Casimir Cascades in Two-Dimensional Turbulence

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Incompressible Turbulence

- *Navier–Stokes* equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \underbrace{(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}}_{\text{advection}} = \underbrace{-\frac{1}{\rho}\boldsymbol{\nabla} P}_{\text{force/mass}} + \underbrace{\nu \nabla^2 \boldsymbol{u}}_{\text{dissipation}} + \underbrace{\boldsymbol{F}}_{\text{external force}},$$

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- Continuity and incompressibility $\Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$.

- If $\boldsymbol{u}$ is continuously differentiable on a simply connected domain (free of holes), it can be expressed in terms of a *vector potential*: $\boldsymbol{\nabla}\cdot\boldsymbol{u} = \mathbf{0} \iff \boldsymbol{u} = \boldsymbol{\nabla}\times\boldsymbol{A}$ with $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ (*Coulomb gauge*).

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- Incompressibility: uniform initial density remains unchanged; choose mass units so that $\rho = 1$.

Energy

- Rewrite the Navier–Stokes equation as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla} u^2 - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = -\boldsymbol{\nabla} P + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F},$$

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- When the flow is inviscid ($\nu = 0$) and forcing is absent ($\boldsymbol{F} = \boldsymbol{0}$), the *energy* (mean-squared velocity) $E \doteq \frac{1}{2}\int u^2\, d\boldsymbol{x}$ is conserved:

$$\frac{dE}{dt} = \int \boldsymbol{u}\cdot\frac{\partial \boldsymbol{u}}{\partial t}\, d\boldsymbol{x} = -\int \boldsymbol{u}\cdot\left[\boldsymbol{\nabla}\left(\frac{u^2}{2} + P\right) - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})\right] d\boldsymbol{x}$$

$$= \int \left(\frac{u^2}{2} + P\right)\boldsymbol{\nabla}\cdot\boldsymbol{u}\, d\boldsymbol{x} = 0,$$

given zero boundary conditions at infinity (or periodic boundary conditions).

Pressure

- Divergence of Navier–Stokes $\Rightarrow$ equation for the pressure P:

$$\nabla^2 P = \boldsymbol{\nabla}\boldsymbol{\cdot}[\boldsymbol{F} - (\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{u}].$$

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$$\Rightarrow \frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{w} = \underbrace{(\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{u}}_{\text{vortex stretching}} + \nu\nabla^2\boldsymbol{w} + \boldsymbol{\nabla}\times\boldsymbol{F}.$$

- Taylor–Proudman theorem: A fluid rotating rapidly about an axis $\hat{\boldsymbol{z}}$ at frequency Ω tends to become uniform along that axis, due to the Coriolis force $\boldsymbol{F} = -2\Omega\, \hat{\boldsymbol{z}} \times \boldsymbol{u}$:

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- The predominant features of the flow are thus confined to 2D planes perpendicular to the axis of rotation.

2D Turbulence

- If $\boldsymbol{u} = (u, v, 0)$, with no dependence on the z coordinate, then $\boldsymbol{w} = \omega\hat{\boldsymbol{z}}$ and the vortex-stretching term $\boldsymbol{w}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}$ will vanish:

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- In 2D there thus exists an additional inviscid invariant, the *enstrophy* (mean-squared vorticity) $Z \doteq \frac{1}{2}\int \omega^2\, d\boldsymbol{x}$:

$$\frac{dZ}{dt} = \int \omega\frac{\partial \omega}{\partial t}\, d\boldsymbol{x} = -\int \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\left(\frac{w^2}{2}\right) d\boldsymbol{x} = \int \left(\frac{\omega^2}{2}\right)\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}\, d\boldsymbol{x} = 0.$$

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- In three dimensions, the helicity may have a nonzero value, but that value is still conserved by the inviscid dynamics.

Stream Function

- In 2D, the vorticity vector is perpendicular to the plane of motion:

$$\omega \hat{\boldsymbol{z}} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = -\nabla^2 \boldsymbol{A}.$$

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- 2D turbulence may thus be cast in terms of a single scalar field ψ, or equivalently, ω.

2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega = \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}$:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = -\nu \nabla^2 \omega + f.$$

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- In Fourier space:

$$\frac{\partial\omega_{\boldsymbol{k}}}{\partial t} = S_{\boldsymbol{k}} - \nu k^2\omega_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

where $S_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} \frac{\hat{\boldsymbol{z}}\times\boldsymbol{p}\cdot\boldsymbol{k}}{p^2}\,\omega^*_{\boldsymbol{p}}\,\omega^*_{-\boldsymbol{k}-\boldsymbol{p}}.$

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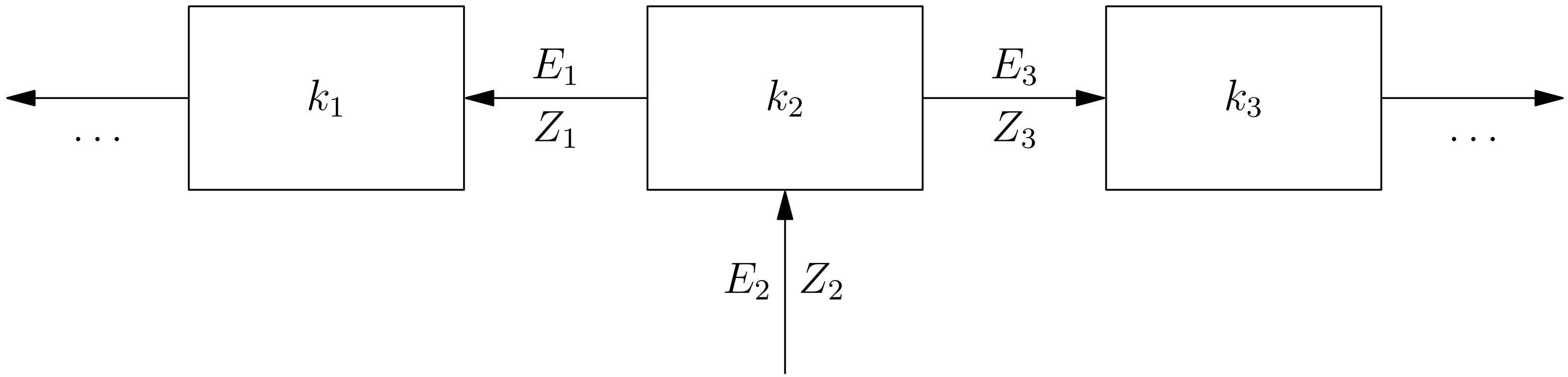
- When $\nu = 0$ and $f_{\boldsymbol{k}} = 0$:

energy $E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ and enstrophy $Z = \frac{1}{2}\sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ are conserved.

Fjørtoft Dual Cascade Scenario



$$E_2 = E_1 + E_3, \qquad Z_2 = Z_1 + Z_3, \qquad Z_i \approx k_i^2 E_i.$$

- When $k_1 = k$, $k_2 = 2k$, and $k_3 = 4k$:

$$E_1 \approx \frac{4}{5} E_2, \quad Z_1 \approx \frac{1}{5} Z_2, \qquad E_3 \approx \frac{1}{5} E_2, \quad Z_3 \approx \frac{4}{5} Z_2.$$

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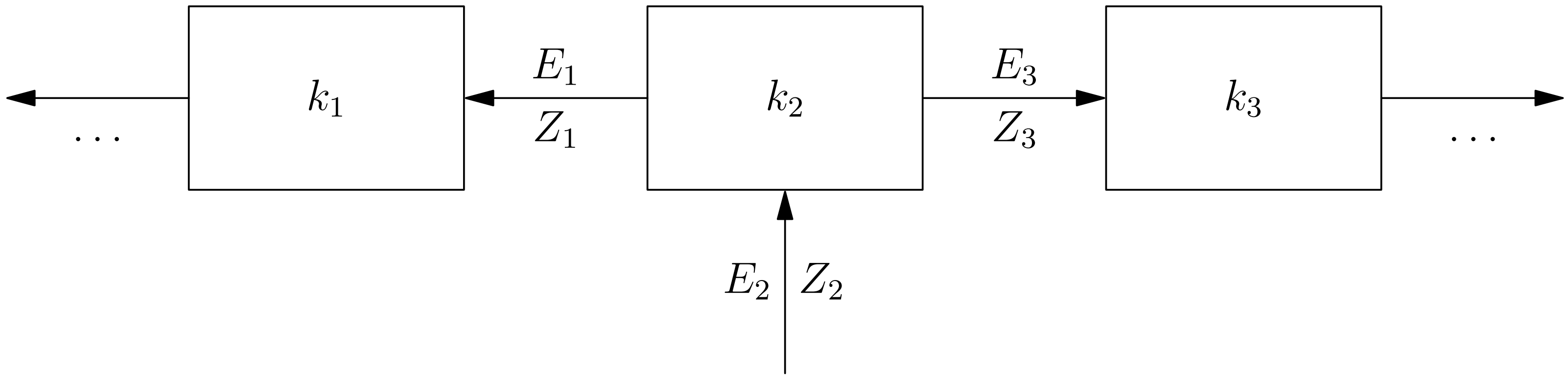

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- Fjørtoft [1953]: energy cascades to large scales and enstrophy cascades to small scales.

