Casimir Cascades in Two-Dimensional Turbulence

John C. Bowman (University of Alberta)

Acknowledgements: Jahanshah Davoudi (University of Toronto)

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www.math.ualberta.ca/∼bowman/talks

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Two-Dimensional Turbulence

- Navier–Stokes equation for vorticity $\omega = \widehat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{u}$:

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega = -\nu\nabla^2\omega + f.$$

- In Fourier space:

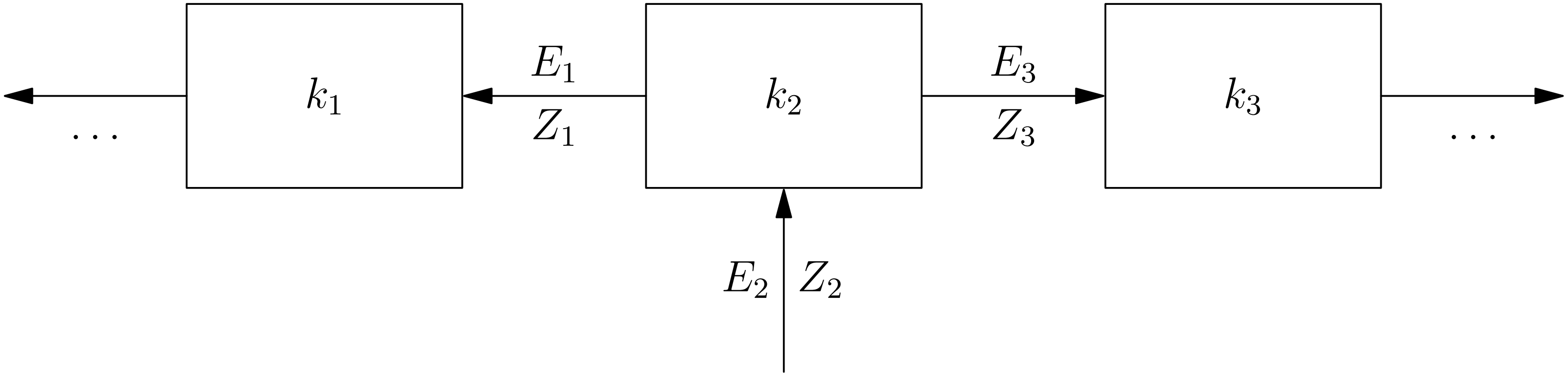
$$\frac{\partial\omega_{\boldsymbol{k}}}{\partial t} = S_{\boldsymbol{k}} - \nu k^2\omega_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

where $S_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} \frac{\widehat{\boldsymbol{z}}\cdot\boldsymbol{p}\times\boldsymbol{k}}{p^2}\,\omega^*_{\boldsymbol{p}}\,\omega^*_{-\boldsymbol{k}-\boldsymbol{p}}.$

- When $\nu = 0$ and $f_{\boldsymbol{k}} = 0$:

energy $E = \frac{1}{2}\sum_{\boldsymbol{k}} \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}$ and enstrophy $Z = \frac{1}{2}\sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2$ are conserved.

Fjørtoft Dual Cascade Scenario



$$E_2 = E_1 + E_3, \qquad Z_2 = Z_1 + Z_3, \qquad Z_i \approx k_i^2 E_i.$$

- When $k_1 = k$, $k_2 = 2k$, and $k_3 = 4k$:

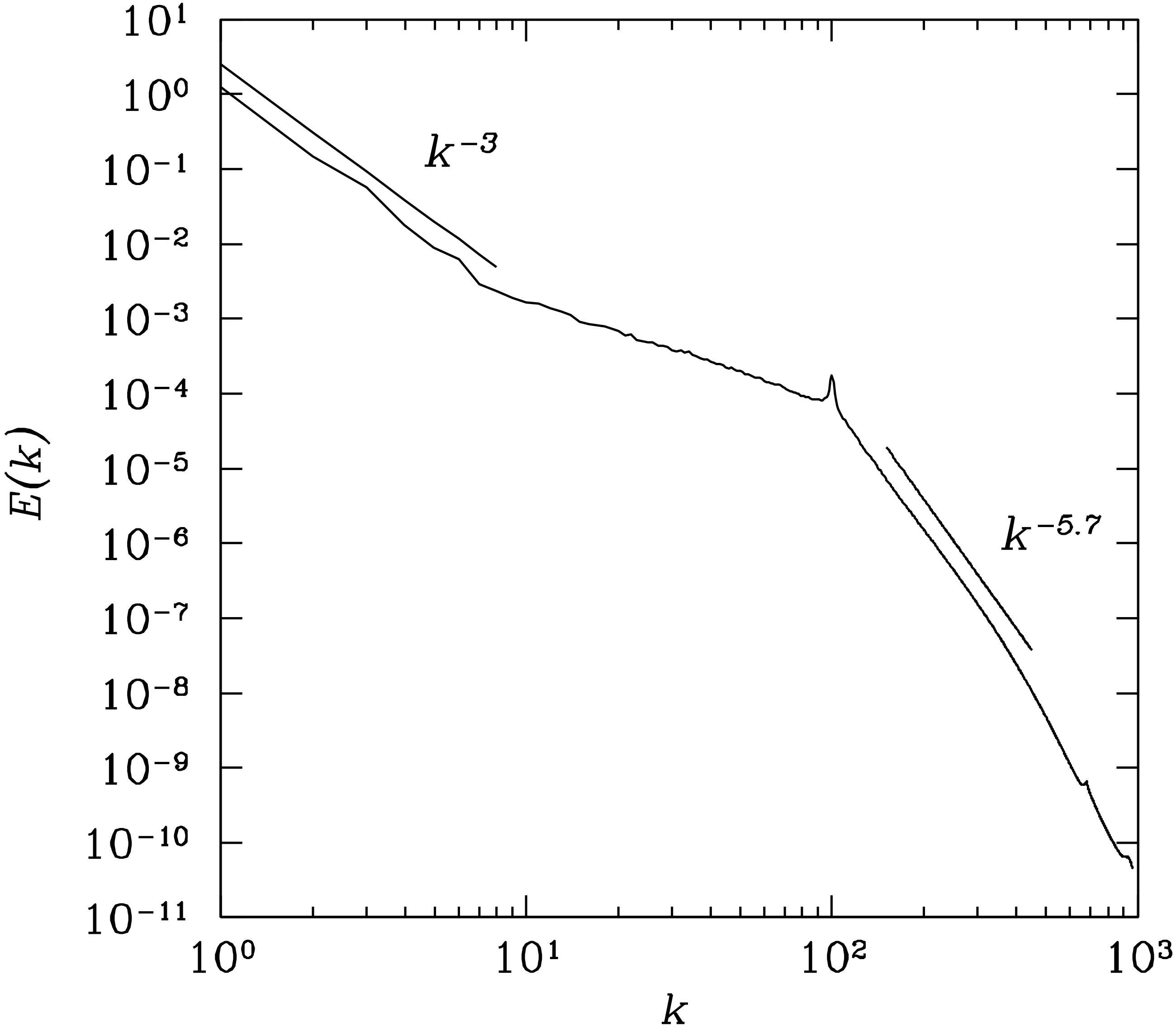
$$E_1 \approx \frac{4}{5} E_2, \quad Z_1 \approx \frac{1}{5} Z_2, \qquad E_3 \approx \frac{1}{5} E_2, \quad Z_3 \approx \frac{4}{5} Z_2.$$

