Realizable Markovian Closures

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Statistical Closures

- Fundamental nonlinear equation:

$$\boxed{\left(\frac{\partial}{\partial t}+\nu_{\boldsymbol{k}}\right)\psi_{\boldsymbol{k}}(t)=\frac{1}{2}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\,d\boldsymbol{q}\,M_{\boldsymbol{kpq}}\psi_{\boldsymbol{p}}{}^{*}\psi_{\boldsymbol{q}}{}^{*},}$$

where $\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\,d\boldsymbol{q} \doteq \int d\boldsymbol{p}\,d\boldsymbol{q}\,\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$.

- Symmetrize the mode-coupling coefficients $M_{\boldsymbol{kpq}}$:

$$M_{\boldsymbol{kpq}}=M_{\boldsymbol{kqp}}.$$

- For certain real time-independent factors $\sigma_{\boldsymbol{k}}$:

$$\sigma_{\boldsymbol{k}}M_{\boldsymbol{kpq}}+\sigma_{\boldsymbol{p}}M_{\boldsymbol{pqk}}+\sigma_{\boldsymbol{q}}M_{\boldsymbol{qkp}}=0.$$

Moments

- Define the two-time correlation

$$C_{\boldsymbol{k}}(t,t') \doteq \left\langle \psi_{\boldsymbol{k}}(t)\psi_{\boldsymbol{k}}{}^*(t') \right\rangle$$

and one-time correlation

$$C_{\boldsymbol{k}}(t) \doteq C_{\boldsymbol{k}}(t,t) = \left\langle \left|\psi_{\boldsymbol{k}}(t)\right|^2 \right\rangle .$$

- The nonlinear Green's function $G_{\boldsymbol{k}}(t,t')$ is the infinitesimal response to an added source function $\bar{\eta}_{\boldsymbol{k}}$:

$$G_{\boldsymbol{k}}(t,t') \doteq \left\langle \frac{\delta\psi_{\boldsymbol{k}}(t)}{\delta\bar{\eta}_{\boldsymbol{k}}(t')} \right\rangle\bigg|_{\bar{\eta}_{\boldsymbol{k}}=0} .$$

Example: $\psi =$ 2D stream function

- Let

$$M_{\boldsymbol{kpq}} = \frac{\hat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{q}}{k^2}(q^2 - p^2)$$

- Conservation of energy

$$E = \frac{1}{2}\sum_{\boldsymbol{k}} k^2 C_{\boldsymbol{k}}(t)$$

and enstrophy

$$Z = \frac{1}{2}\sum_{\boldsymbol{k}} k^4 C_{\boldsymbol{k}}(t)$$

follow, using $\sigma_{\boldsymbol{k}} = k^2$ and $\sigma_{\boldsymbol{k}} = k^4$ in

$$\sigma_{\boldsymbol{k}} M_{\boldsymbol{kpq}} + \sigma_{\boldsymbol{p}} M_{\boldsymbol{pqk}} + \sigma_{\boldsymbol{q}} M_{\boldsymbol{qkp}} = 0.$$

Closure Tutorial

- Fundamental equation

$$\frac{\partial \psi}{\partial t} + \nu \psi = M \psi \psi .$$

- Second moment:

$$\frac{\partial \langle \psi \psi \rangle}{\partial t} = 2 \left\langle \frac{\partial \psi}{\partial t} \psi \right\rangle = -2\nu \langle \psi \psi \rangle + 2M \langle \psi \psi \psi \rangle .$$

- Gaussian approximation $\langle \psi \psi \psi \rangle = 0$ $\Rightarrow$ linear theory!
- Instead, formulate the equation for $\langle \psi \psi \psi \rangle$:

$$\frac{\partial}{\partial t} \langle \psi \psi \psi \rangle + 3\nu \langle \psi \psi \psi \rangle = 3M \langle \psi \psi \psi \psi \rangle .$$

Quasinormal Closure

- Adopt Gaussian initial conditions at $t = 0$. Letting $\bar{\psi} \doteq \psi(\bar{t})$,

$$\langle\psi\psi\psi\rangle = 3M \int_0^t d\bar{t}\, e^{-3\nu(t-\bar{t})} \left\langle\bar{\psi}\bar{\psi}\bar{\psi}\bar{\psi}\right\rangle,$$

- Make the quasinormal approximation:

$$\left\langle\bar{\psi}\bar{\psi}\bar{\psi}\bar{\psi}\right\rangle = 3 \left\langle\bar{\psi}\bar{\psi}\right\rangle \left\langle\bar{\psi}\bar{\psi}\right\rangle$$

  For Gaussian statistics, this holds exactly.

- We arrive at the quasinormal closure:

$$\boxed{\frac{\partial}{\partial t}\langle\psi\psi\rangle + 2\nu\langle\psi\psi\rangle = 18MM \int_0^t d\bar{t}\, e^{-3\nu(t-\bar{t})} \left\langle\bar{\psi}\bar{\psi}\right\rangle \left\langle\bar{\psi}\bar{\psi}\right\rangle.}$$

- [Ogura 1963], [Orszag 1977]: the quasinormal closure can incorrectly predict negative energies!

Renormalization

- Better to renormalize: replace the linear Green's function

$$G^{(0)}(t,\bar{t}) \equiv e^{-3\nu(t-\bar{t})}H(t-\bar{t}),$$

where H is the Heaviside step function, by the statistical mean G of the perturbed nonlinear Green's function $\widetilde{G}$:

$$\frac{\partial}{\partial t}\widetilde{G} + \nu\widetilde{G} - 2M\psi\widetilde{G} = \delta(t-\bar{t}).$$

- The equations for $C \doteq \langle\psi\psi\rangle$ and G take on the form

$$\frac{\partial}{\partial t}C + 2\nu C = 18MM\int_0^t d\bar{t}\, G\, C\, C,$$

$$\frac{\partial}{\partial t}G + \nu G = 9MM\int_0^t d\bar{t}\, G\, C\, G + \delta(t-\bar{t}).$$