Kraichnan–Leith–Batchelor Theory

- In an infinite domain [Kraichnan 1967], [Leith 1968], [Batchelor 1969]:
  - large-scale $k^{-5/3}$ energy cascade;
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  - large-scale $k^{-5/3}$ energy cascade;
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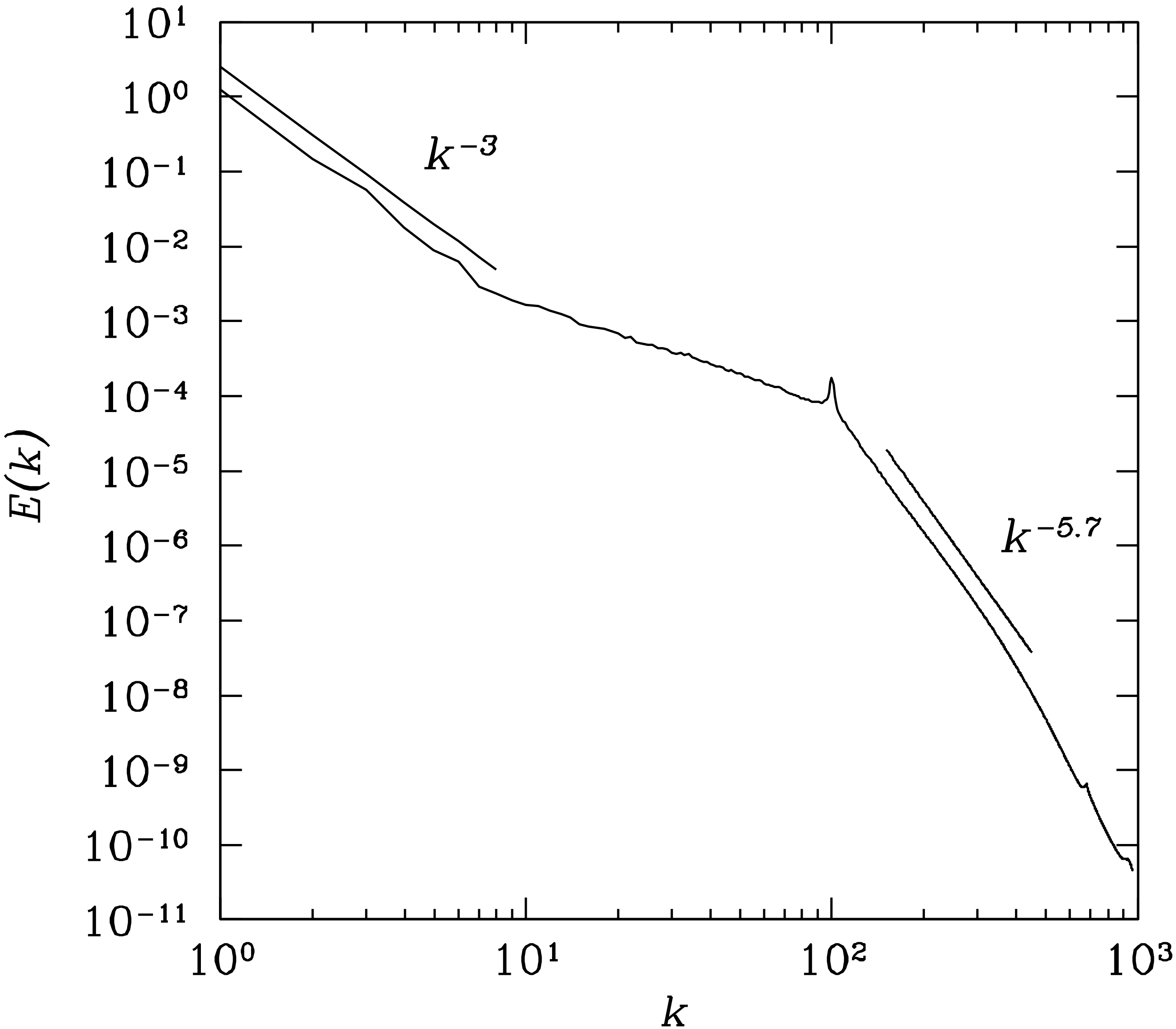
- In a bounded domain, the situation may be quite different…

Long-Time Behaviour in a Bounded Domain



Tran and Bowman, PRE 69, 036303, 1–7 (2004).

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- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit cascades?

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- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit cascades?
- Polyakov [1992] has suggested that the higher-order Casimir invariants cascade to large scales, while Eyink [1996] suggests that they might cascade to small scales.

High-Wavenumber Truncation

- Only the quadratic invariants survive high-wavenumber truncation (Montgomery calls them rugged invariants).

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*.$$

where $\epsilon_{\boldsymbol{kpq}} = (\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q})\, \delta(\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q})$.

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- Enstrophy evolution:

$$\frac{d}{dt} \sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2 = \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{k}}^* \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* = 0.$$

- Invariance of $Z_3 = \int \omega^3 \, dx$ follows from:

$$0 = \sum_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{s}} \left[\sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* \omega_{\boldsymbol{r}}^* \omega_{\boldsymbol{s}}^* + 2 \text{ other similar terms} \right].$$

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- We find that this is indeed the case.

Enstrophy Balance

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu k^2 \omega_{\boldsymbol{k}} = S_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

- Multiply by $\omega_{\boldsymbol{k}}^*$ and integrate over wavenumber angle $\Rightarrow$ enstrophy spectrum $Z(k)$ evolves as:

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k),$$

where $T(k)$ and $G(k)$ are the corresponding angular averages of $\mathrm{Re}\,\langle S_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* \rangle$ and $\mathrm{Re}\,\langle f_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* \rangle$.

Nonlinear Enstrophy Transfer Function

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$$\Pi(k) \doteq 2 \int_k^{\infty} T(p)\, dp$$

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- Integrate from k to ∞:

$$\frac{d}{dt} \int_k^\infty Z(p)\, dp = \Pi(k) - \epsilon_Z(k),$$

where $\epsilon_Z(k) \doteq 2\nu \int_k^\infty p^2 Z(p)\, dp - \int_k^\infty G(p)\, dp$ is the total enstrophy transfer, via dissipation and forcing, out of wavenumbers higher than k.

