell=0}^{r-1} f_{rK+\ell} = \frac{1}{r^2 M}\sum_{\ell=0}^{r-1}\sum_{J=0}^{M-1}\sum_{s=0}^{r-1}\zeta_N^{-(rK+\ell)(sM+J)}\hat{f}_{sM+J}$$

$$= \frac{1}{r^2 M}\sum_{J=0}^{M-1}\zeta_M^{-KJ}\sum_{s=0}^{r-1} S^*_{J,s}\hat{f}_{sM+J}.$$

1D Coarse-Grained Convolution

- The coarse-grained convolution $\langle f * g\rangle_K$ of f and g can then be computed as

$$\langle f * g\rangle_K \doteq \frac{1}{r}\sum_{\ell=0}^{r-1}(f*g)_{rK+\ell} = \frac{1}{r^2M}\sum_{J=0}^{M-1}\zeta_M^{-KJ}\sum_{s=0}^{r-1}S_{J,s}^{*}\hat{f}_{sM+J}\hat{g}_{sM+J}$$
$$= \frac{1}{r^2M}\sum_{J=0}^{M-1}\zeta_M^{-KJ}W_J\hat{F}_J\hat{G}_J,$$

in terms of the spatial weight factors $W_J \doteq \sum_{s=0}^{r-1}\left|S_{J,s}\right|^2 S_{J,s}$.

- Similarly, the bin-averaged Fourier transform of F_K weighted by ℓ is given by

$$\hat{f}_{sM+J} = \sum_{K=0}^{M-1} \sum_{\ell=0}^{r-1} \zeta_N^{(sM+J)(rK+\ell)} \ell F_K = T_{J,s} \hat{F}_J,$$

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$$T_{J,s} \doteq \sum_{\ell=0}^{r-1} \ell \zeta_N^{J\ell} \zeta_r^{s\ell}.$$

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$$T_{J,s} \doteq \sum_{\ell=0}^{r-1} \ell \zeta_N^{J\ell} \zeta_r^{s\ell}.$$

- Let $W'_J \doteq \sum_{s=0}^{r-1} |S_{J,s}|^2 T_{J,s}$.

Pseudospectral reduction

- In terms of $F^0 \doteq K_x\Omega_{\boldsymbol{K}}$, $F^1 \doteq K_y\Omega_{\boldsymbol{K}}$, $F^2 \doteq \Omega_{\boldsymbol{K}}$, $G^0 \doteq K_xK^{-2}\Omega_{\boldsymbol{K}}$, $G^1 \doteq K_yK^{-2}\Omega_{\boldsymbol{K}}$, and $G^2 \doteq K^{-2}\Omega_{\boldsymbol{K}}$:

$$\sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \langle \delta_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{k}}(p_xq_y - p_yq_x)\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}\Omega_{\boldsymbol{Q}}$$

$$= \frac{1}{r^2}\sum_{\boldsymbol{\ell}} \Big([(rK_x+\ell_x)\Omega_{\boldsymbol{K}}] * \big[(rK_y+\ell_y)K^{-2}\Omega_{\boldsymbol{K}}\big]\Big)_{r\boldsymbol{K}+\boldsymbol{\ell}}$$

$$- \frac{1}{r^2}\sum_{\boldsymbol{\ell}} \Big([(rK_y+\ell_y)\Omega_{\boldsymbol{K}}] * \big[(rK_x+\ell_x)K^{-2}\Omega_{\boldsymbol{K}}\big]\Big)_{r\boldsymbol{K}+\boldsymbol{\ell}}$$

$$= \frac{1}{r^4M^2}\sum_{\boldsymbol{J}} \zeta_M^{-\boldsymbol{K}\cdot\boldsymbol{J}} \Big[r^2 W_{J_x}W_{J_y}(\hat{F}^0_{\boldsymbol{J}}\hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}}\hat{G}^0_{\boldsymbol{J}})$$

$$+ rW'_{J_x}W_{J_y}(\hat{F}^2_{\boldsymbol{J}}\hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}}\hat{G}^2_{\boldsymbol{J}}) + rW_{J_x}W'_{J_y}(\hat{F}^0_{\boldsymbol{J}}\hat{G}^2_{\boldsymbol{J}} - \hat{F}^2_{\boldsymbol{J}}\hat{G}^0_{\boldsymbol{J}})\Big].$$

Pseudospectral reduction

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$$- \frac{1}{r^2}\sum_{\boldsymbol{\ell}} \Big([(rK_y+\ell_y)\Omega_{\boldsymbol{K}}] * \big[(rK_x+\ell_x)K^{-2}\Omega_{\boldsymbol{K}}\big]\Big)_{r\boldsymbol{K}+\boldsymbol{\ell}}$$