General form of a closure:

$$\left(\frac{\partial}{\partial t}+\nu_{\boldsymbol{k}}\right)C_{\boldsymbol{k}}(t,t')+\overbrace{\int_0^t d\bar{t}\,\Sigma_{\boldsymbol{k}}(t,\bar{t})C_{\boldsymbol{k}}(\bar{t},t')}^{\text{nonlinear (eddy) damping}} = \overbrace{\int_0^{t'} d\bar{t}\,\mathcal{F}_{\boldsymbol{k}}(t,\bar{t})G^*_{\boldsymbol{k}}(t',\bar{t})}^{\text{nonlinear noise}},$$

$$\left(\frac{\partial}{\partial t}+\nu_{\boldsymbol{k}}\right)G_{\boldsymbol{k}}(t,t')+\int_{t'}^t d\bar{t}\,\Sigma_{\boldsymbol{k}}(t,\bar{t})G_{\boldsymbol{k}}(\bar{t},t')=\delta(t-t').$$

- Direct-interaction approximation (DIA):

$$\Sigma_{\boldsymbol{k}}(t,\bar{t})=-\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\,d\boldsymbol{q}\,M_{\boldsymbol{kpq}}M^*_{\boldsymbol{pqk}}G^*_{\boldsymbol{p}}(t,\bar{t})C^*_{\boldsymbol{q}}(t,\bar{t}),$$

$$\mathcal{F}_{\boldsymbol{k}}(t,\bar{t})=\frac{1}{2}\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\,d\boldsymbol{q}\,M_{\boldsymbol{kpq}}M^*_{\boldsymbol{kpq}}C^*_{\boldsymbol{p}}(t,\bar{t})C^*_{\boldsymbol{q}}(t,\bar{t}).$$

Advantages of the DIA

- Reduces correctly to perturbation theory.
- Produces two-time spectral information.
- The DIA can be formally written as

$$\boxed{C_{\boldsymbol{k}} = G_{\boldsymbol{k}}\, \mathcal{F}_{\boldsymbol{k}}\, G^{\dagger}_{\boldsymbol{k}},}$$

- Whenever $\mathcal{F}_{\boldsymbol{k}}$ is a positive definite matrix $\left\langle f_{\boldsymbol{k}}(t) f^{*}_{\boldsymbol{k}}(t')\right\rangle$, the DIA is the exact statistical solution for the generalized Langevin equation

$$\boxed{\left(\frac{\partial}{\partial t} + \nu_{\boldsymbol{k}}\right)\psi_{\boldsymbol{k}}(t) + \int_0^t d\bar{t}\, \Sigma_{\boldsymbol{k}}(t,\bar{t})\, \psi_{\boldsymbol{k}}(\bar{t}) = f_{\boldsymbol{k}}(t),}$$

Disadvantages of the DIA

- Kramer, Majda, and Vanden-Eijnden [2003] have apparently found a case of passive scalar advection with a fluctuating random sweep where realizability is violated despite the fact that [Kraichnan 1958b] claims the DIA is the exact solution to a random coupling model.
- Violates random Galilean invariance.
- Predicts an energy range $E(k) \sim k^{-3/2}$ instead of $k^{-5/3}$.
- Predicts a 2D enstrophy range $E(k) \sim k^{-5/2}$ instead of k^{-3}.
- Contains time-history integrals: nontrivial to compute.
- Only handles second-order statistics; mistreats higher-order coherent structures.

Eddy-Damped QuasiNormal Markovian closure

- The EDQNM approximates the DIA time-history convolutions in favour of a triad interaction time.

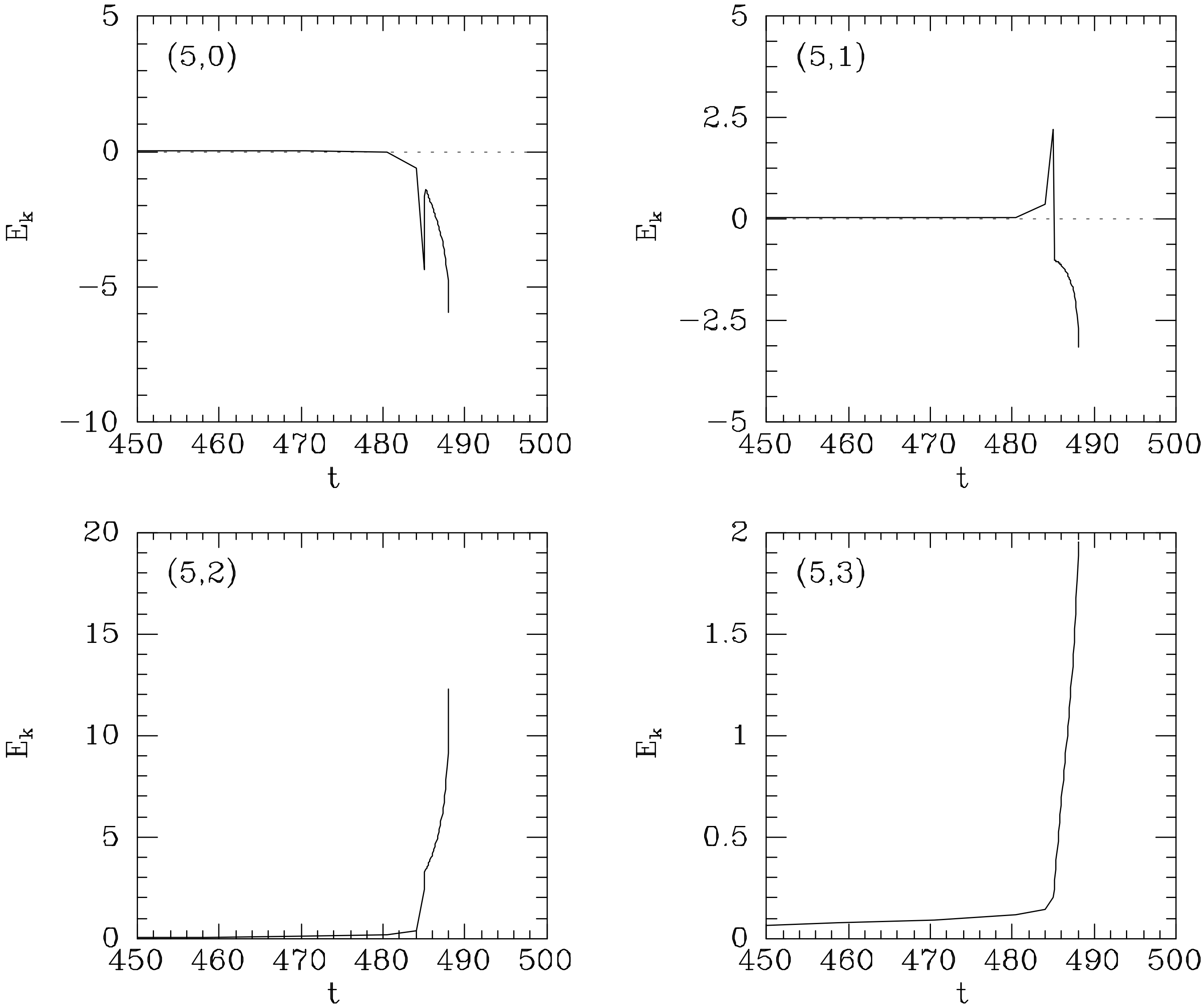
Advantages:

- Much faster than DIA.
- In the absence of wave phenomena, it is realizable: it predicts the exact statistics of an underlying Langevin equation.