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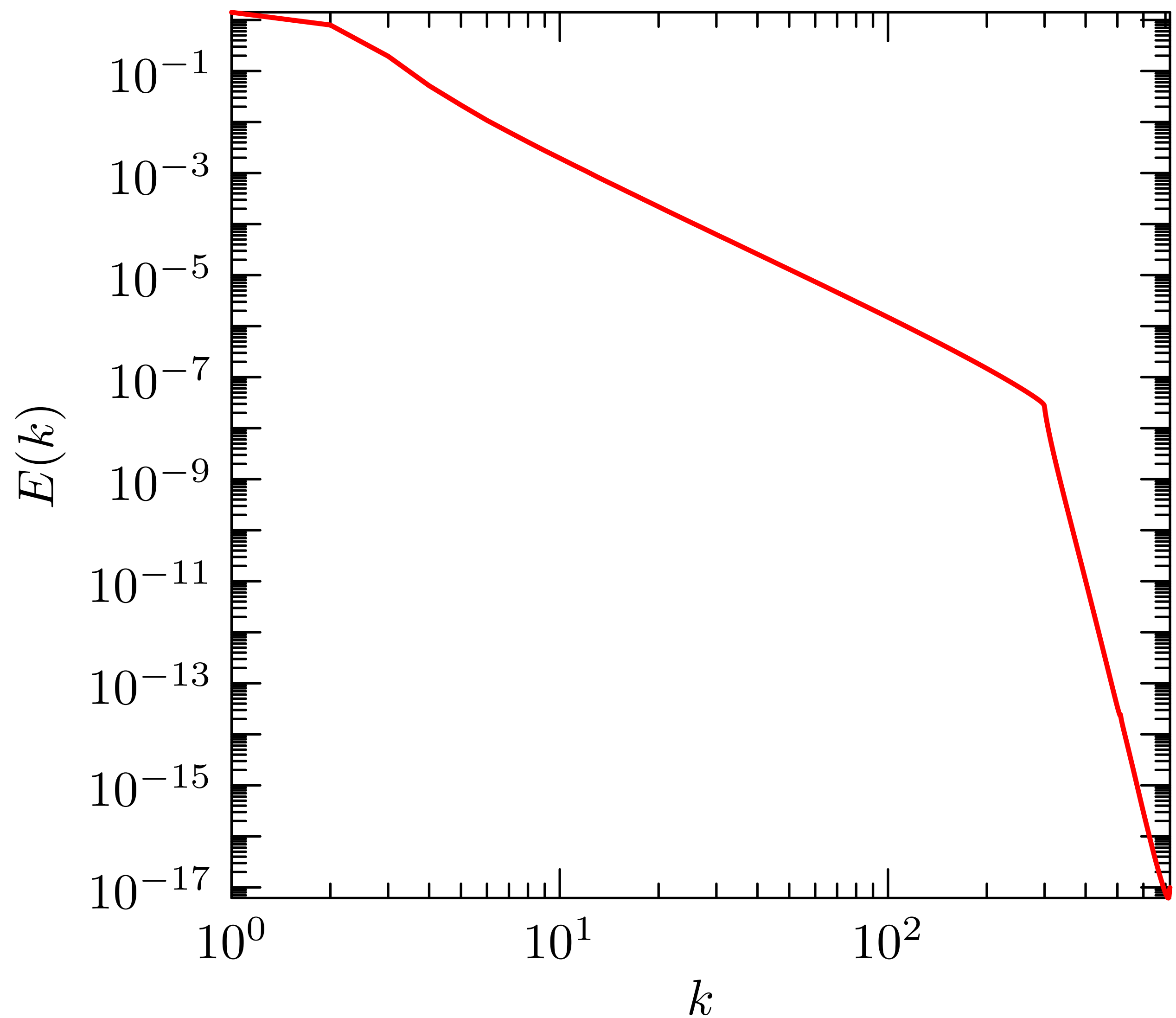
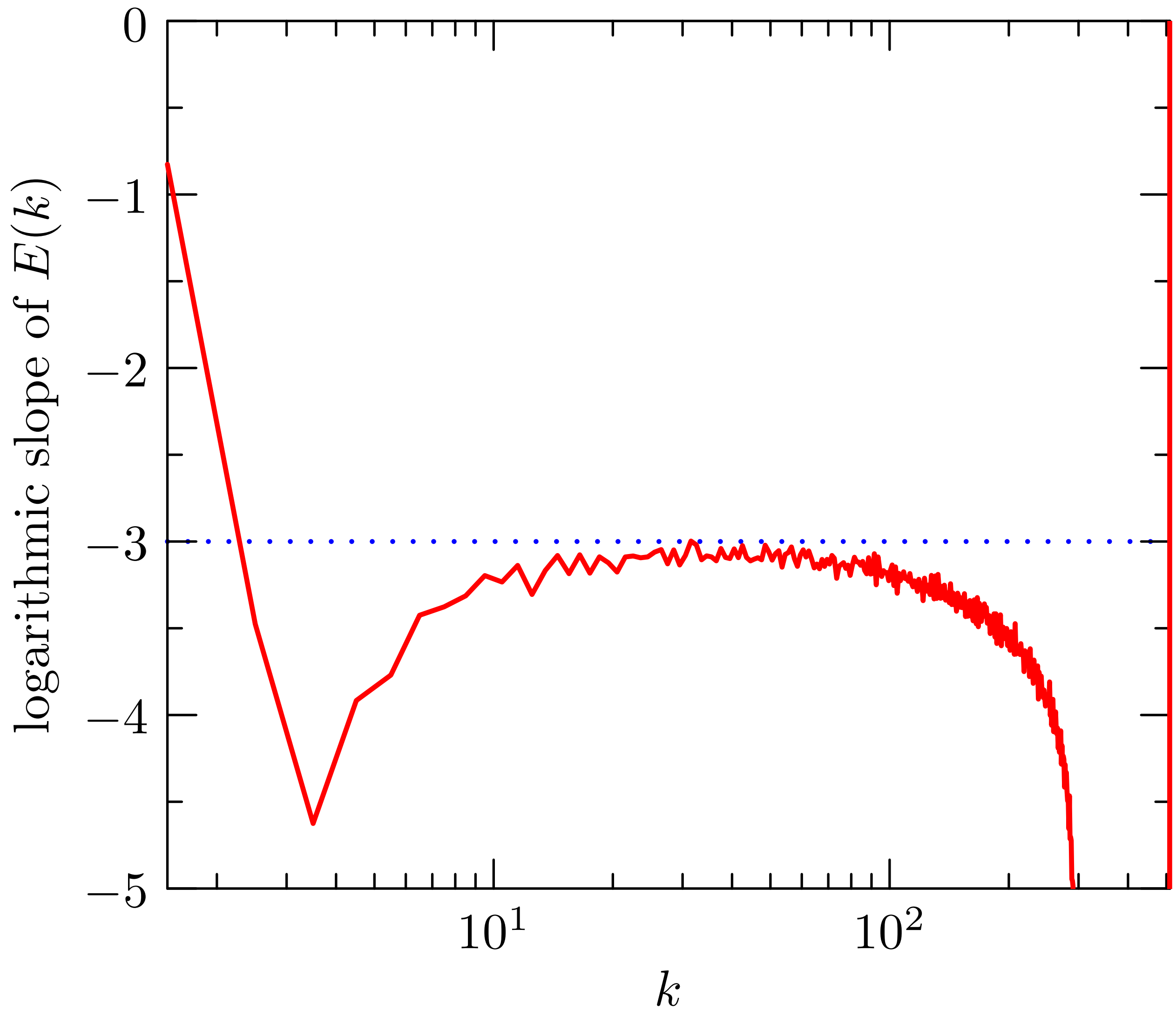
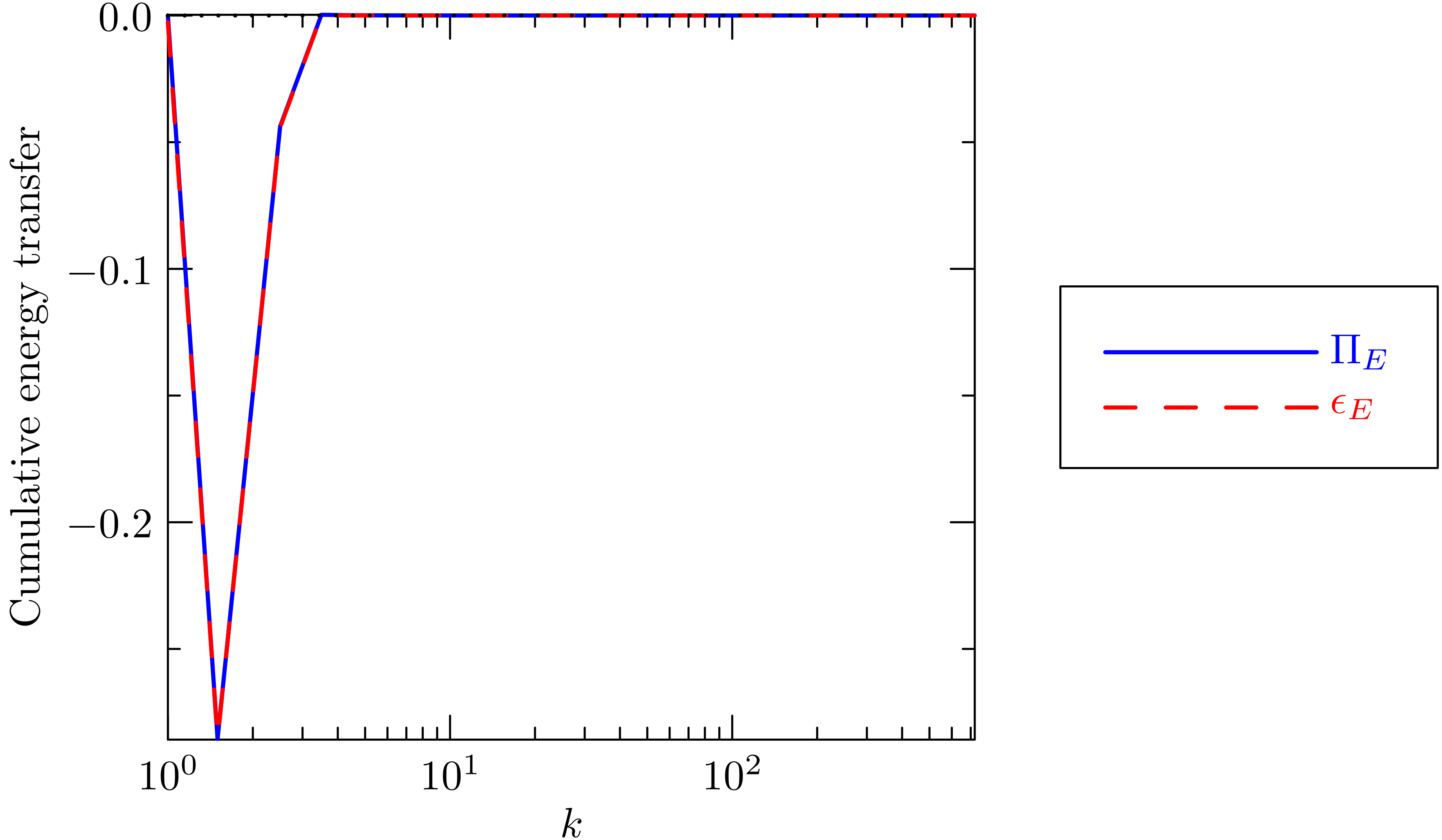
- This provides an excellent numerical diagnostic for when a steady state has been reached.

Forcing at $k = 2$, friction for $k < 3$, viscosity for $k \geq k_H = 300$ (1023×1023 dealiased modes)