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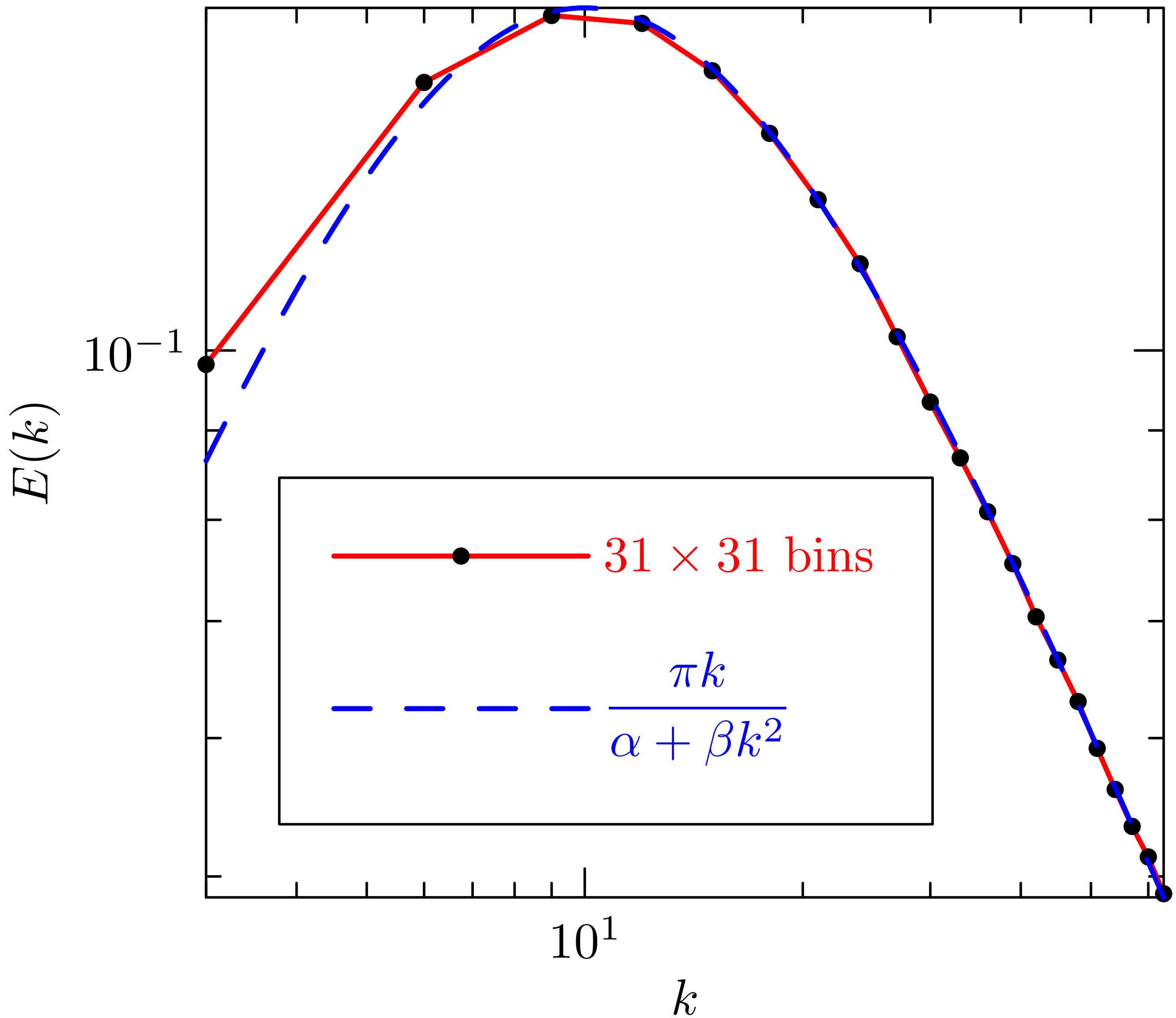
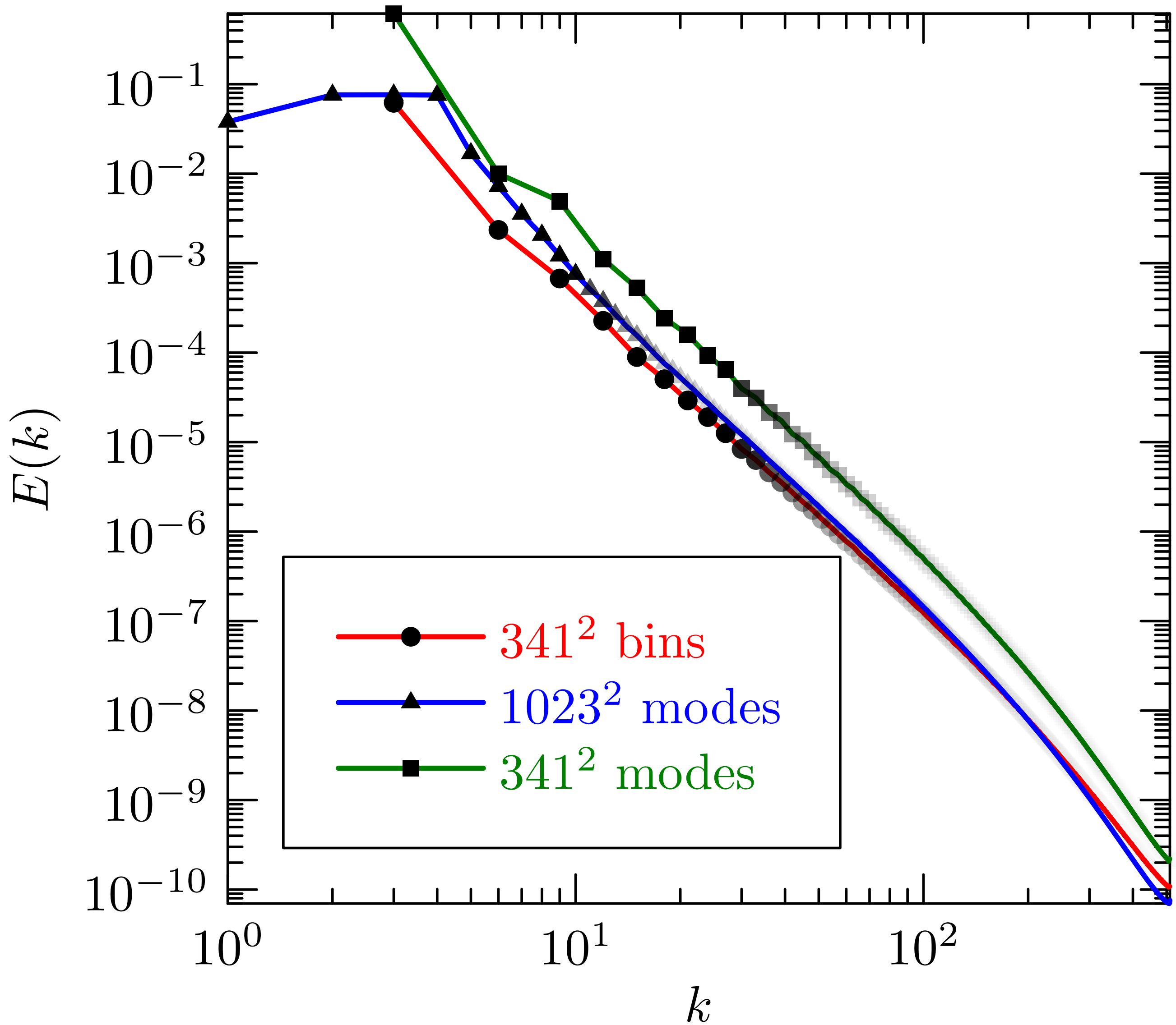
$$+ rW'_{J_x}W_{J_y}(\hat{F}^2_{\boldsymbol{J}}\hat{G}^1_{\boldsymbol{J}} - \hat{F}^1_{\boldsymbol{J}}\hat{G}^2_{\boldsymbol{J}}) + rW_{J_x}W'_{J_y}(\hat{F}^0_{\boldsymbol{J}}\hat{G}^2_{\boldsymbol{J}} - \hat{F}^2_{\boldsymbol{J}}\hat{G}^0_{\boldsymbol{J}})\Big].$$