Disadvantages:

- Assumes a Fluctuation–Dissipation relation.
- Only predicts one-time spectral information.
- Does not take account of time-history effects accurately.
- Proof of realizability breaks down in the presence of Rossby or drift waves.
- No general multiple-field formulation.

Nonrealizability of the EDQNM



The DIA-based EDQNM

- The DIA equation for the one-time correlation function still contains unknown two-time correlation functions in $\mathcal{F}_{\boldsymbol{k}}$.
- In thermal equilibrium, the Fluctuation–Dissipation (FD) theorem holds:

$$C_{\boldsymbol{k}}(t,t') = G_{\boldsymbol{k}}(t,t')C_{\boldsymbol{k}}(\infty) \qquad (t > t').$$

- In thermal equilibrium, statistical quantities are stationary, so $C_{\boldsymbol{k}}(t,t') = \mathcal{C}_{\boldsymbol{k}}(t-t')$. Hence $C_{\boldsymbol{k}}(t) = \mathcal{C}_{\boldsymbol{k}}(0) = C_{\boldsymbol{k}}(t')$.
- So we must replace the FD theorem by either

$$C_{\boldsymbol{k}}(t,t') = G_{\boldsymbol{k}}(t,t')C_{\boldsymbol{k}}(t) \qquad (t > t')$$

or

$$C_{\boldsymbol{k}}(t,t') = G_{\boldsymbol{k}}(t,t')C_{\boldsymbol{k}}(t') \qquad (t > t').$$

- EDQNM adopts the 1st form: unlike the 2nd form this leads to a realizable closure [Orszag 1977] in the absence of waves.

- In terms of the triad interaction time

$$\theta_{\boldsymbol{kpq}}(t) \doteq \int_0^t d\bar{t}\, G_{\boldsymbol{k}}(t,\bar{t})\, G_{\boldsymbol{p}}(t,\bar{t})\, G_{\boldsymbol{q}}(t,\bar{t}).$$

the Markovianized DIA can be written in the compact form

$$\left(\frac{\partial}{\partial t} + 2\,\mathrm{Re}\,\nu_{\boldsymbol{k}}\right) C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\,\widehat{\eta}_{\boldsymbol{k}}(t)\, C_{\boldsymbol{k}}(t) = 2F_{\boldsymbol{k}}(t)$$

by defining a nonlinear damping rate,

$$\widehat{\eta}_{\boldsymbol{k}}(t) \doteq -\int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{q}}(t),$$

and a nonlinear noise term,

$$F_{\boldsymbol{k}}(t) \doteq \frac{1}{2}\,\mathrm{Re} \int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\, \left|M_{\boldsymbol{kpq}}\right|^2 \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{p}}(t)\, C_{\boldsymbol{q}}(t).$$

EDQNM

- Replace the Green's function equation by the Markovian form

$$\frac{\partial}{\partial t} G_{\boldsymbol{k}}(t,t') + \eta_{\boldsymbol{k}}(t)\, G_{\boldsymbol{k}}(t,t') = \delta(t-t'),$$

where $\eta_{\boldsymbol{k}}(t) \doteq \nu_{\boldsymbol{k}} + \widehat{\eta}_{\boldsymbol{k}}(t)$. What results is the EDQNM:

$$\frac{\partial}{\partial t} C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\,\eta_{\boldsymbol{k}}(t)\, C_{\boldsymbol{k}}(t) = 2F_{\boldsymbol{k}}(t),$$

$$\eta_{\boldsymbol{k}}(t) \doteq \nu_{\boldsymbol{k}} - \int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\; M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{q}}(t),$$

$$F_{\boldsymbol{k}}(t) \doteq \frac{1}{2}\,\mathrm{Re} \int_{\Delta_{\boldsymbol{k}}} d\boldsymbol{p}\, d\boldsymbol{q}\; \left|M_{\boldsymbol{kpq}}\right|^2 \theta^*_{\boldsymbol{kpq}}(t)\, C_{\boldsymbol{p}}(t)\, C_{\boldsymbol{q}}(t),$$

$$\frac{\partial}{\partial t}\theta_{\boldsymbol{kpq}} + (\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}})\theta_{\boldsymbol{kpq}} = 1, \qquad \theta_{\boldsymbol{kpq}}(0) = 0.$$

The computational scaling of this system with time T is $\mathcal{O}T$, a vast improvement over the $\mathcal{O}T^3$ scaling of the DIA.

Test-Field Model (TFM)

- The test-field model [Kraichnan 1971, Kraichnan 1972] also approximates the DIA time-history convolutions.