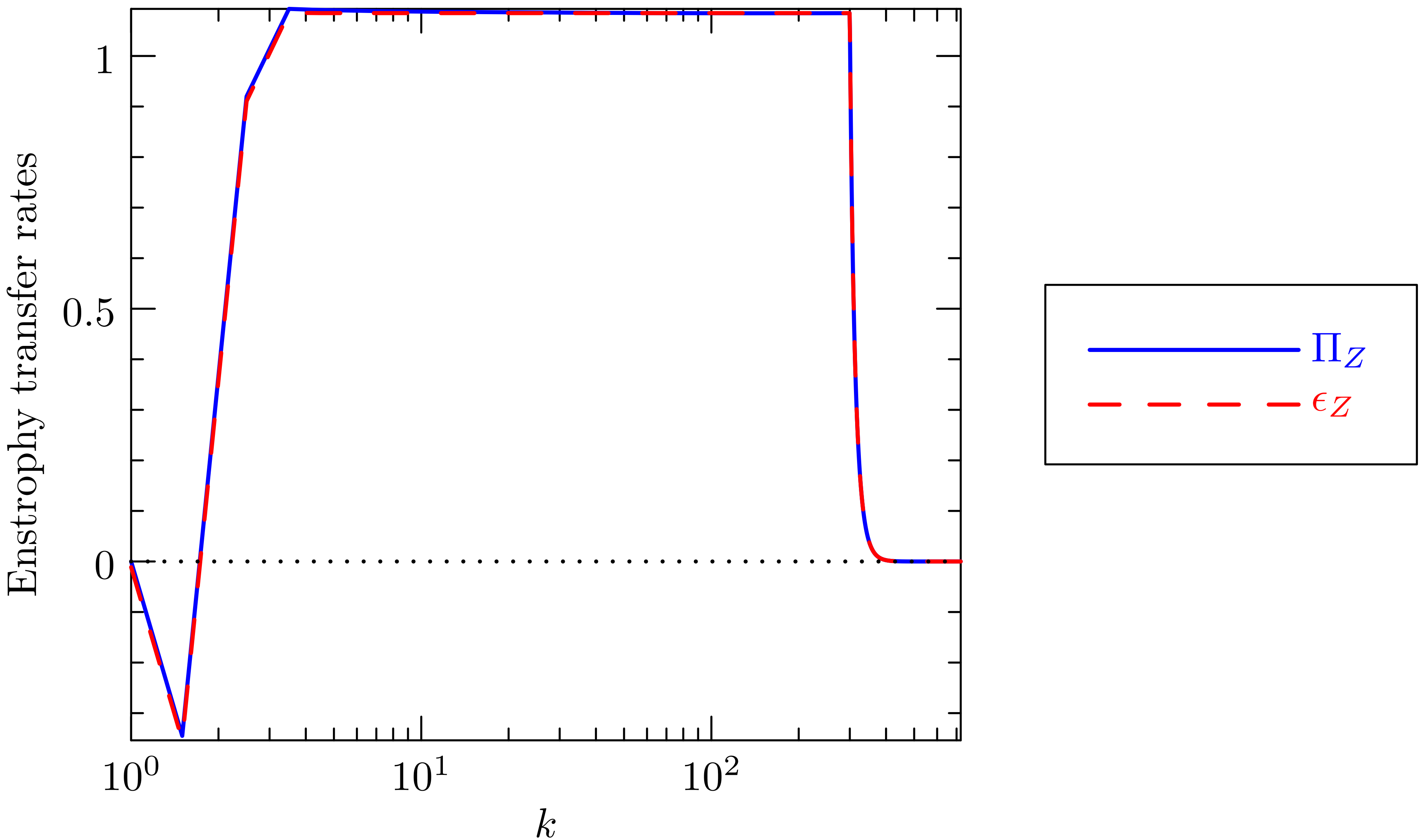




0
−1
−2
−3
−4
−5
logarithmic slope of $E(k)$
10^1
10^2
k



Cutoff viscosity $(k \geq k_H = 300)$



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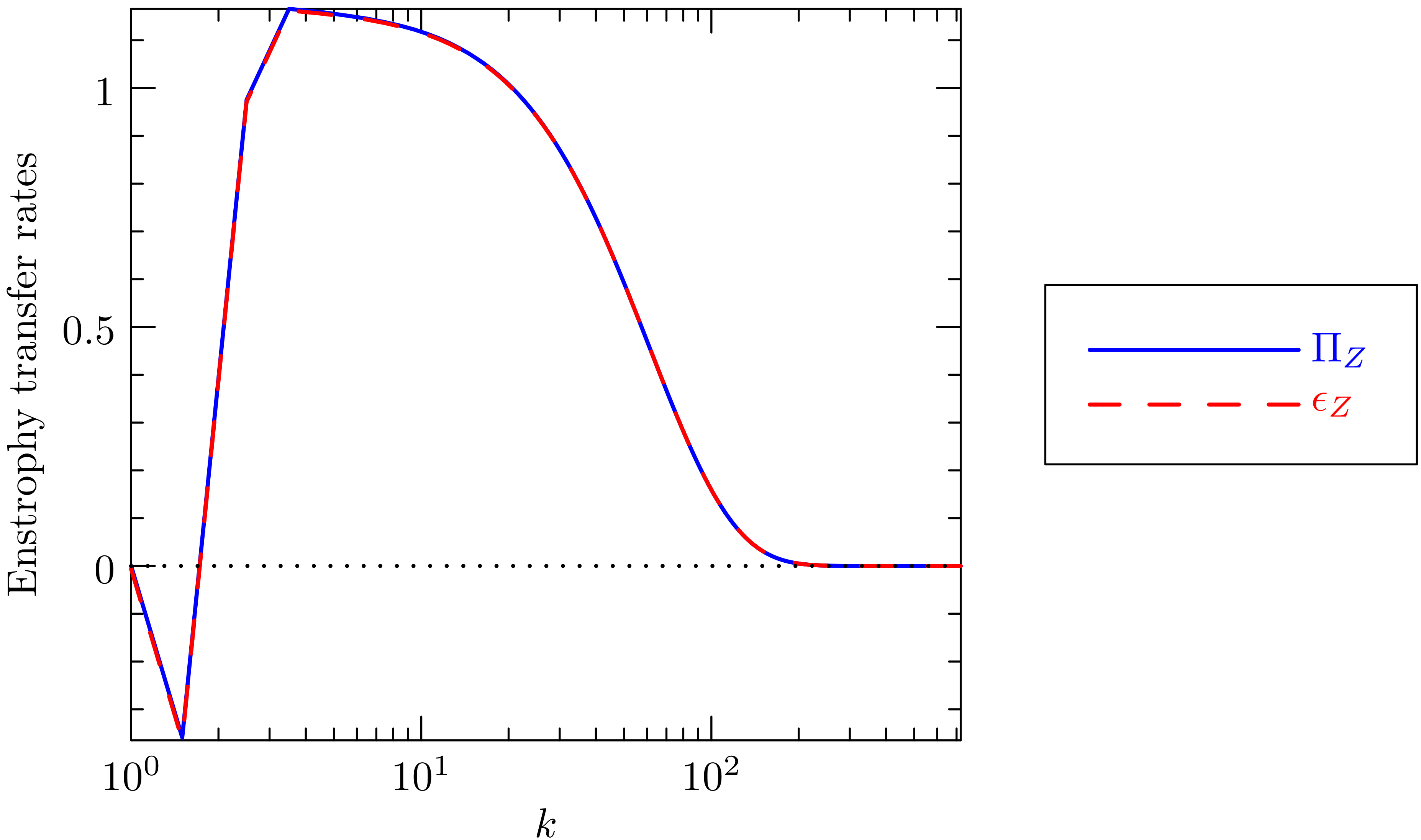
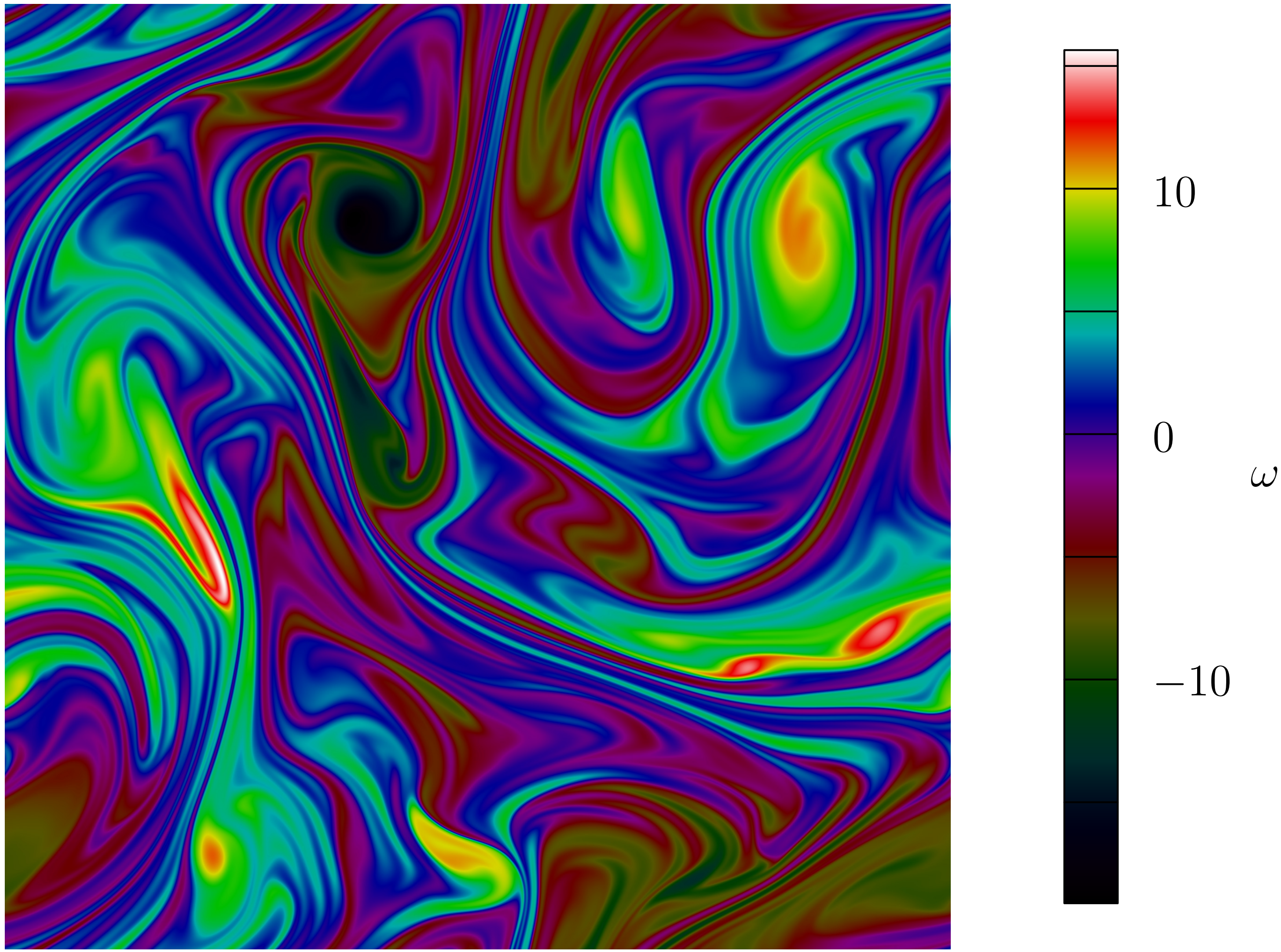
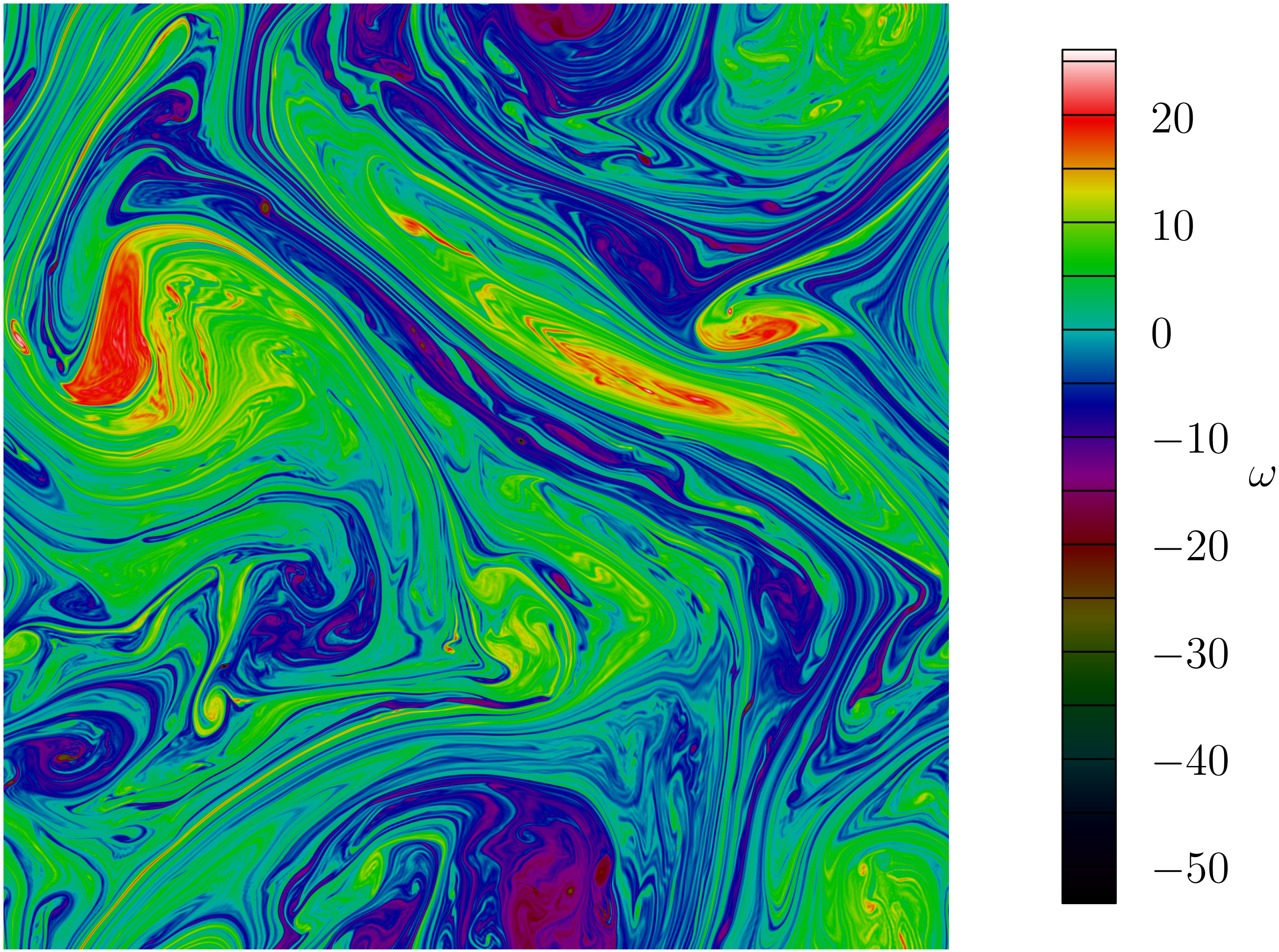