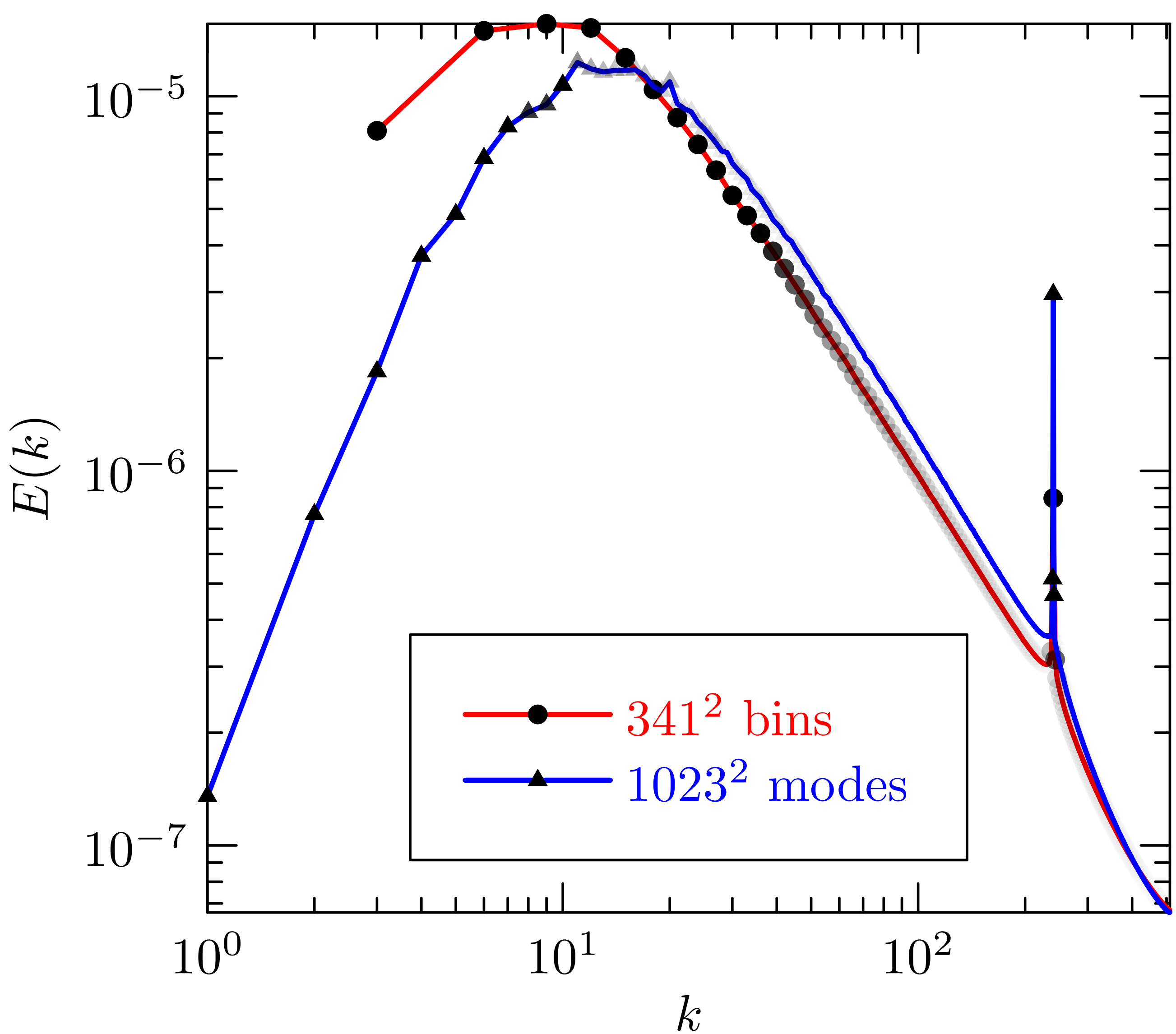
- Computational complexity is $\mathcal{O}(N \log N)$, with a coefficient $7/5 = 1.4$ times greater that for pseudospectral collocation.



Inviscid equipartition of a 31×31 pseudospectrally reduced simulation with radix $r = 3$.



Direct cascade.



Inverse cascade.

Conclusions

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Conclusions

- Spectral reduction affords a dramatic reduction in the number of degrees of freedom that must be explicitly evolved in turbulence simulations.
- One can evolve a turbulent system for thousands of eddy turnover times to obtain energy spectra smooth enough to compare with theory.
- Recognizing that spectral reduction yields correct inviscid equipartition spectra only with uniform binning and restricting our attention to this case only, an efficient FFT-based implementation, which we call pseudospectral reduction, is proposed.
- Even with uniform binning, the resulting energy spectrum is much closer to the predictions of the full dynamics than, say, the spectrum obtained by simply using a smaller spatial domain (larger mode spacing).

- We have recently generalized our efficient **FFTW++** [Bowman & Roberts 2011] library to support implicitly dealiased 2D coarse-grained Hermitian convolutions:

  http://fftwpp.sourceforge.net

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- Spectral reduction could be used to develop a reliable dynamic subgrid model: Malcolm Roberts' recent Ph.D. thesis (2011) explores ways to couple a pseudospectrally reduced subgrid model to a large-eddy simulation.

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