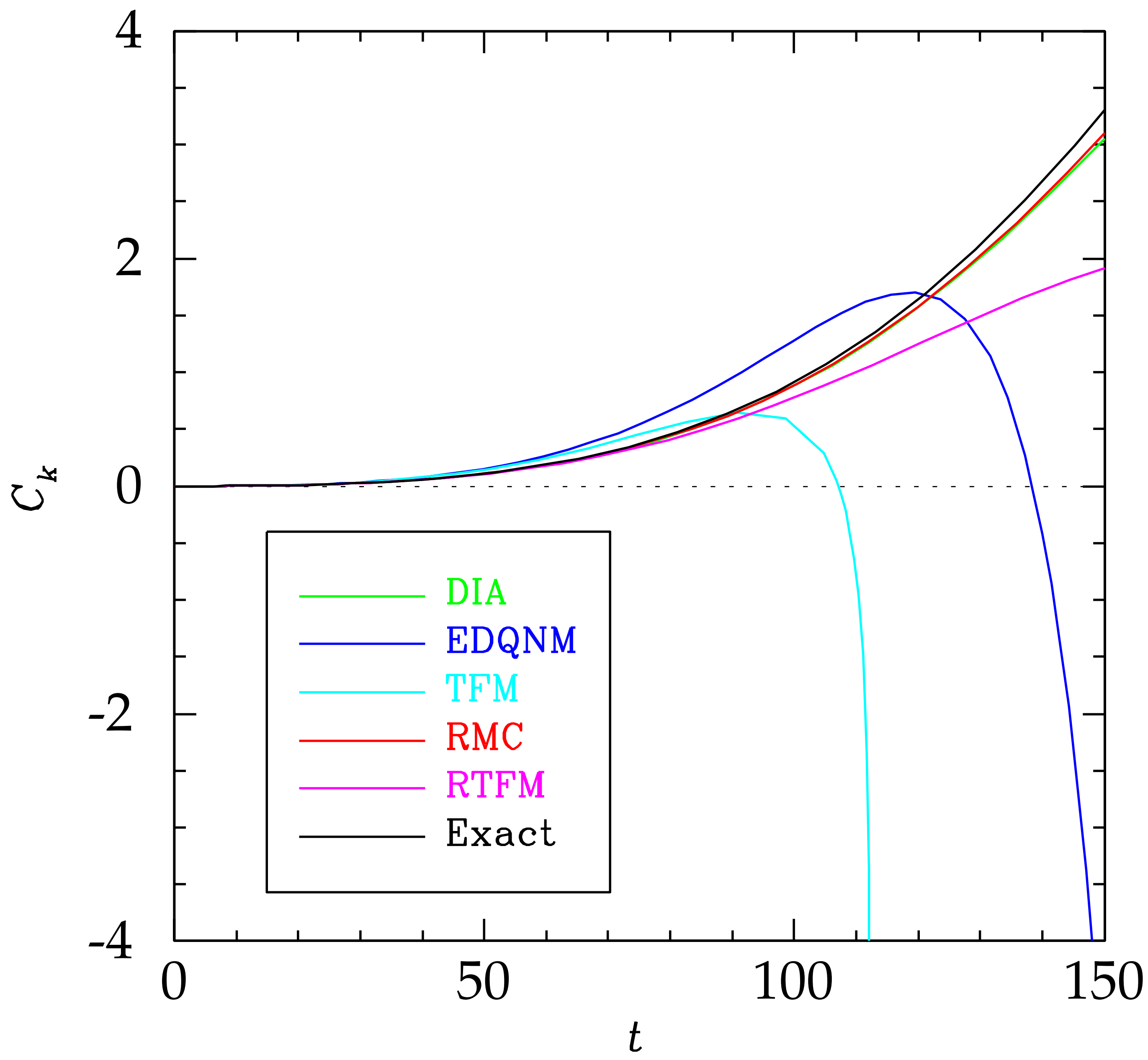
Advantages:

- Invariant to random Galilean transformations.
- Predicts a 2D enstrophy range spectrum k^{-3}.
- Much faster than DIA.

Disadvantages:

- Heuristic construction.
- Only predicts one-time spectral information.
- Does not take account of time-history effects accurately.
- Assumes a Fluctuation–Dissipation relation.
- Can predict negative energies if wave effects are present!

Nonrealizability of the EDQNM and TFM



Realizable Markovian Closure (RMC)

- Goal: Replace the FD Ansatz with a relation that reduces to the FD theorem in a steady state.
- EDQNM FD Ansatz:

$$\frac{C_{\boldsymbol{k}}(t,t')}{C_{\boldsymbol{k}}(t)} = G_{\boldsymbol{k}}(t,t') \qquad (t \geq t'),$$

- Langevin statistics:

$$\frac{C_{\boldsymbol{k}}(t,t')}{C_{\boldsymbol{k}}(t')} = G_{\boldsymbol{k}}(t,t') \qquad (t \geq t').$$

- Thermal equilibrium:

$$\frac{C_{\boldsymbol{k}}(t,t')}{C_{\boldsymbol{k}}(\infty)} = G_{\boldsymbol{k}}(t,t') \qquad (t \geq t').$$

Modified Fluctuation–Dissipation Ansatz

- In a non stationary state, (19) should be restated as a balance between the correlation coefficient and the response function (for $t \geq t'$):

$$\boxed{\underbrace{\overbrace{\frac{C_{\boldsymbol{k}}(t,t')}{C_{\boldsymbol{k}}^{1/2}(t)\,C_{\boldsymbol{k}}^{1/2}(t')}}^{\text{finite amplitude}}}_{\text{correlation coefficient}} = \underbrace{\overbrace{G_{\boldsymbol{k}}(t,t')}^{\text{infinitesimal}}}_{\text{response function}} .}$$

- Time scales of amplitude decorrelation and decay of infinitesimal disturbances should be equal, since these processes both occur by interaction with the turbulent background.

Realizability

- For unrestricted time arguments t and t':

$$\boxed{C_{\boldsymbol{k}}(t,t') = C_{\boldsymbol{k}}^{1/2}(t)\left[G_{\boldsymbol{k}}(t,t') + G_{\boldsymbol{k}}^{*}(t',t)\right]C_{\boldsymbol{k}}^{1/2*}(t').}$$

- $C_{\boldsymbol{k}}(t,t')$ positive-semidefinite $\iff$
$G_{\boldsymbol{k}}^{h}(t,t') \doteq G_{\boldsymbol{k}}(t,t') + G_{\boldsymbol{k}}^{*}(t',t)$ is positive-semidefinite.
We employ the following theorem [Bowman *et al.* 1993]:

Theorem 1: *Let* $\operatorname{Re}\eta_{\boldsymbol{k}}(t)$ *be continuous almost everywhere. The Hermitian function* $G_{\boldsymbol{k}}^{h}$ *defined by*

$$G_{\boldsymbol{k}}^{h}(t,t') \doteq \begin{cases} e^{-\int_{t'}^{t}\eta_{\boldsymbol{k}}(\bar{t})d\bar{t}}, & \text{for } t \ge t'; \\ e^{-\int_{t}^{t'}\eta_{\boldsymbol{k}}^{*}(\bar{t})d\bar{t}}, & \text{for } t < t', \end{cases}$$

is positive-semidefinite $\iff$ $\operatorname{Re}\eta_{\boldsymbol{k}}(t) \ge 0$ *for almost all* t.