Molecular viscosity ($k \geq k_H = 0$)

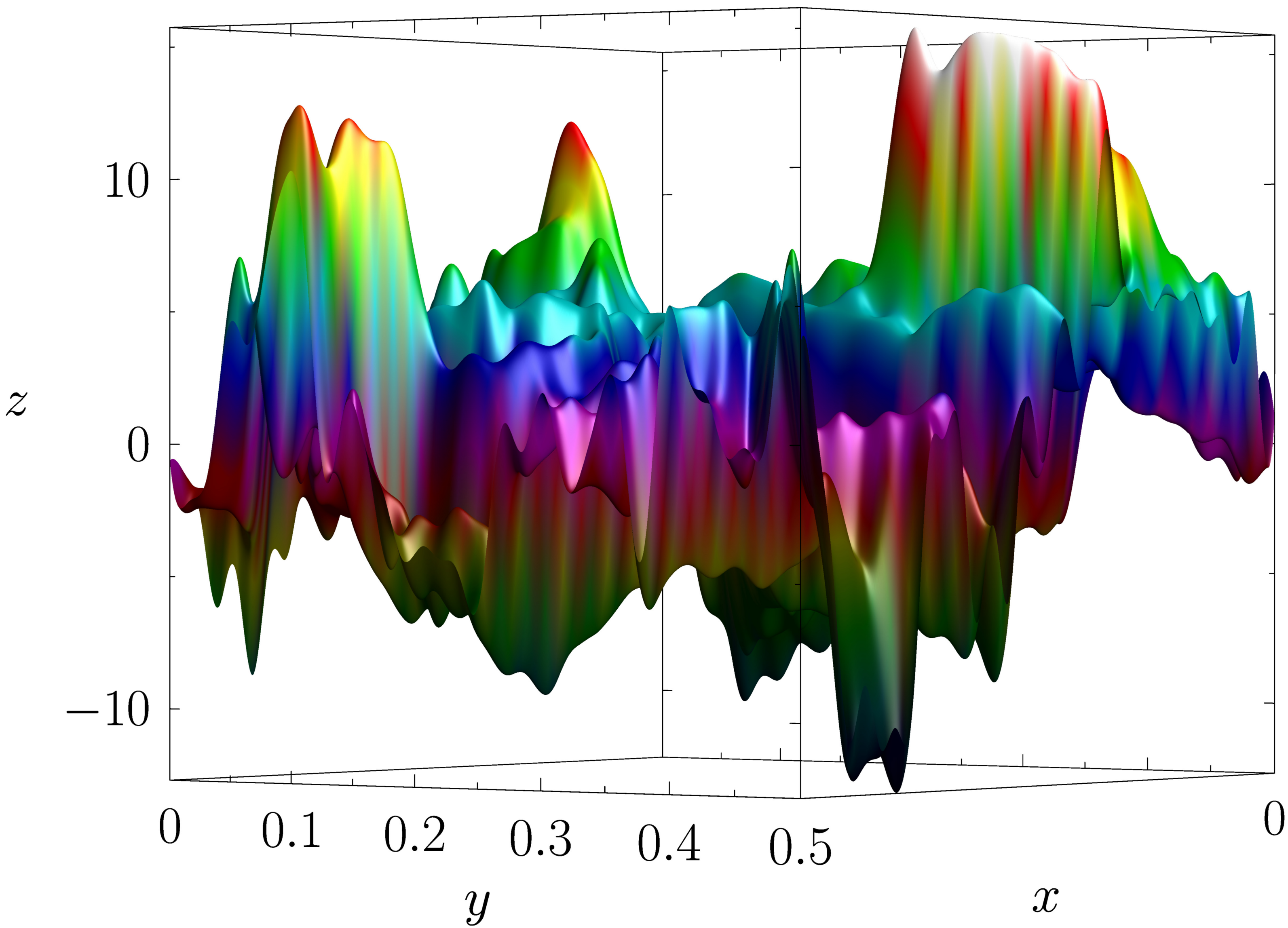
Vorticity Field with Molecular Viscosity



Vorticity Field with Viscosity Cutoff



Vorticity Surface Plot with Molecular Viscosity



Nonlinear Casimir Transfer

- Fourier decompose the fourth-order Casimir invariant $Z_4 = N^3 \sum_{\boldsymbol{j}} \omega^4(x_{\boldsymbol{j}})$ in terms of N spatial collocation points $x_{\boldsymbol{j}}$:

$$Z_4 = \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \omega_{\boldsymbol{k}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}}.$$

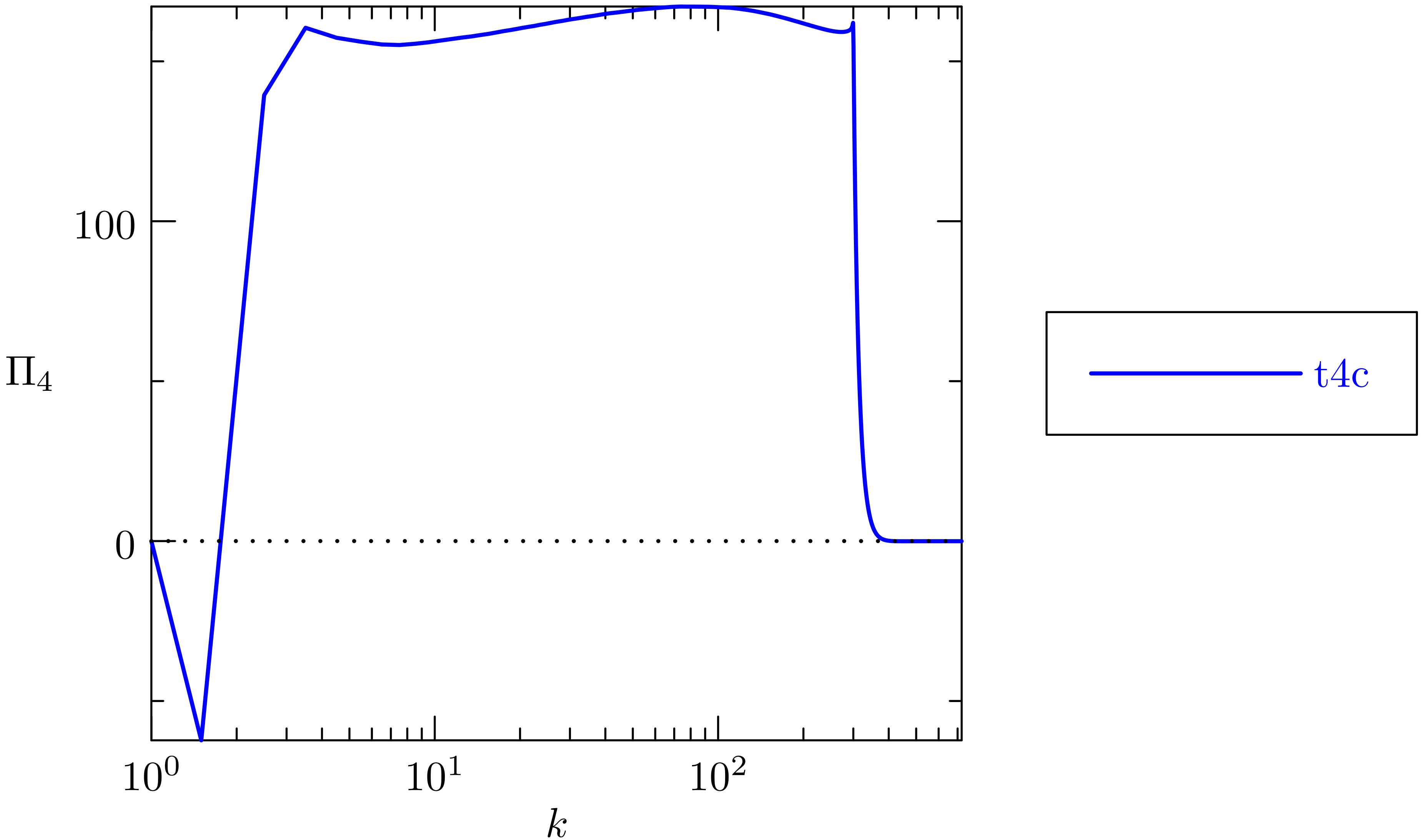
$$\frac{d}{dt} Z_4 = \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{p},\boldsymbol{q}} \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{p},\boldsymbol{q}} S_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \right]$$

$$\frac{d}{dt} Z_4 = N^2 \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{j}} \omega^3(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{j}} S(x_{\boldsymbol{j}}) \omega^2(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} \right]$$

$$\doteq \sum_{k} T_4(k).$$

Here $S_{\boldsymbol{k}}$ is the nonlinear source term in $\frac{\partial}{\partial t}\omega_{\boldsymbol{k}}$.