- Fjørtoft [1953]: energy cascades to large scales and enstrophy cascades to small scales.

Kraichnan–Leith–Batchelor Theory

- In an infinite domain:
  - large scale $k^{-5/3}$ energy cascade
  - small scale k^{-3} enstrophy cascade
- In a bounded domain, the situation may be quite different...

Long-Time Behaviour in a Bounded Domain



Tran and Bowman, PRE 69, 036303, 1–7 (2004).

Casimir Invariants

- Inviscid unforced two dimensional turbulence has uncountably many other Casimir invariants.
- Any continuously differentiable function of the (scalar) vorticity is conserved by the nonlinearity:

$$\frac{d}{dt}\int f(\omega)\,d\boldsymbol{x} = \int f'(\omega)\frac{\partial\omega}{\partial t}\,d\boldsymbol{x} = -\int f'(\omega)\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega\,d\boldsymbol{x}$$
$$= -\int \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla} f(\omega)\,d\boldsymbol{x} = \int f(\omega)\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}\,d\boldsymbol{x} = 0.$$

- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit cascades?
- Polyakov has suggested that the higher-order Casimir invariants cascade to large scales, while Eyink suggests that they might cascade to small scales.

- What is certain is that only the quadratic invariants survive high-wavenumber truncation (Montgomery calls them rugged invariants).

High-Wavenumber Truncation

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \, \omega_{\boldsymbol{q}}^*.$$

where $\epsilon_{\boldsymbol{kpq}} = (\widehat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{p} \times \boldsymbol{q}) \, \delta(\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q})$.

- Enstrophy evolution:

$$\frac{d}{dt} \sum_{\boldsymbol{k}} |\omega_{\boldsymbol{k}}|^2 = \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{k}}^* \omega_{\boldsymbol{p}}^* \, \omega_{\boldsymbol{q}}^* = 0.$$

- Invariance of $Z_3 = \int \omega^3 \, dx$ follows from:

$$0 = \sum_{\boldsymbol{k},\boldsymbol{r},\boldsymbol{s}} \left[\sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \, \omega_{\boldsymbol{q}}^* \omega_{\boldsymbol{r}}^* \omega_{\boldsymbol{s}}^* + 2 \text{ other similar terms} \right].$$

- The absence of an explicit $\omega_{\boldsymbol{k}}$ in the first term means that setting $\omega_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} > K$ will make the summations no longer symmetric!
- However, since the missing terms involve $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ for p and q higher than the truncation wavenumber K, one might expect that a very well-resolved simulation would lead to almost exact invariance of Z_3.
- We will show that this is indeed the case.

Enstrophy Balance

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu k^2 \omega_{\boldsymbol{k}} = S_{\boldsymbol{k}} + f_{\boldsymbol{k}},$$

- Multiply by $\omega_{\boldsymbol{k}}^*$ and integrate over wavenumber angle $\Rightarrow$ enstrophy spectrum $Z(k)$ evolves as:

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k),$$

where $T(k)$ and $G(k)$ are the corresponding angular averages of $\mathrm{Re}\,\langle S_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* \rangle$ and $\mathrm{Re}\,\langle f_{\boldsymbol{k}} \omega_{\boldsymbol{k}}^* \rangle$.

Nonlinear Enstrophy Transfer Function

$$\frac{\partial}{\partial t}Z(k) + 2\nu k^2 Z(k) = 2T(k) + G(k).$$

- Let

$$\Pi(k) \doteq 2\int_k^\infty T(p)\,dp$$

represent the nonlinear transfer of enstrophy into $[k,\infty)$.

- Integrate from k to ∞:

$$\frac{d}{dt}\int_k^\infty Z(p)\,dp = \Pi(k) - \epsilon_Z(k),$$

where $\epsilon_Z(k) \doteq 2\nu\int_k^\infty p^2 Z(p)\,dp - \int_k^\infty G(p)\,dp$ is the total enstrophy transfer, via dissipation and forcing, *out* of wavenumbers higher than k.

- A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than k.
- When $\nu = 0$ and $f_{\boldsymbol{k}} = 0$:

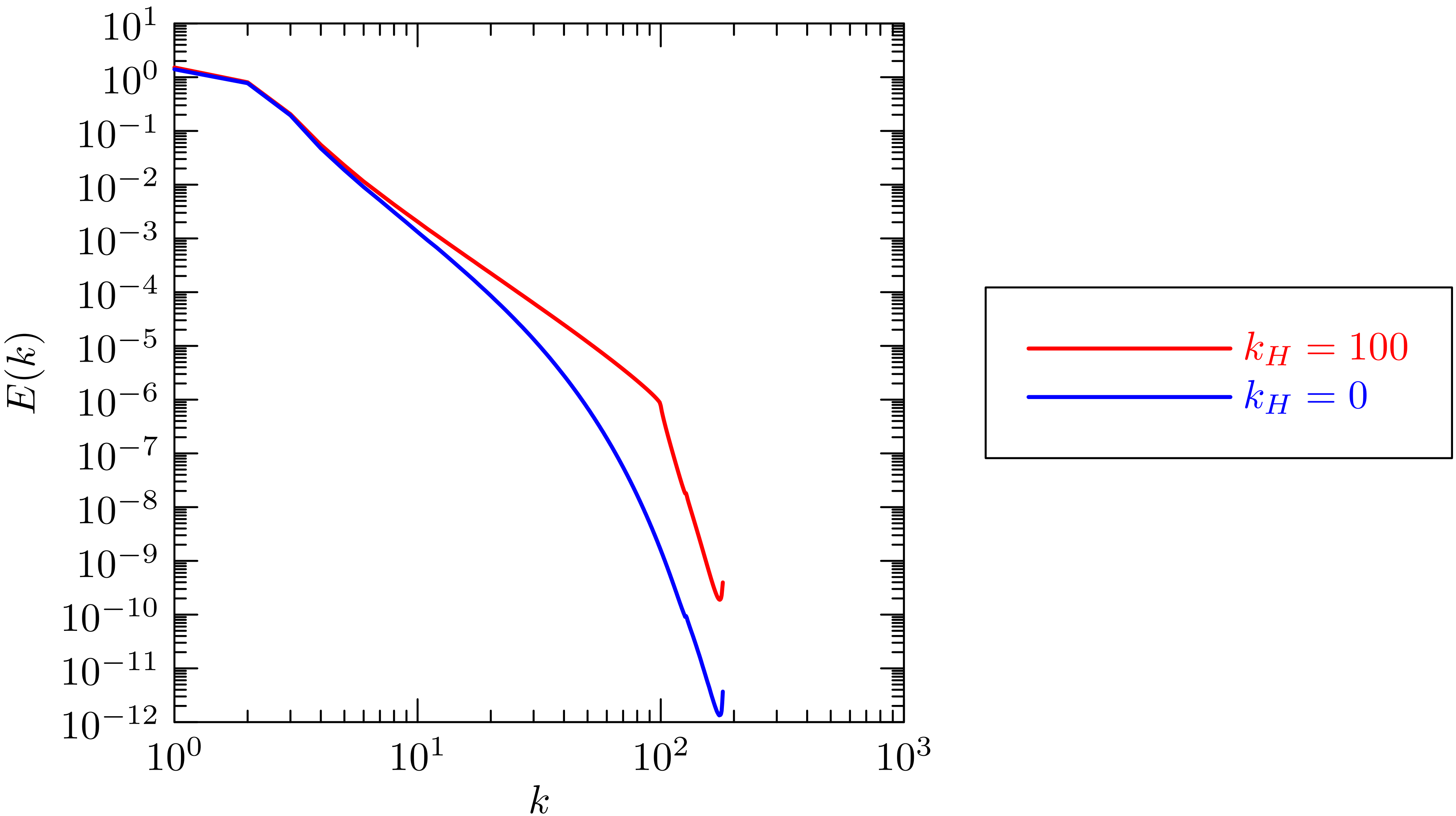
$$0 = \frac{d}{dt}\int_0^\infty Z(p)\,dp = 2\int_0^\infty T(p)\,dp,$$