- Subject to the restriction $\mathrm{Re}\,\eta_{\boldsymbol{k}} \geq 0$, it follows that

$$C_{\boldsymbol{k}}(t,t) = \int d\bar{t}\, d\bar{\bar{t}}\, G_{\boldsymbol{k}}(t,\bar{t})\, \mathcal{F}_{\boldsymbol{k}}(\bar{t},\bar{\bar{t}})\, G^{*}_{\boldsymbol{k}}(t,\bar{\bar{t}}) \geq 0,$$

is real and non-negative, provided that the initial condition is non-negative.

Realizable Markovian Closure (RMC)

- Applying the modified FD Ansatz yields the RMC:

$$\frac{\partial}{\partial t} C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\,\eta_{\boldsymbol{k}}(t)\, C_{\boldsymbol{k}}(t) = 2F_{\boldsymbol{k}}(t)$$

$$\eta_{\boldsymbol{k}}(t) \doteq \nu_{\boldsymbol{k}} - \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \Theta^*_{\boldsymbol{pqk}}(t)\, C_{\boldsymbol{q}}^{1/2}(t) C_{\boldsymbol{k}}^{-1/2}(t)$$

$$2F_{\boldsymbol{k}}(t) \doteq \mathrm{Re} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \left| M_{\boldsymbol{kpq}} \right|^2 \Theta_{\boldsymbol{kpq}}(t) C_{\boldsymbol{p}}^{1/2}(t) C_{\boldsymbol{q}}^{1/2}(t)$$

$$\frac{\partial}{\partial t}\Theta_{\boldsymbol{kpq}} + [\eta_{\boldsymbol{k}} + \mathcal{P}(\eta_{\boldsymbol{p}}) + \mathcal{P}(\eta_{\boldsymbol{q}})]\Theta_{\boldsymbol{kpq}} = C_{\boldsymbol{p}}^{1/2} C_{\boldsymbol{q}}^{1/2},$$

  where $\mathcal{P}(\eta) \doteq \mathrm{Re}\,\eta\, H(\mathrm{Re}\,\eta) + i\,\mathrm{Im}\,\eta$ and H is the Heaviside unit step function.

- Although the steady-state EDQNM and RMC equations are identical, the RMC provides a realizable evolution to this state.

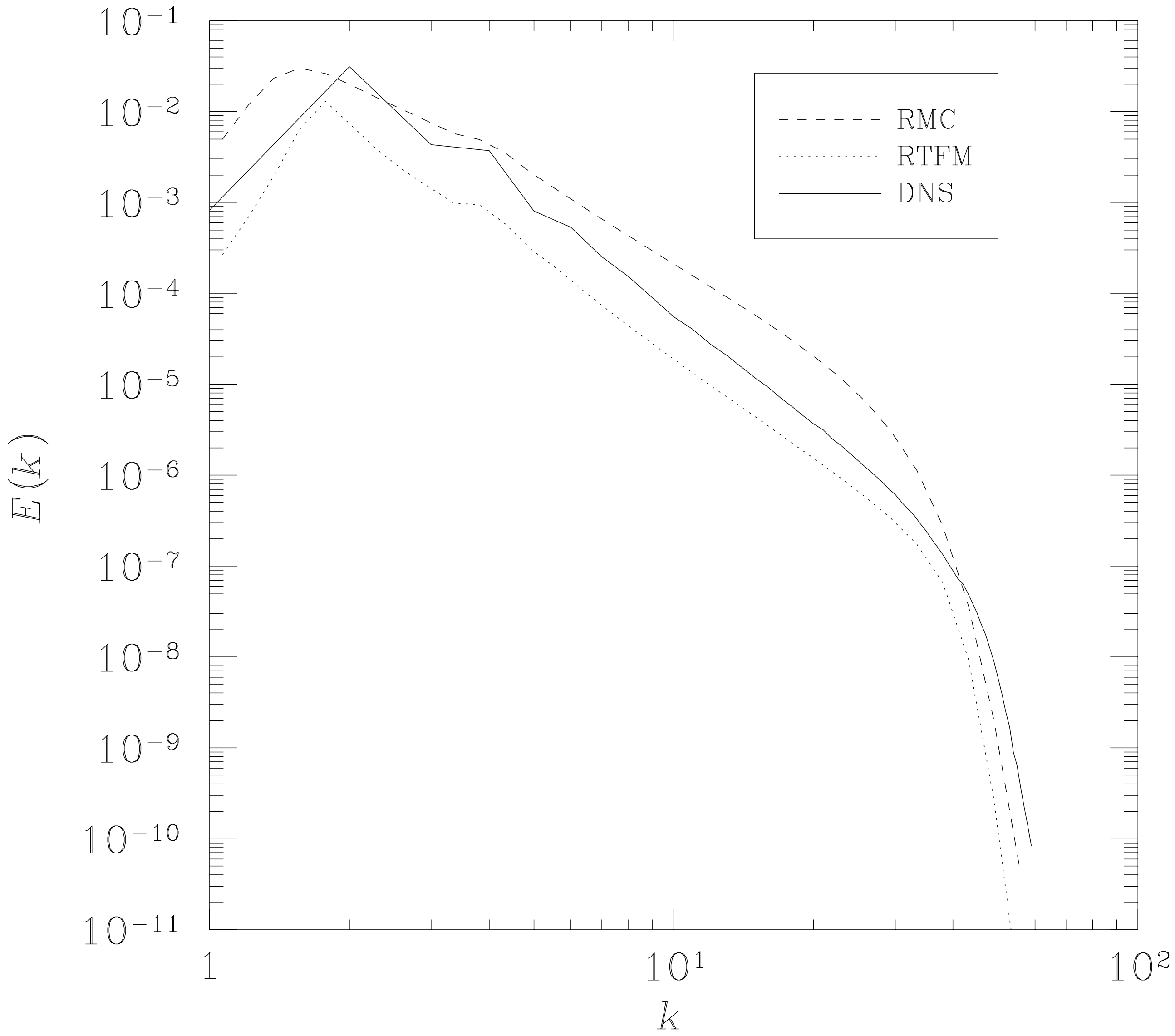
Realizable Test-Field Model (RTFM)

- Similarly, we have constructed a Realizable Test-Field Model [Bowman & Krommes 1997].
- Even for wave-free turbulence, the RMC and RTFM appear to be more representative of the true dynamics than the EDQNM and TFM.
- The RMC and RTFM possess underlying Langevin equations:

$$\frac{\partial}{\partial t}\psi + \eta\psi = f,$$

  which, unlike the EDQNM, does not assume δ-correlated statistics.
- It is also possible to design multiple-rate Markovian closures that allow for different decorrelation and infinitesimal perturbation decay rates; this may afford a more accurate treatment of non-white noise effects.