Downscale Transfer of Z_4



Nonlinear transfer Π_4 of Z_4 averaged over $t \in [250, 740]$.

Dealiasing: Explicit 2/4 Zero Padding

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  $\Rightarrow$ even though a 2048×2048 pseudospectral simulation was used, the maximum physical wavenumber retained in each direction was 512.

- Instead, use *implicit padding*: roughly twice as fast, 1/2 of the memory required by conventional explicit padding.

Discrete Convolutions

- Discrete linear convolution sums based on the fast Fourier transform (FFT) algorithm [Gauss 1866], [Cooley & Tukey 1965] have become important tools for:
  - image filtering;
  - digital signal processing;
  - correlation analysis;
  - pseudospectral simulations.

Discrete Cyclic Convolution

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$$\zeta_N = \exp\left(\frac{2\pi i}{N}\right).$$

- The fast Fourier transform method exploits the properties that $\zeta_N^r = \zeta_{N/r}$ and $\zeta_N^N = 1$.

- The unnormalized *backwards discrete Fourier transform* of $\{F_k : k = 0, \ldots, N\}$ is

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$$\sum_{j=0}^{N-1} \zeta_N^{\ell j} = \begin{cases} N & \text{if } \ell = sN \text{ for } s \in \mathbb{Z}, \\ \dfrac{1-\zeta_N^{\ell N}}{1-\zeta_N^{\ell}} = 0 & \text{otherwise.} \end{cases}$$

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- The pseudospectral method requires a *linear convolution*.

Convolution Theorem

$$\sum_{j=0}^{N-1} f_j g_j \zeta_N^{-jk} = \sum_{j=0}^{N-1} \zeta_N^{-jk} \left(\sum_{p=0}^{N-1} \zeta_N^{jp} F_p \right) \left(\sum_{q=0}^{N-1} \zeta_N^{jq} G_q \right)$$

$$= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} F_p G_q \sum_{j=0}^{N-1} \zeta_N^{(-k+p+q)j}$$

$$= N \sum_{s} \sum_{p=0}^{N-1} F_p G_{k-p+sN}.$$

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- This can be achieved by choosing $N \geq 2m-1$.

- That is, one must *zero pad* input data vectors of length m to length $N \geq 2m - 1$:

$\{F_k\}_{k=0}^{m-1}$

$\{G_k\}_{k=0}^{m-1}$

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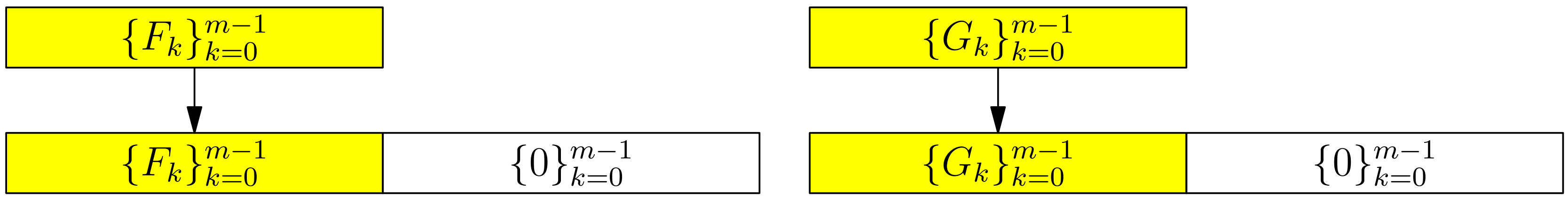

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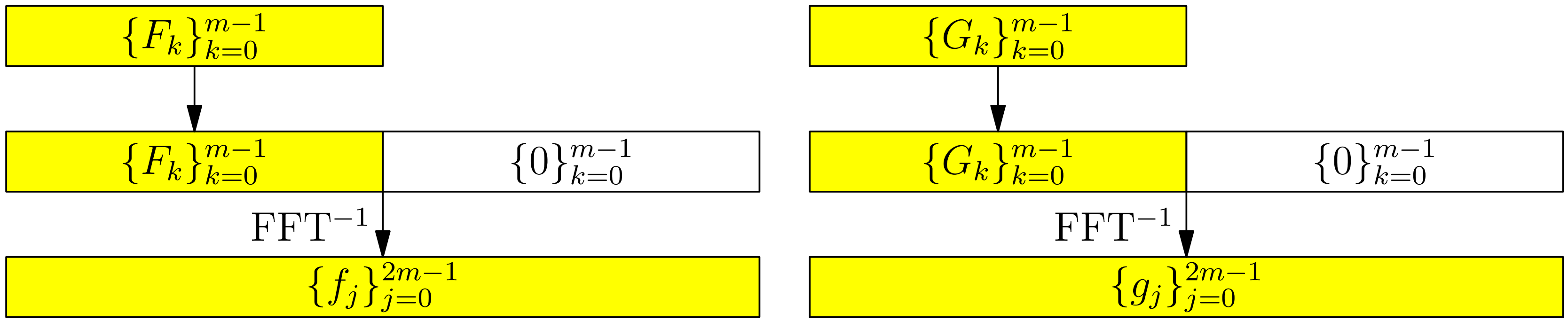

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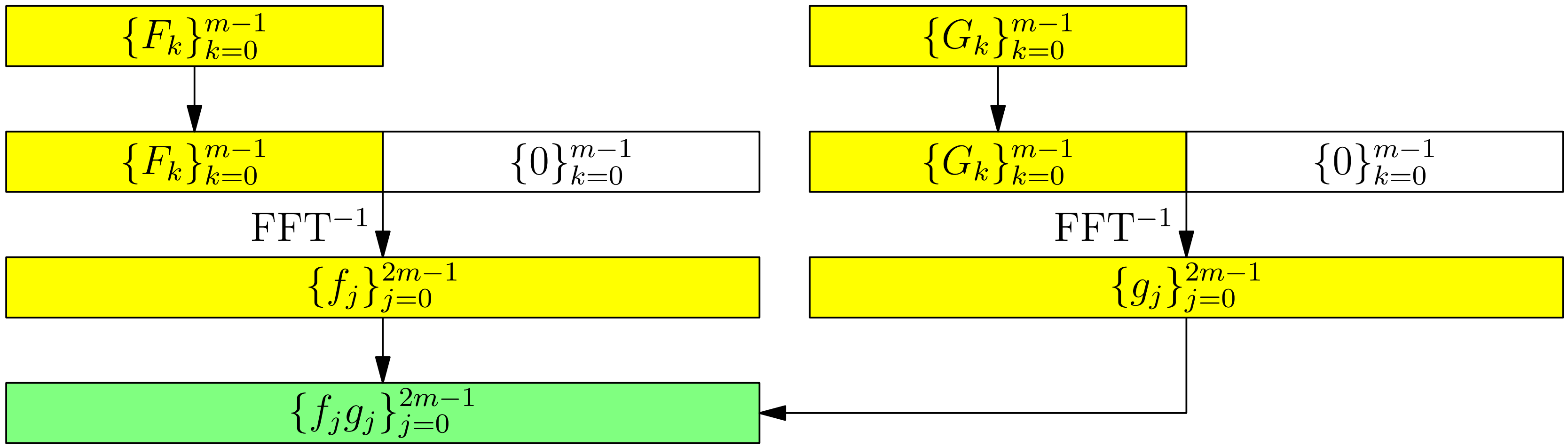

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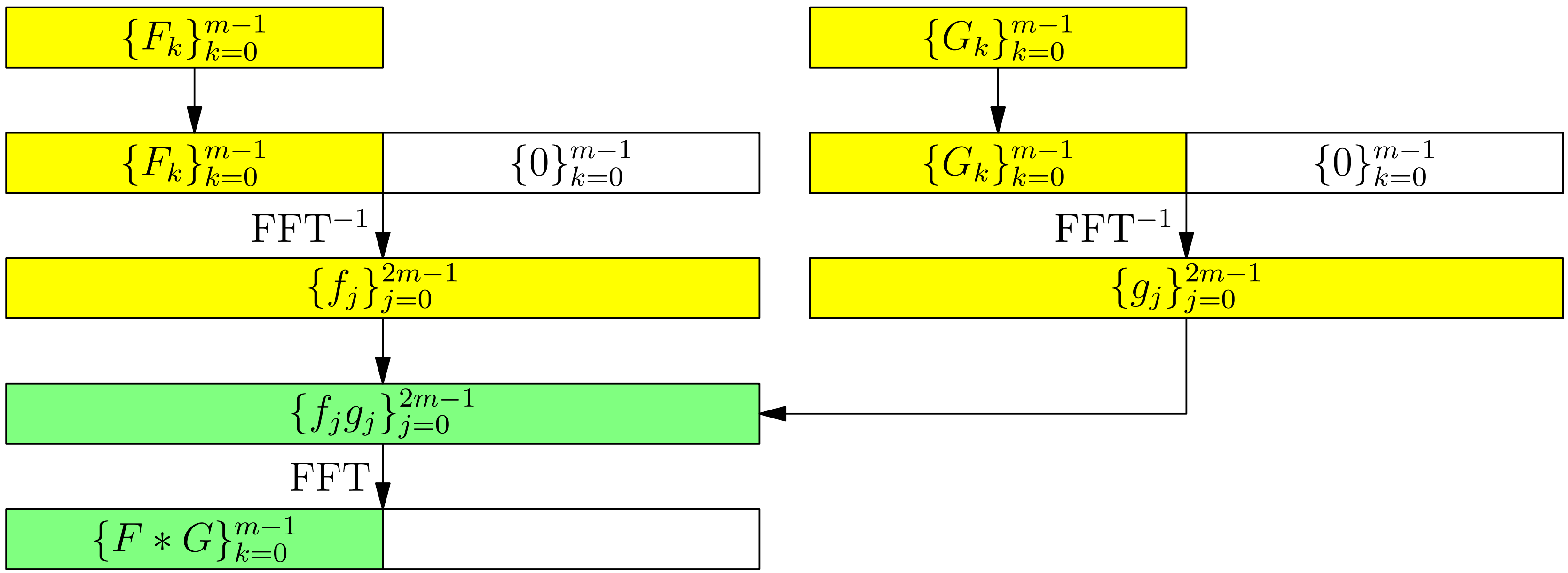