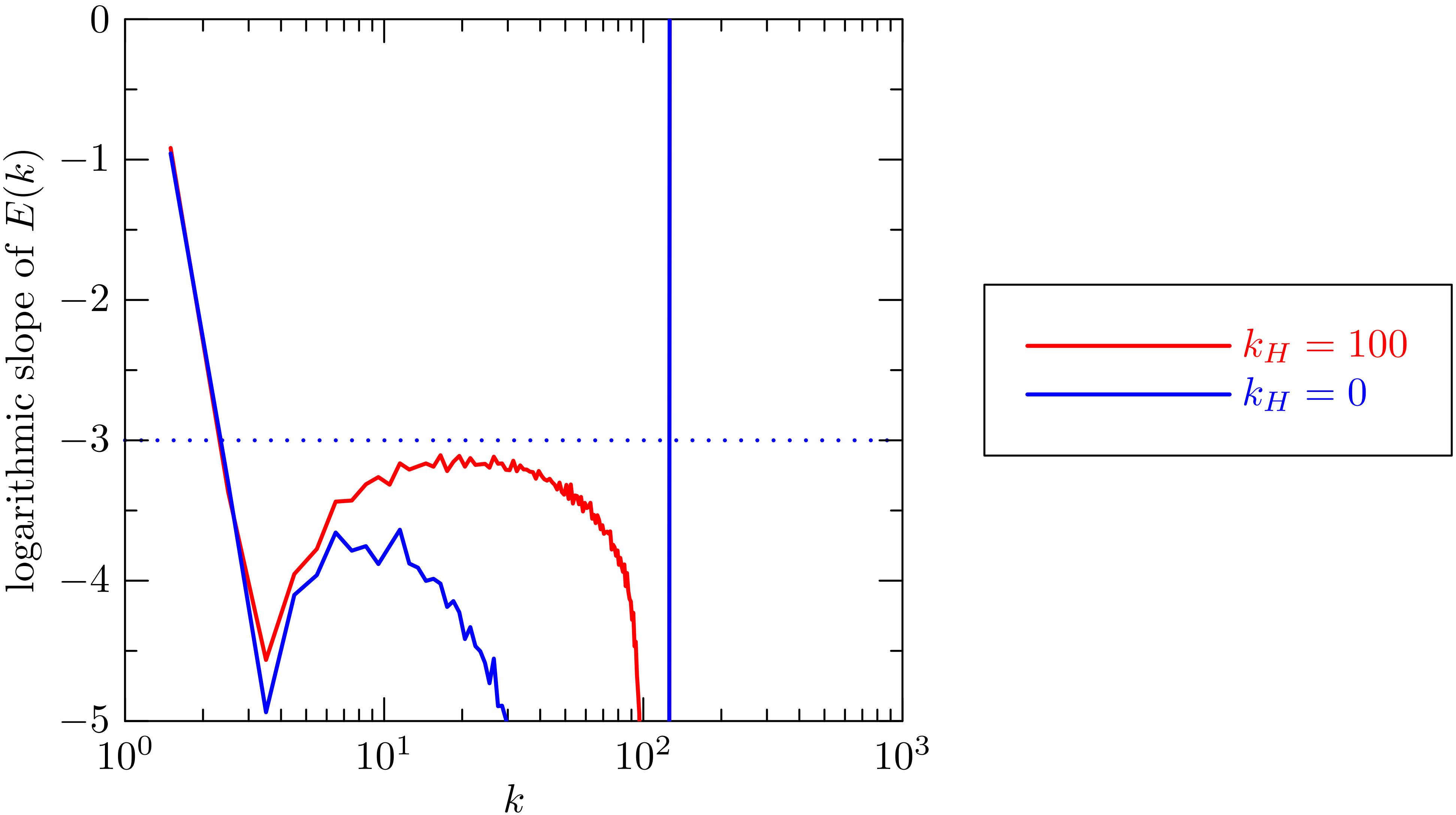
so that

$$\Pi(k) = 2\int_k^\infty T(p)\,dp = -2\int_0^k T(p)\,dp.$$

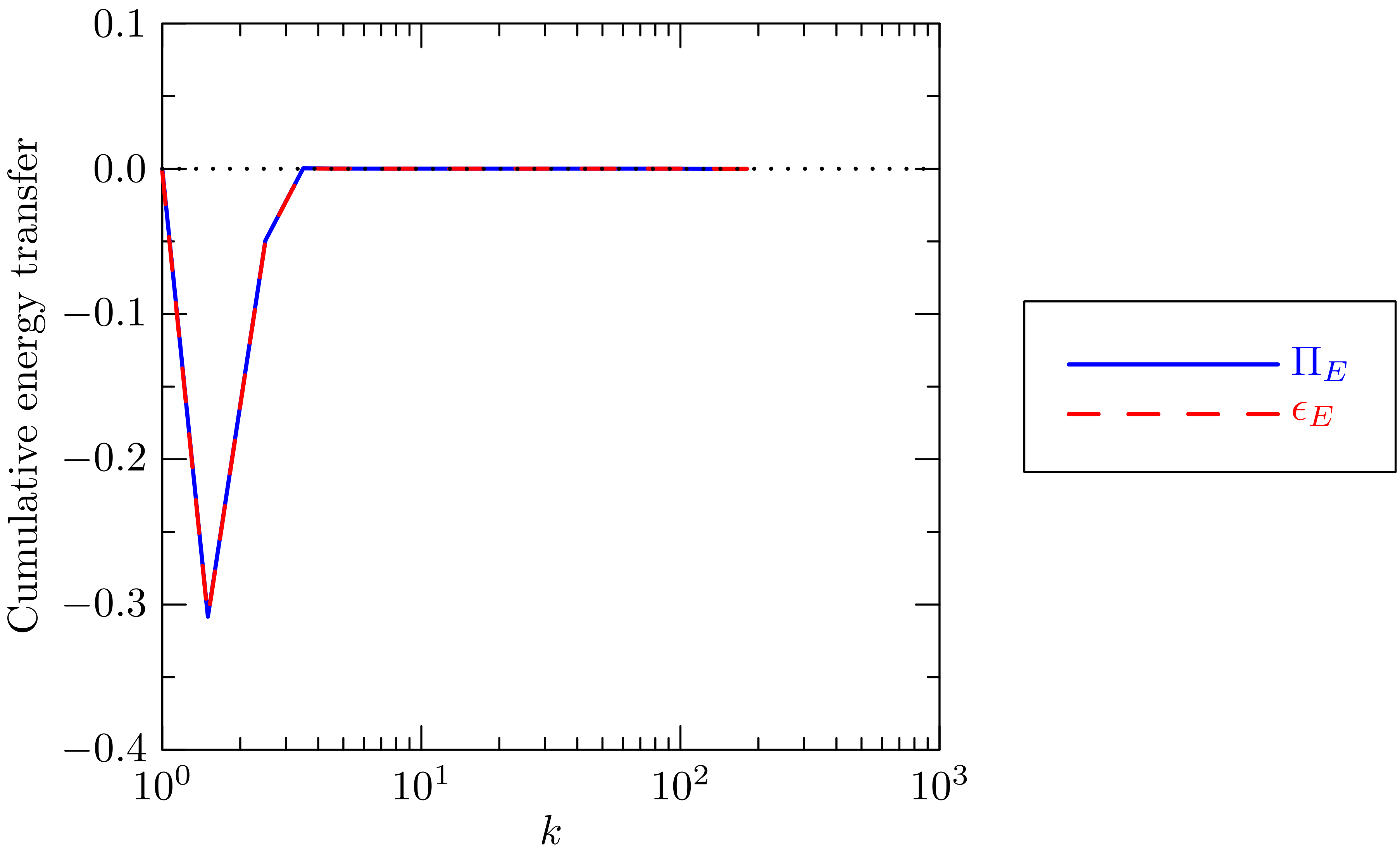
- Note that $\Pi(0) = \Pi(\infty) = 0$.
- In a steady state, $\Pi(k) = \epsilon_Z(k)$.
- This provides an excellent numerical diagnostic for when a steady state has been reached.

Forcing at $k = 2$, friction for $k < 3$, viscosity for $k \geq k_H = 100$ (255×255 dealiased modes)