Comparison of RMC and RTFM with DNS

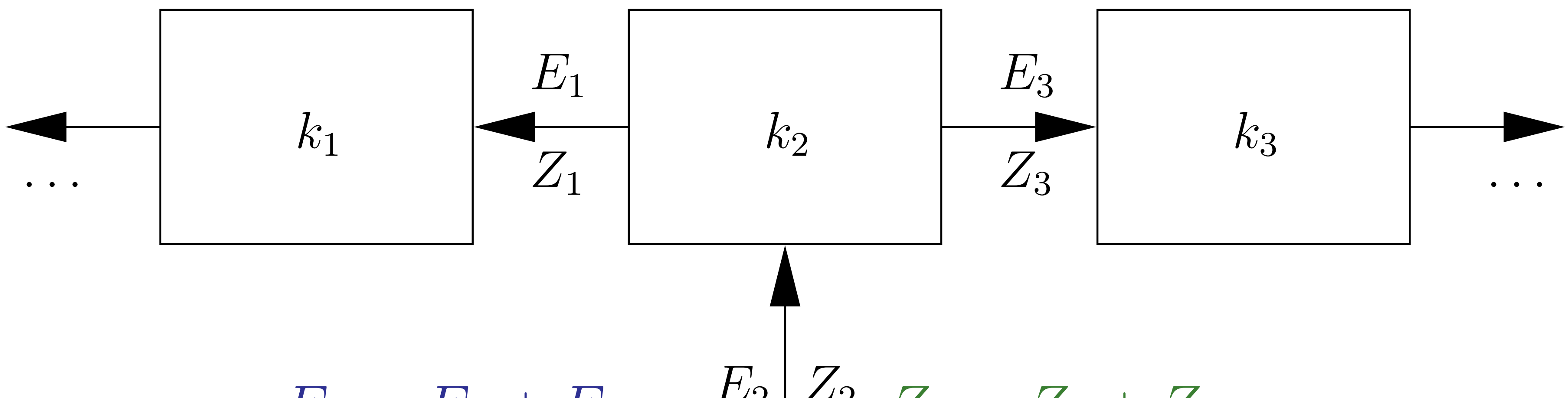


Alternatives

- Mapping Closures
- Kaneda's Lagrangian Renormalized Approximation (LRA) [Kaneda 1981]
- McComb's Local Energy Theory (LET) [McComb 1990]
- Direct Numerical Simulation
- Dynamic Subgrid Models
- Renormalization Group Theory
- Reduced Models:
  - Decimation
  - Empirical Orthogonal Eigenfunctions
  - Spectral Reduction: Bowman, Shadwick, Morrison [1999]
  - Stochastic Models

2D Turbulence

- Energy $E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ and enstrophy $Z = \frac{1}{2}\sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ are conserved.



$$E_2 = E_1 + E_3, \qquad Z_2 = Z_1 + Z_3.$$

- [Fjørtoft 1953]: energy cascades to large scales and enstrophy cascades to small scales.

- [Kraichnan 1967], [Leith 1968], and [Batchelor 1969] (KLB):
  $k^{-5/3}$ inverse energy cascade at large scales,
  k^{-3} direct enstrophy cascade at small scales.

- Let $s^2 = \overline{\sum_{\boldsymbol{k}} f_{\boldsymbol{k}}\omega^*_{\boldsymbol{k}}} \,/\, \overline{\sum_{\boldsymbol{k}} f_{\boldsymbol{k}}\frac{\omega^*_{\boldsymbol{k}}}{k^2}}$ be the ratio of mean enstrophy to energy injection.
- Typically, s will lie within the band of forced wavenumbers.
- Multiply the energy equation

$$\frac{1}{2k^2}\frac{\partial\left|\omega_{\boldsymbol{k}}\right|^2}{\partial t} + D_k\frac{\left|\omega_{\boldsymbol{k}}\right|^2}{k^2} = S_{\boldsymbol{k}}\frac{\omega^*_{\boldsymbol{k}}}{k^2} + f_{\boldsymbol{k}}\frac{\omega^*_{\boldsymbol{k}}}{k^2}$$

by s^2 and subtract the enstrophy equation

$$\frac{1}{2}\frac{\partial\left|\omega_{\boldsymbol{k}}\right|^2}{\partial t} + D_k\left|\omega_{\boldsymbol{k}}\right|^2 = S_{\boldsymbol{k}}\omega^*_{\boldsymbol{k}} + f_{\boldsymbol{k}}\omega^*_{\boldsymbol{k}}$$

$\Rightarrow$ steady-state balance equation [Tran & Bowman 2003]