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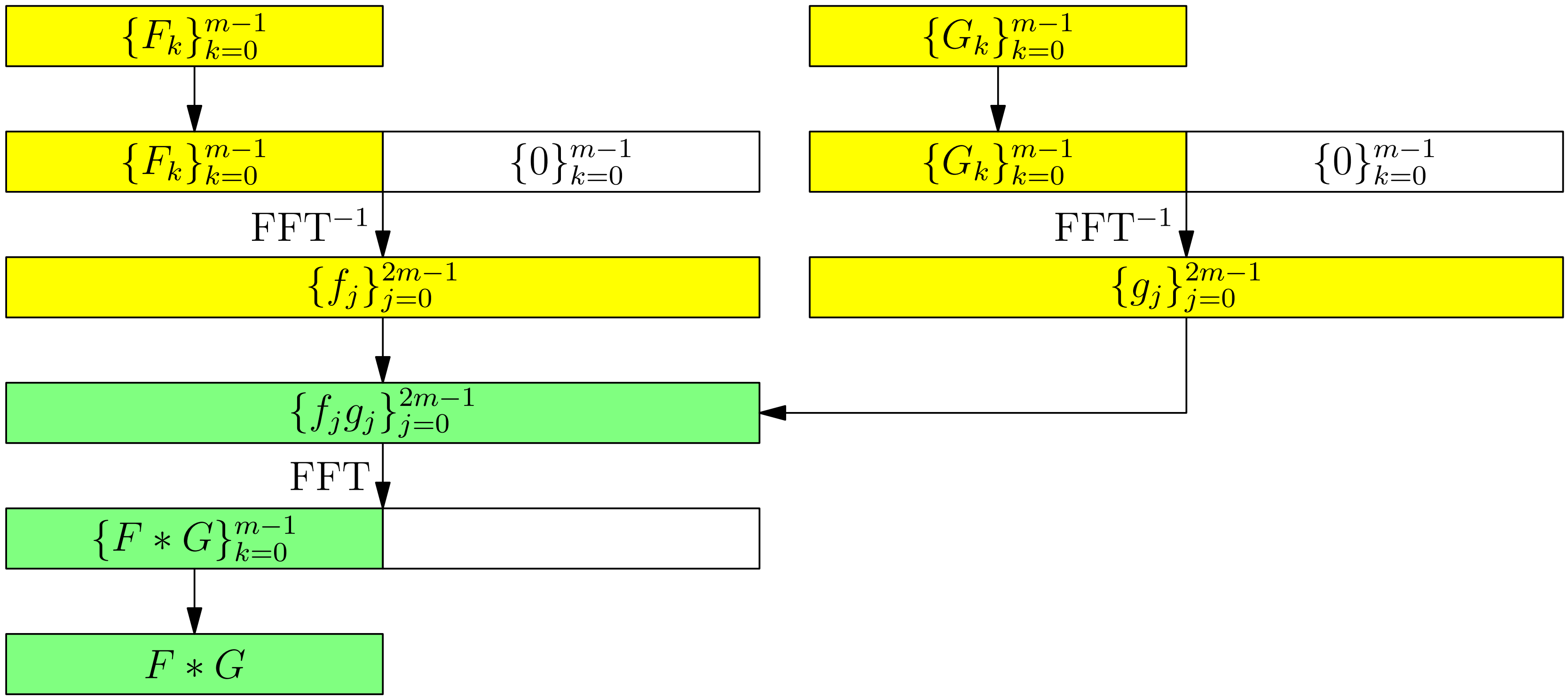

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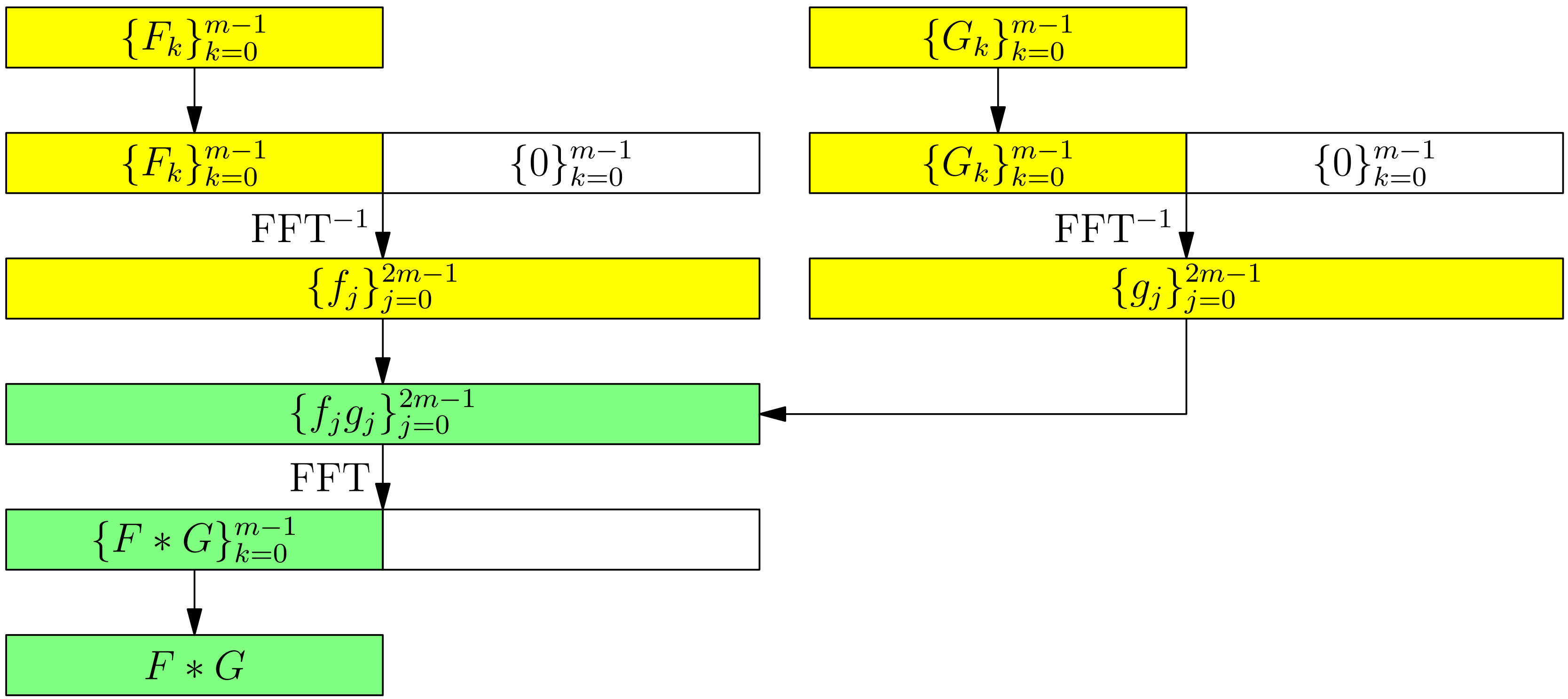

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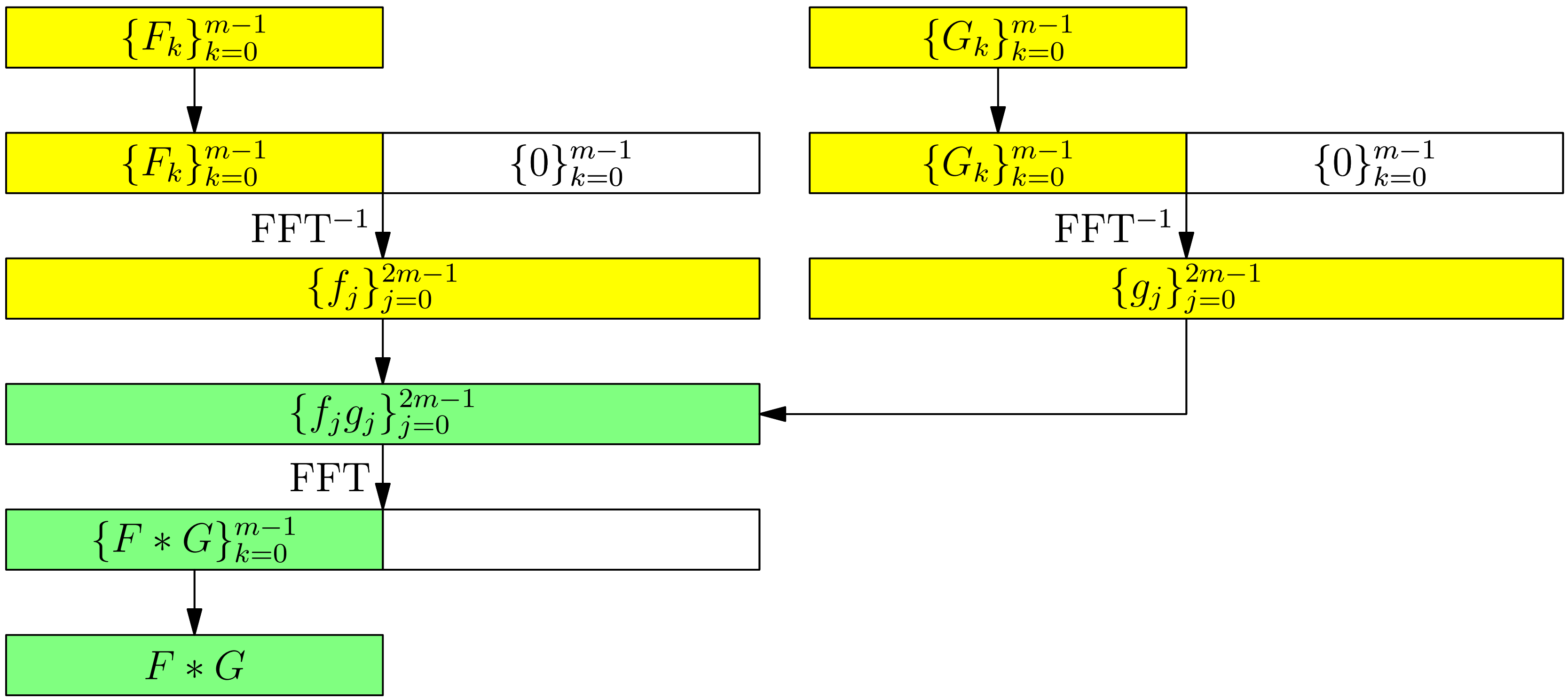

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- Since FFT sizes with small prime factors in practice yield the most efficient implementations, the padding is normally extended to $N = 2m$.