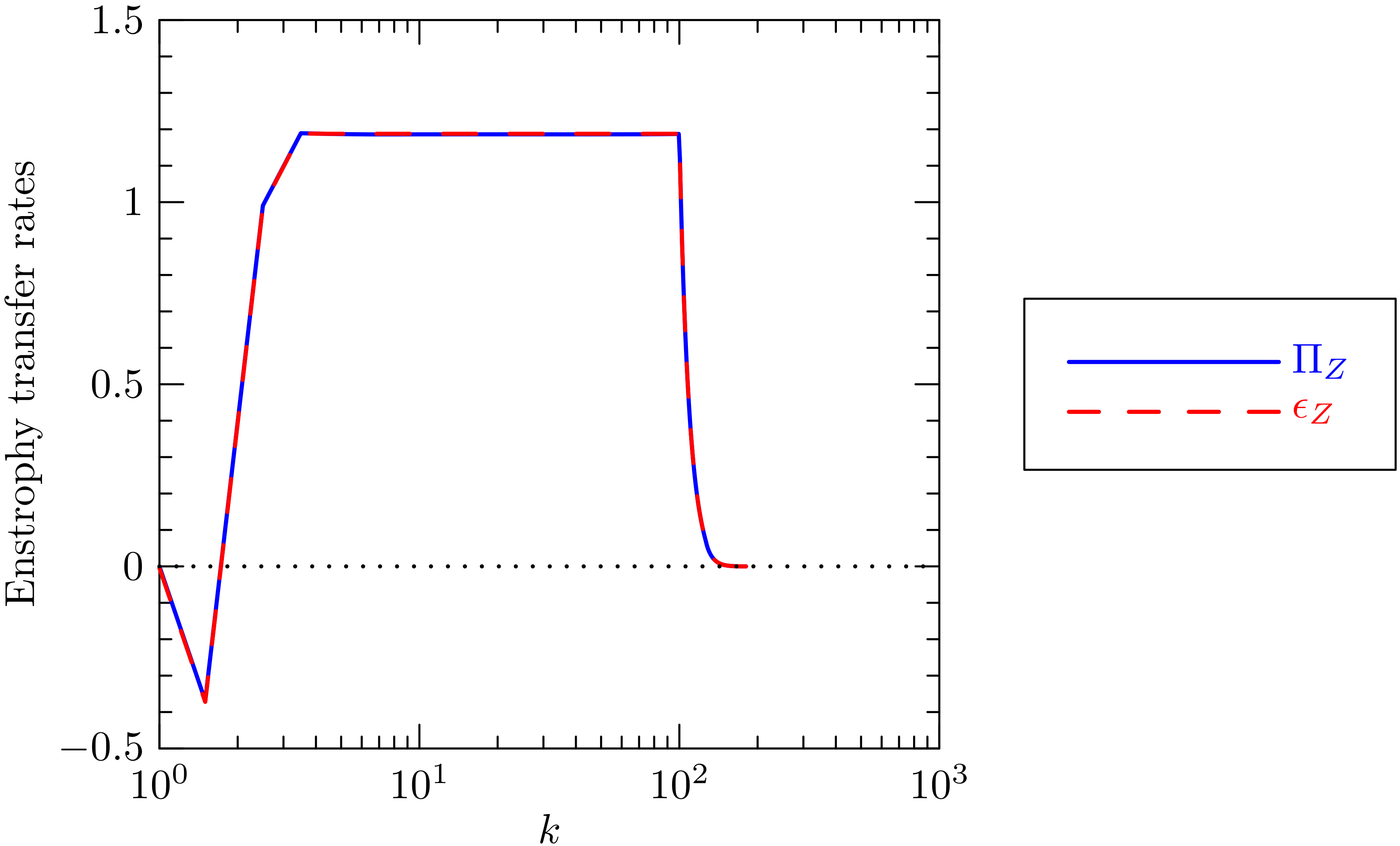




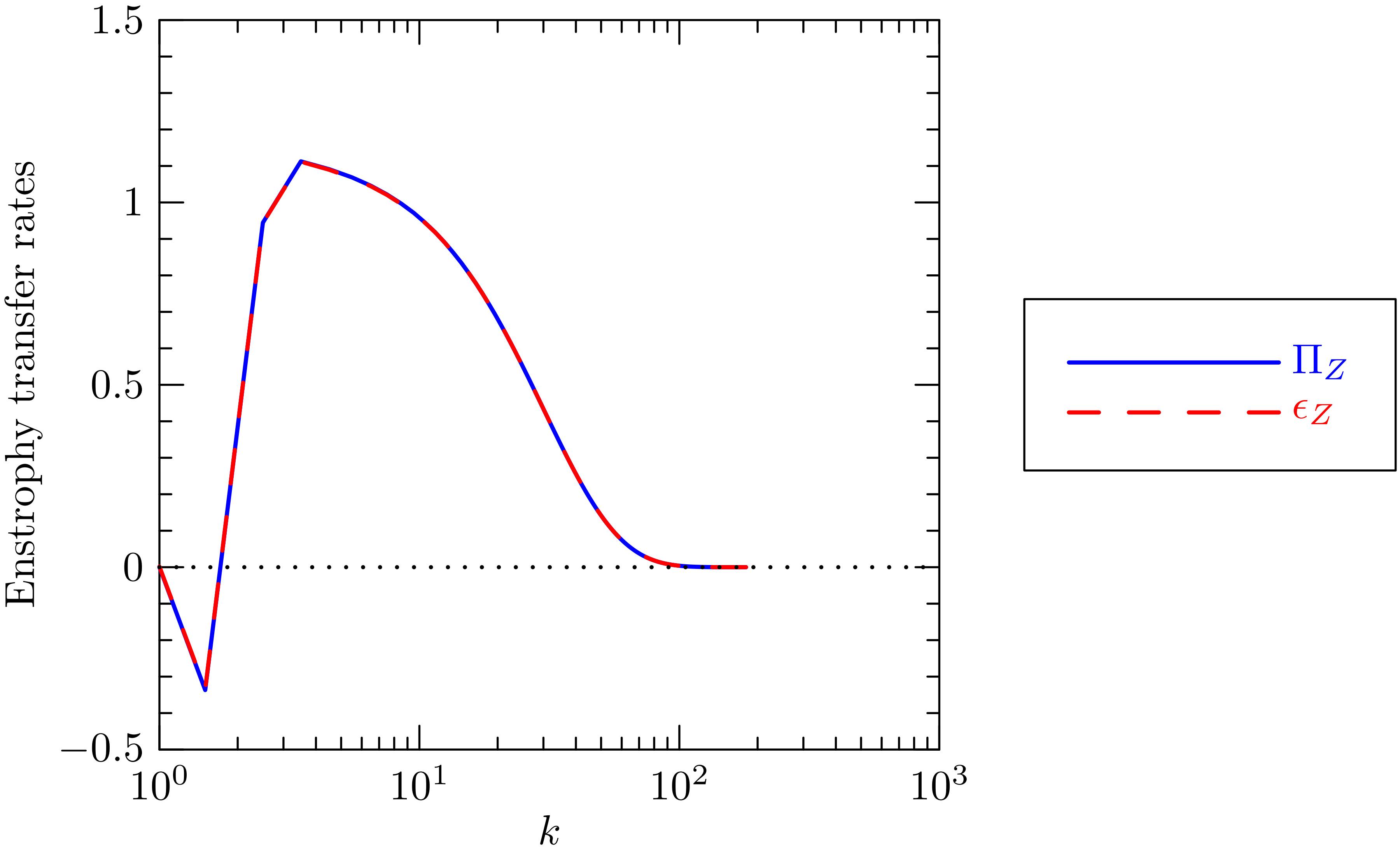
0
−1
−2
−3
−4
−5
logarithmic slope of $E(k)$
10^0
10^1
10^2
10^3
k
$k_H = 100$
$k_H = 0$