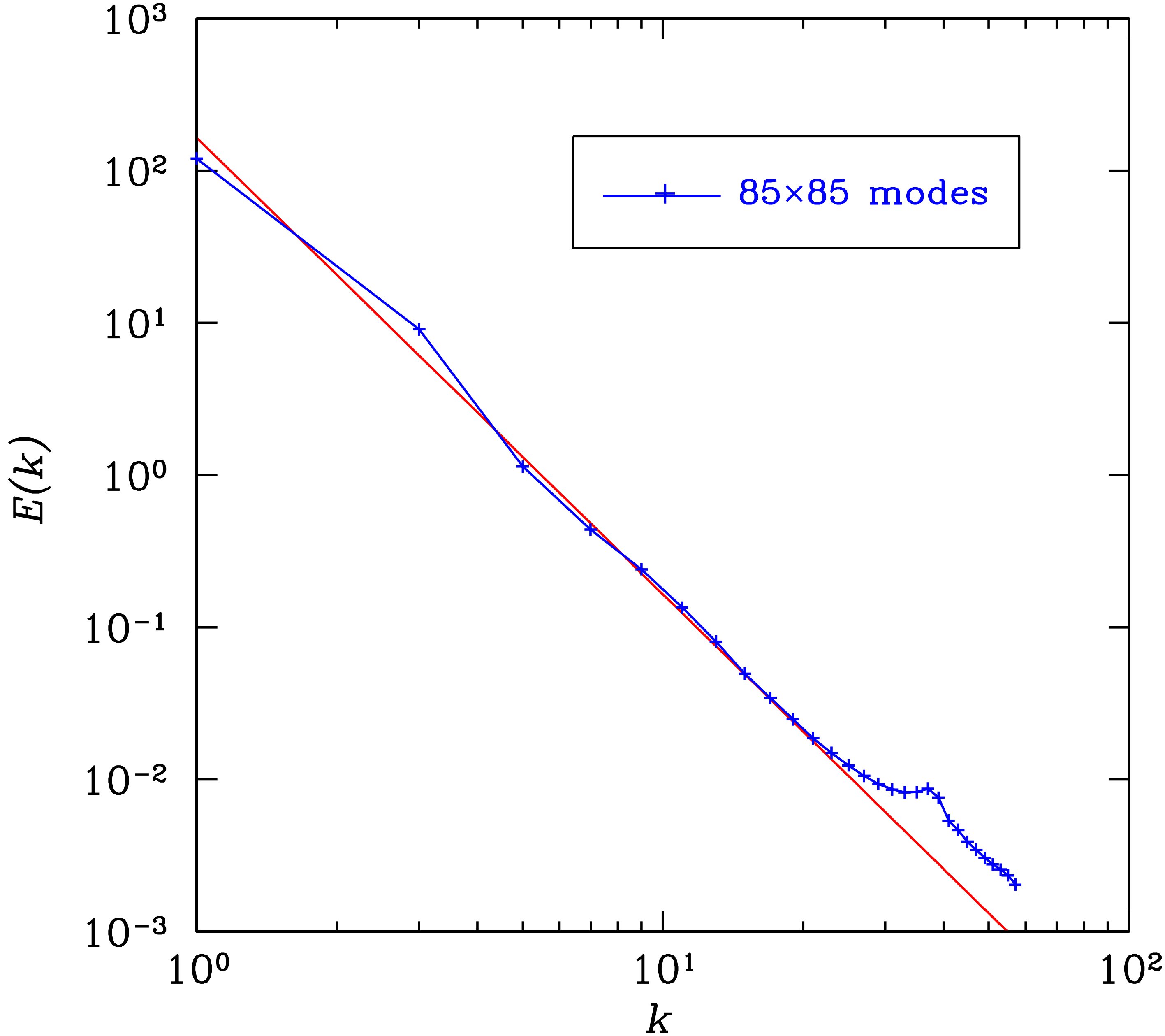
$$\sum_{k=1}^{s}(s^2-k^2)D_kE(k) = \sum_{k=s}^{\infty}(k^2-s^2)D_kE(k).$$

Balance Equation

- Small and large scale dynamics are intricately coupled:

$$\sum_{k=1}^{s}(s^2-k^2)D_kE(k)=\sum_{k=s}^{\infty}(k^2-s^2)D_kE(k).$$

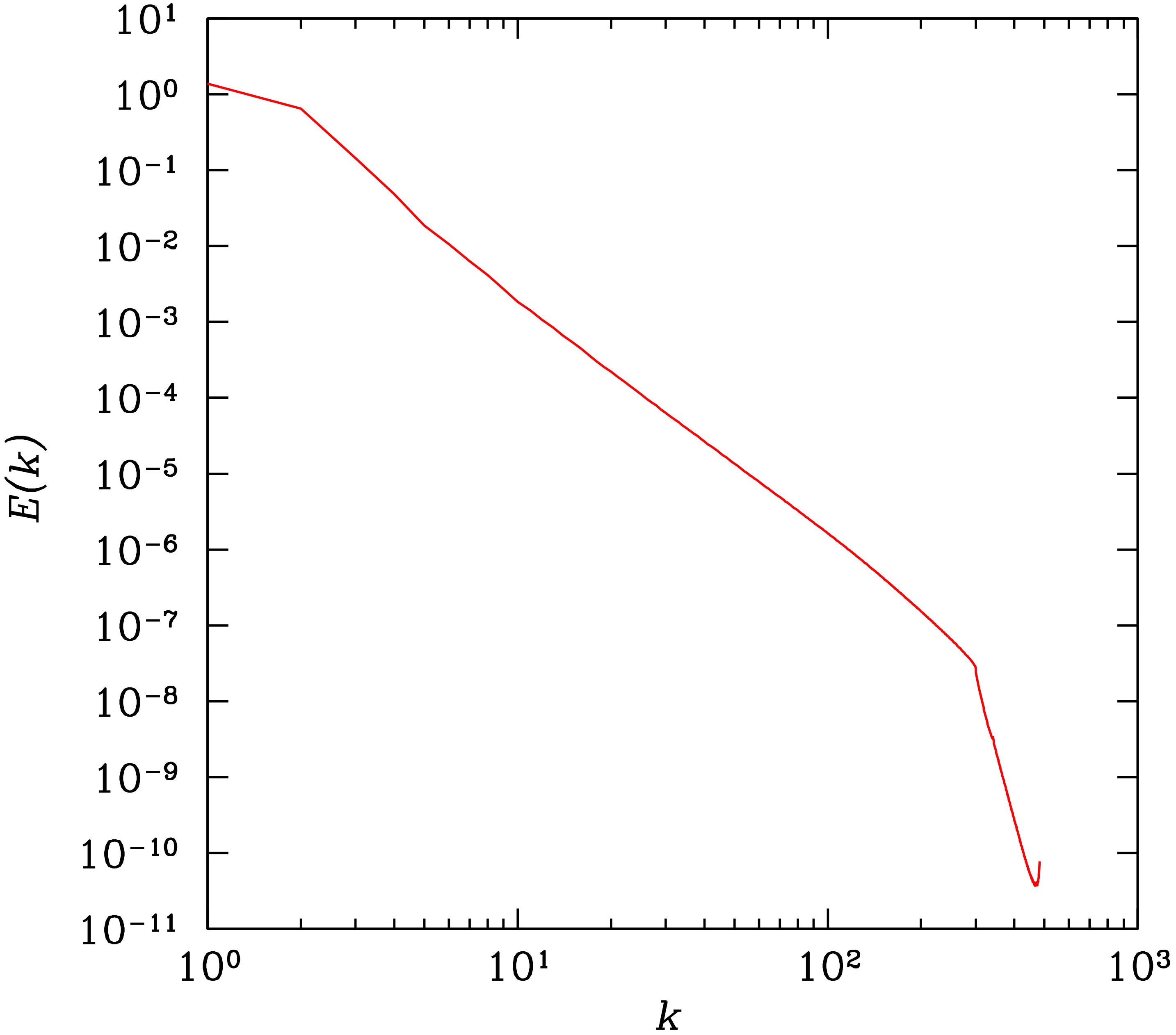
- Can be used to explain the discrepancy between the KLB prediction $E(k)\sim k^{-3}$ and the steep $\sim k^{-5}$ enstrophy-range spectrum typically seen in numerical simulations.
- Unbounded domain: everlasting inverse energy cascade.
- Bounded domain: upscale energy cascade is halted at the lowest wavenumber.
- The effect of this lower spectral boundary may be understood by replacing it with an external forcing.