Pruned FFTs

- Although explicit padding seems like an obvious waste of memory and computation, the conventional wisdom on avoiding this waste is well summed up by Steven G. Johnson, coauthor of the **FFTW** ("Fastest Fourier Transform in the West") library [Frigo & Johnson]:

> *The most common case where people seem to want a pruned FFT is for zero-padded convolutions, where roughly 50% of your inputs are zero (to get a linear convolution from an FFT-based cyclic convolution). Here, a pruned FFT is hardly worth thinking about, at least in one dimension. In higher dimensions, matters change (e.g. for a 3d zero-padded array about 1/8 of your inputs are non-zero, and one can fairly easily save a factor of two or so simply by skipping 1d sub-transforms that are zero).*

Implicit Padding

- Let $N = 2m$. For $j = 0, \ldots, 2m-1$ we want to compute

$$f_j = \sum_{k=0}^{2m-1} \zeta_{2m}^{jk} F_k.$$

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- If $F_k = 0$ for $k \geq m$, one can easily avoid looping over the unwanted zero Fourier modes by decimating in wavenumber:

$$f_{2\ell} = \sum_{k=0}^{m-1} \zeta_{2m}^{2\ell k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} F_k,$$

$$f_{2\ell+1} = \sum_{k=0}^{m-1} \zeta_{2m}^{(2\ell+1)k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} \zeta_{2m}^{k} F_k, \qquad \ell = 0, 1, \ldots m-1.$$

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- This requires computing two subtransforms, each of size m, for an overall computational scaling of order $2m \log_2 m = N \log_2 m$.

- Odd and even terms of the convolution can then be computed separately, multiplied term-by-term, and transformed again to Fourier space:

$$2mF_k = \sum_{j=0}^{2m-1} \zeta_{2m}^{-kj} f_j = \sum_{\ell=0}^{m-1} \zeta_{2m}^{-k2\ell} f_{2\ell} + \sum_{\ell=0}^{m-1} \zeta_{2m}^{-k(2\ell+1)} f_{2\ell+1}$$

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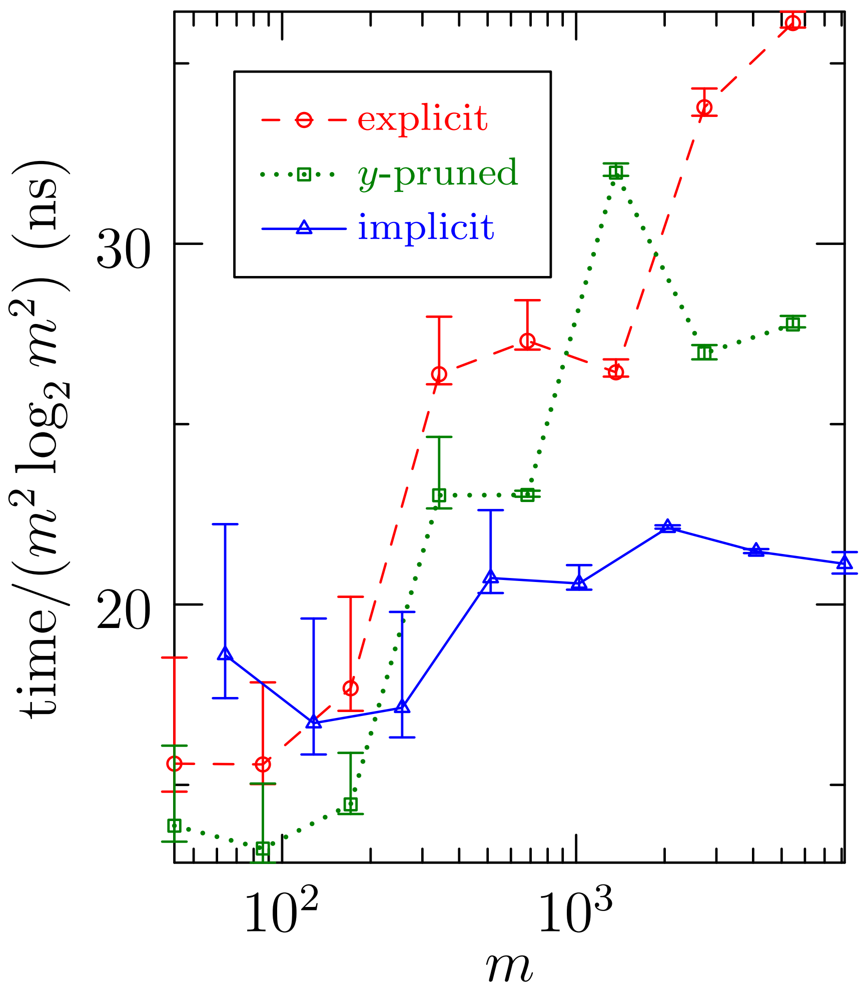
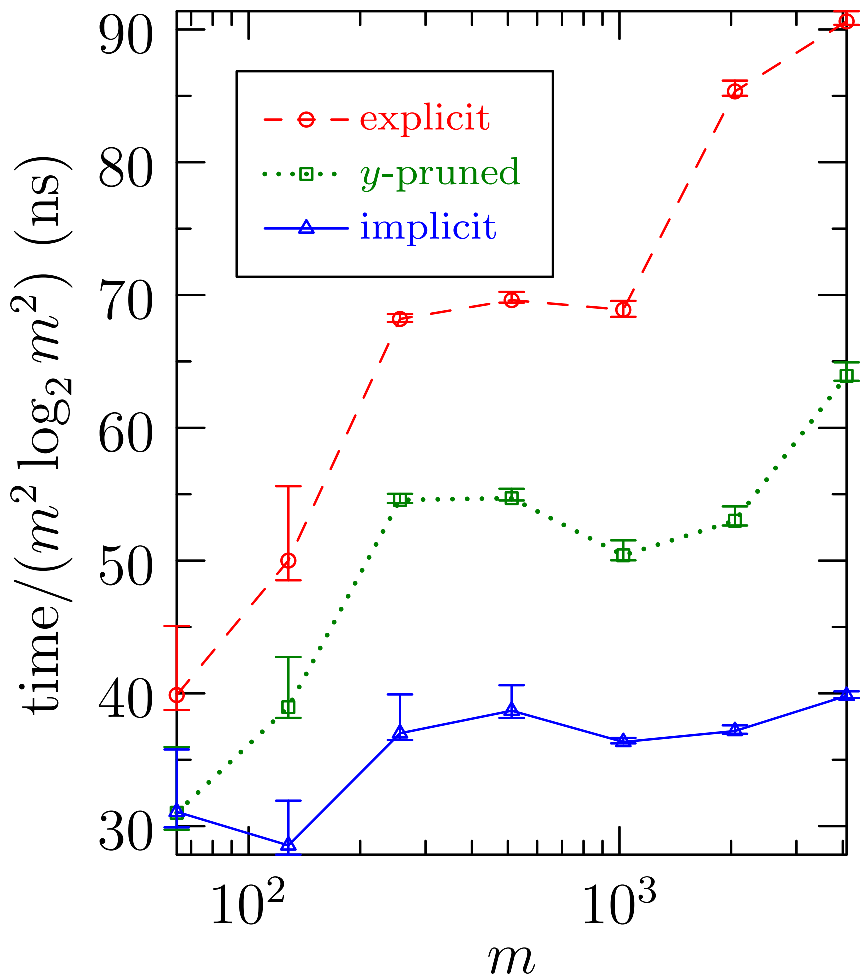
- This in-place convolution requires six out-of-place transforms, thereby avoiding bit reversal at all levels.

```
Input: vector f, vector g
Output: vector f
u ← fft^{-1}(f);
v ← fft^{-1}(g);
u ← u * v;
for k = 0 to m - 1 do
  | f[k] ← ζ_{2m}^{k} f[k];
  | g[k] ← ζ_{2m}^{k} g[k];
end
v ← fft^{-1}(f);
f ← fft^{-1}(g);
v ← v * f;
f ← fft(u);
u ← fft(v);
for k = 0 to m - 1 do
  | f[k] ← f[k] + ζ_{2m}^{-k} u[k];
end
return f/(2m);
```

2D Binary Convolution: Implicit 2/3 Zero Padding



2D Ternary Convolution: Implicit 2/4 Zero Padding



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- Highly optimized versions of these routines have been implemented as a software layer **FFTW++** on top of the **FFTW** library and released under the Lesser GNU Public License.

Conclusions

- Even though higher-order Casimir invariants do not survive wavenumber truncation, it is possible, with sufficiently well resolved simulations, to check whether they cascade to large or small scales.
- Numerical evidence suggests that in the enstrophy inertial range there is a direct cascade of the globally integrated ω^4 inviscid invariant to small scales.
- Memory savings: in d dimensions implicit padding asymptotically uses $(2/3)^{d-1}$ or $(1/2)^{d-1}$ of the memory require by conventional explicit padding.
- Highly optimized versions of these routines have been implemented as a software layer `FFTW++` on top of the `FFTW` library and released under the Lesser GNU Public License.
- With the advent of this `FFTW++` library, writing a high-performance dealiased pseudospectral code is now a relatively straightforward exercise.

Asymptote: 2D & 3D Vector Graphics Language

Andy Hammerlindl, John C. Bowman, Tom Prince

`http://asymptote.sf.net`

(freely available under the GNU public license)

Asymptote Lifts TEX to 3D

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$$

`http://asymptote.sf.net`

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