Cutoff viscosity $(k \geq k_H = 100)$



Cutoff viscosity $(k \geq k_H = 100)$



Molecular viscosity ($k \geq k_H = 0$)

Nonlinear Casimir Transfer

- Fourier decompose the fourth-order Casimir invariant $Z_4 = N^3 \sum_{\boldsymbol{j}} \omega^4(x_{\boldsymbol{j}})$ in terms of N spatial collocation points $x_{\boldsymbol{j}}$:

$$Z_4 = \sum_{\boldsymbol{k},\boldsymbol{p}} \omega_{\boldsymbol{k}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}}.$$

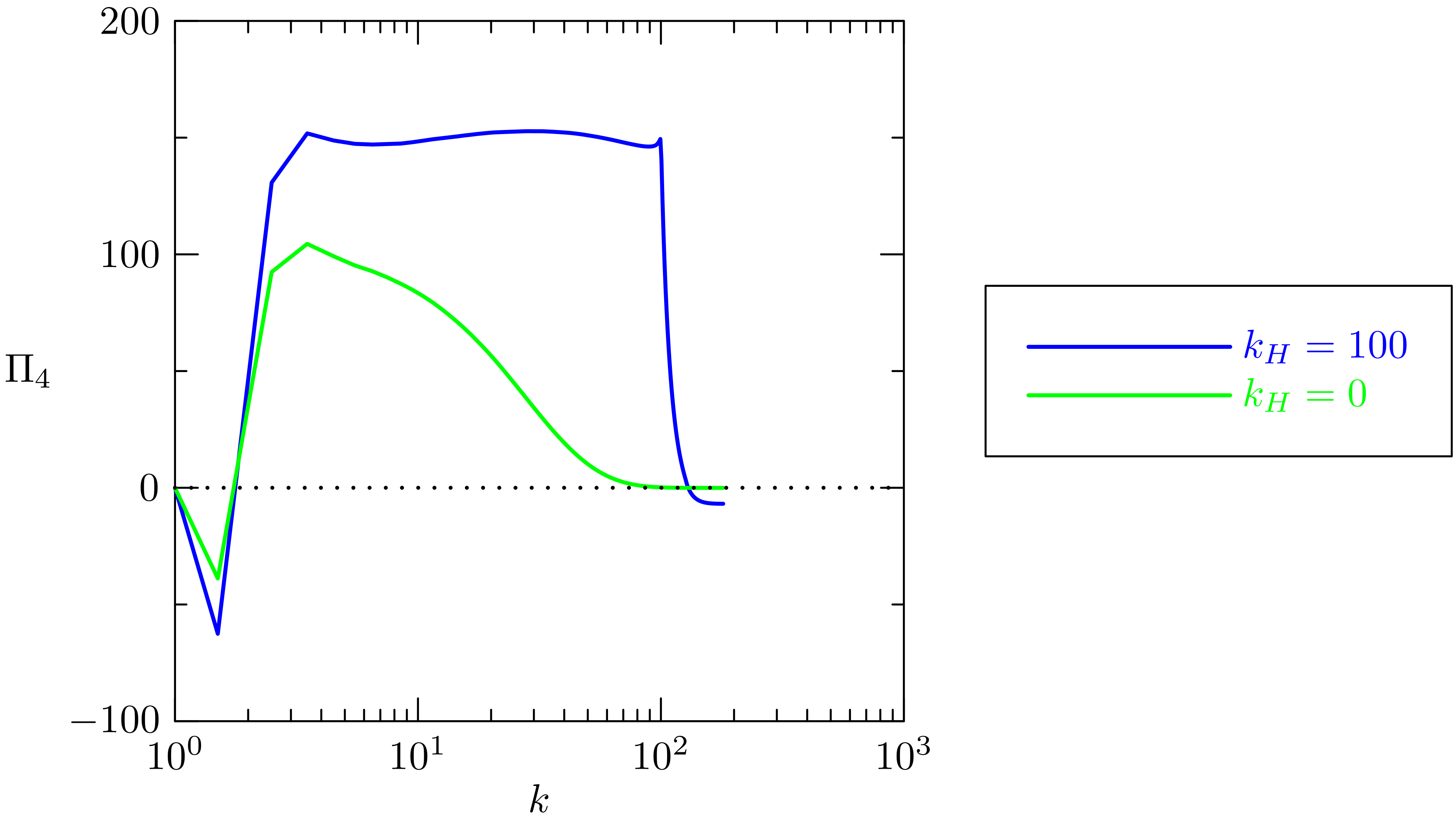
$$\frac{d}{dt} Z_4 = \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{p}} \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{p}} S_{\boldsymbol{p}}\, \omega_{\boldsymbol{q}}\, \omega_{-\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \right]$$

$$\frac{d}{dt} Z_4 = N^2 \sum_{\boldsymbol{k}} \left[S_{\boldsymbol{k}} \sum_{\boldsymbol{j}} \omega^3(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} + 3\omega_{\boldsymbol{k}} \sum_{\boldsymbol{j}} S(x_{\boldsymbol{j}}) \omega^2(x_{\boldsymbol{j}}) e^{2\pi i \boldsymbol{j}\cdot\boldsymbol{k}/N} \right]$$