Large-scale direct cascade (zero dissipation for $k < 40$)?

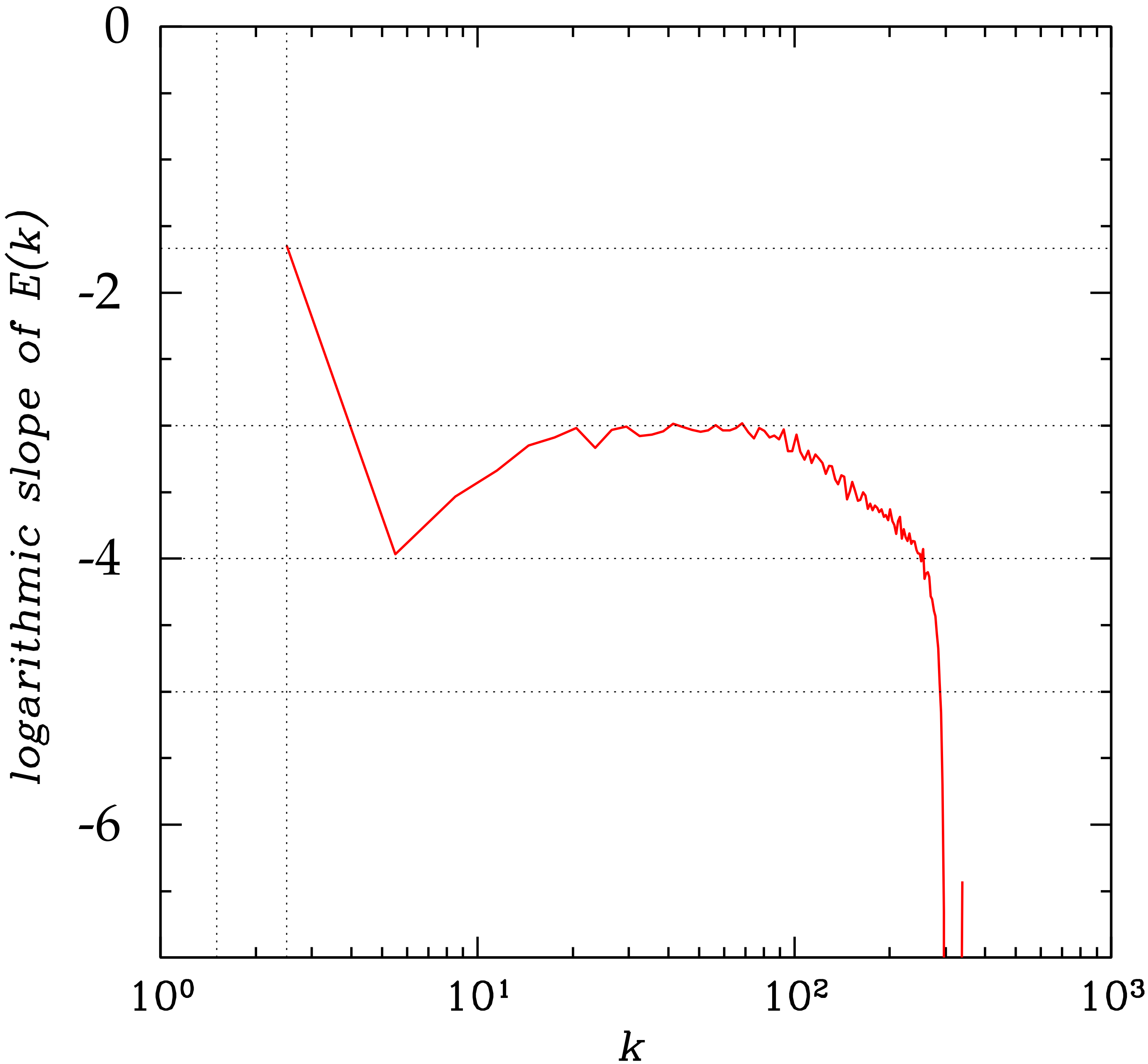
- Energetic reflections at the lower spectral boundary eventually lead to a large-scale direct cascade.
- This would agree with the large-scale k^{-3} spectra seen numerically by [Borue 1994] and observed in the atmosphere [Lilly & Peterson 1983].
- [Tran & Bowman 2003]: In a bounded domain, the two inertial range exponents must sum to -8 (high Reynolds number).
- Large-scale k^{-3} spectrum $\Rightarrow$ a small-scale k^{-5} spectrum.
- Consistent with rigorous [Tran & Shepherd 2002] constraint: the spectrum must be at least as steep as k^{-5}.

Direct k^{-3} enstrophy cascade



Zero dissipation for $3 < k < 300$.

Logarithmic spectral slope



Zero dissipation for $3 < k < 300$.

Conclusions

- Realizability ensures physically reasonable behaviour.
- The EDQNM closure can predict negative energies in the presence of non-hermitian effects such as wave phenomena.
- The unrealizability of the EDQNM closure arises from an improper Fluctuation–Dissipation Ansatz.
- Correcting this difficulty has led to the realizable Markovian closure.
- A realizable test-field model, invariant to random Galilean transformations, has been implemented for two-dimensional Navier–Stokes turbulence.

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