$$\doteq \sum_{k} T_4(k).$$

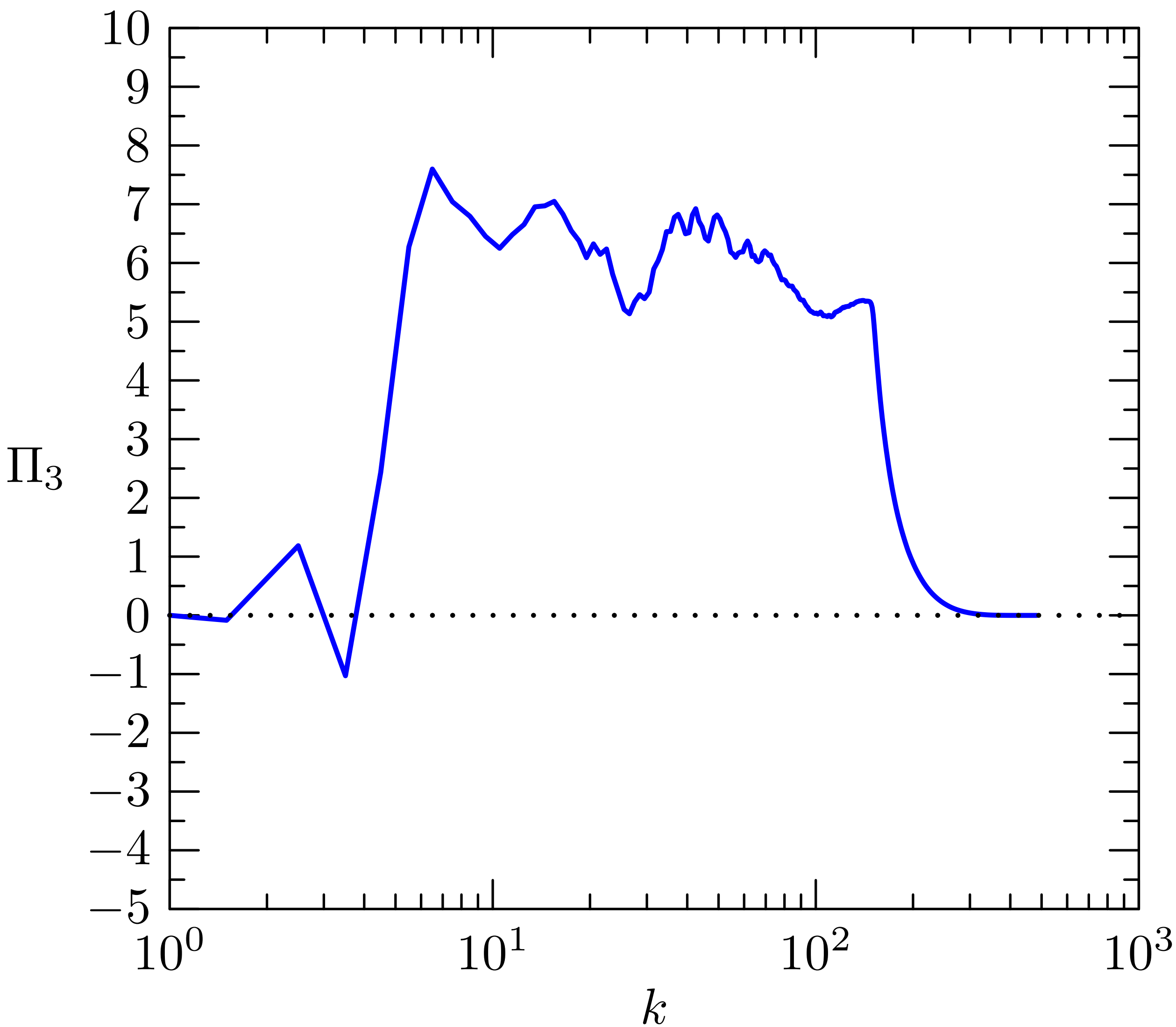
Here $S_{\boldsymbol{k}}$ is the nonlinear source term in $\frac{\partial}{\partial t}\omega_{\boldsymbol{k}}$.

Downscale Transfer of Z_4



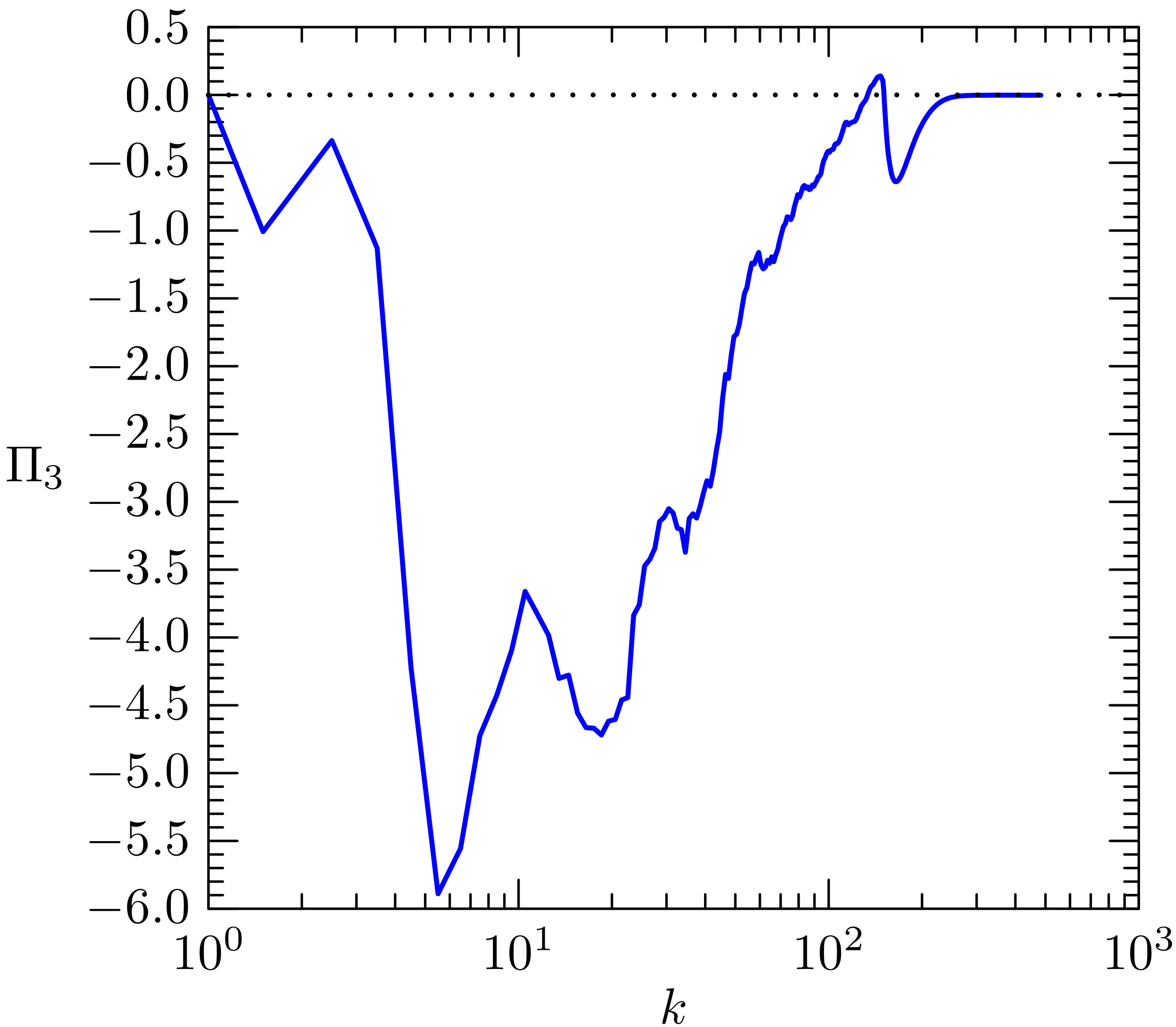
Nonlinear transfer Π_4 of Z_4 averaged over $t \in [15, 55]$.

Third-order Casimir Transfer Function



Nonlinear transfer Π_3 of Z_3 averaged over $t \in [7, 12]$.

No Cascade of Z_3



Nonlinear transfer Π_3 of Z_3 averaged over $t \in [12, 17]$.

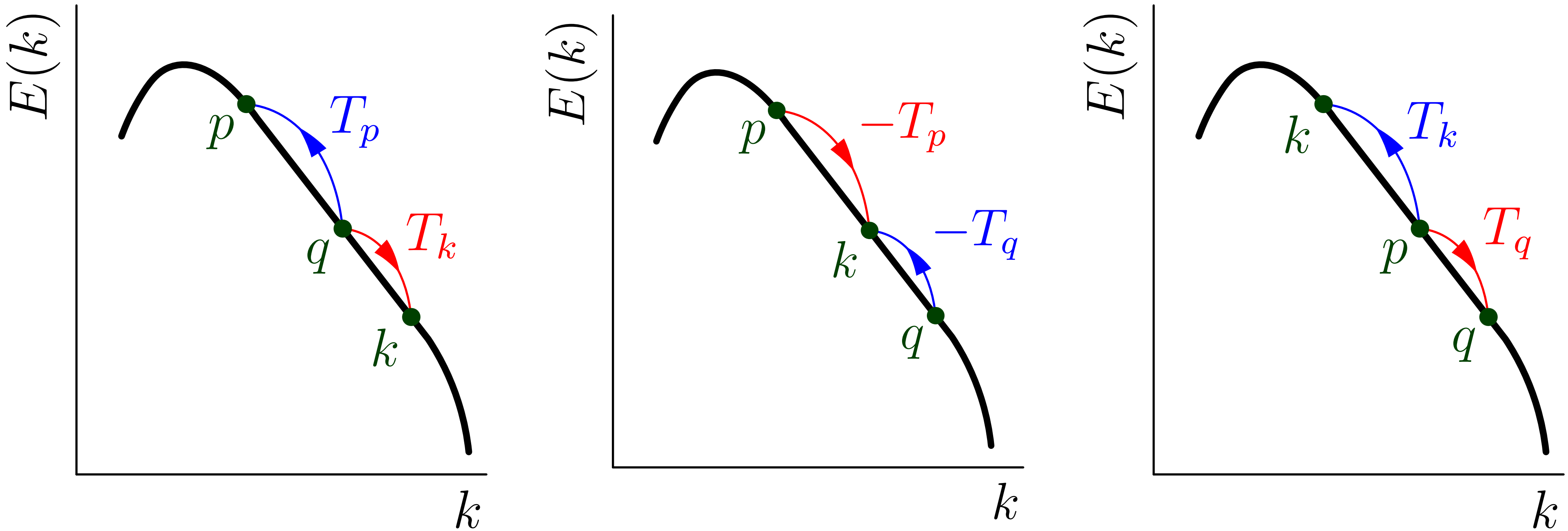
Transfer vs. Flux

- Distinguish between transfer and flux.
- The mean rate of enstrophy transfer to $[k, \infty)$ is given by

$$\Pi(k) = \int_k^\infty T(k)\, dk = -\int_0^k T(k)\, dk.$$

- In a steady state, $\Pi(k)$ will trivially be constant within a true inertial range.
- In contrast, the enstrophy flux through a wavenumber k is the amount of enstrophy transferred to small scales *via* triad interactions involving mode k.

Flux Decomposition for a Single $(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ Triad



$L_k = T_k$
$S_k = 0$

$L_k = -T_p$
$S_k = -T_q$

$L_k = 0$
$S_k = T_k$

- Note that energy is conserved: $L_k + S_k = T_k = -T_p - T_q$. Thus

$$L_k = \text{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|<k}} M_{\boldsymbol{k},\boldsymbol{p}}\, \omega_{\boldsymbol{p}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{k}}^* - \text{Re} \sum_{\substack{|\boldsymbol{k}|=k \\ |\boldsymbol{p}|<k \\ |\boldsymbol{k}-\boldsymbol{p}|>k}} M_{\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{k}}\, \omega_{\boldsymbol{k}-\boldsymbol{p}}\, \omega_{\boldsymbol{p}}^*.$$

Conclusions

- Even though higher-order Casimir invariants do not survive wavenumber truncation, it is possible, with sufficiently well resolved simulations, to check whether they cascade to large or small scales.

- We computed the transfer function of the globally integrated ω^4 inviscid invariant.

- Numerical evidence suggests that in the enstrophy inertial range there is a direct cascade of this invariant to small scales.

- However, for the globally integrated ω^3 inviscid invariant, we found no systematic cascade: it appears to slosh back and forth between the large and small scales. This is expected since ω^3 does not have a definite sign.

- One should distinguish between nonlocal transfer and flux. To compute this decomposition efficiently, one needs to develop a restricted Fast Fourier transform.

Asymptote: The Vector Graphics Language

`http://asymptote.sf.net`

(freely available under the GNU public license)