Links between dissipation, intermittency, and helicity in the GOY shell model

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Outline

- GOY Shell Model
  - Intermittency-Length Scale Paradox
  - Intermediate Dissipation Range
  - Modal Truncation
  - Intermittency vs. Helicity [Kadanoff *et al.* 1995]
- Conclusions

GOY Shell Model

- Complex version of the Gledzer [1973] model for shell velocities u_n proposed by Yamada and Ohkitani [1987]:

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i k_n \left(\alpha u_{n+1}^* u_{n+2}^* + \frac{\beta}{\lambda} u_{n-1}^* u_{n+1}^* + \frac{\gamma}{\lambda^2} u_{n-1}^* u_{n-2}^*\right) + F_n.$$

- Shell wavenumbers $k_n = \lambda^n$ scale geometrically.
- Periodic or zero Dirichlet wavenumber "boundary conditions":

$\alpha + \beta + \gamma = 0 \Rightarrow$ nonlinearity conserves energy $E \doteq \frac{1}{2}\sum_n |u_n|^2$.

- Set $\alpha = 1$ by rescaling time $\Rightarrow$ one free parameter δ:

$$\alpha = 1, \qquad \beta = -\delta, \qquad \gamma = \delta - 1.$$

- Nonlinearity conserves second invariant $\frac{1}{2}\sum_n k_n^{-\log_\lambda \gamma} |u_n|^2$.

- [2D Turbulence:] For $\delta = 5/4$, $\lambda = 2$, the second invariant $\frac{1}{2}\sum_n k_n^2 |u_n|^2$ is the enstrophy.
- [3D Turbulence:] For $\delta = 1/2$, $\lambda = 2$, the second invariant $\frac{1}{2}\sum_n (-1)^n k_n |u_n|^2$ has dimensions and indefiniteness of helicity.
- When $\nu = F_n = 0$, the GOY model has an unstable fixed point, corresponding to the Kolmogorov-like power law

$$u_n \sim k_n^{-1/3},$$

with energy spectrum

$$E(k) = C\epsilon^{2/3} k^{-5/3}.$$

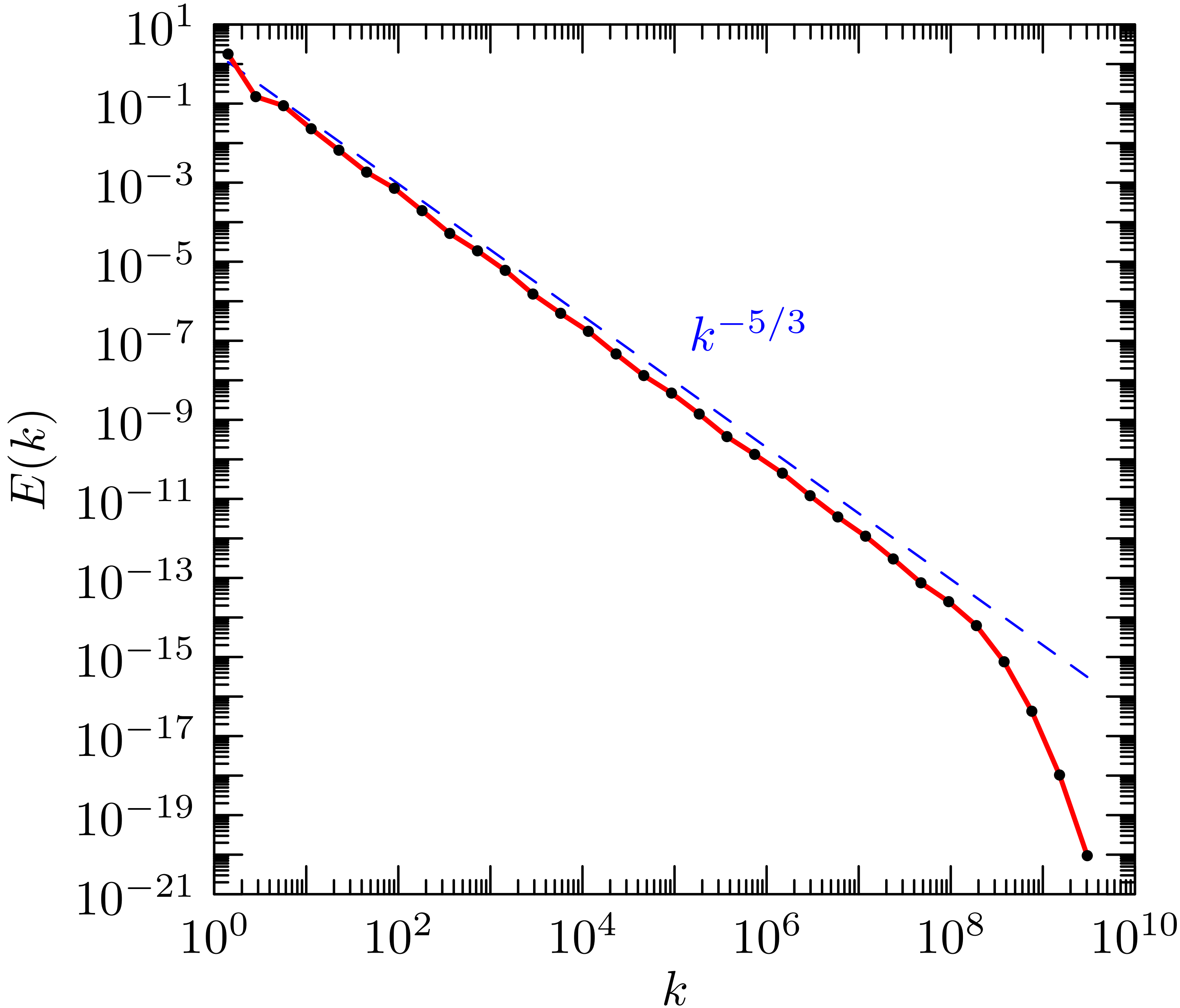
Forced-Dissipative GOY turbulence

- Adding forcing on first shell and small-scale dissipation perturbs the system away from the (unstable) fixed point.
- The energy spectrum now exhibits intermittency corrections:

$$E(k) = C\epsilon^{2/3}k^{-5/3}(k\ell)^{-\delta_2},$$

where ϵ is the energy injection rate and ℓ is the intermittency length scale.

Kolmogorov Law



Intermittency-Length Scale Paradox

- Q. What is the length scale ℓ?
- A. Balance the energy injection and dissipation between the largest scale L and some dissipation scale η_d:

$$\epsilon = 2\nu \int_{2\pi/L}^{2\pi/\eta_d} k^2 E(k)\, dk = 2C\nu\epsilon^{2/3}\ell^{-\delta_2} \int_{2\pi/L}^{2\pi/\eta_d} k^{1/3-\delta_2}\, dk$$

$$\sim \nu\epsilon^{2/3}\ell^{-\delta_2} \left(\frac{1}{\eta_d}\right)^{4/3-\delta_2} \qquad (\eta_d \ll L).$$

- for $\delta < 4/3$ (upper integration limit $1/\eta$ dominates).
- If one takes η_d to be the Kolmogorov scale $\eta = (\nu^3/\epsilon)^{1/4}$,

$$\epsilon \sim \nu\epsilon^{2/3} \left(\frac{\eta}{\ell}\right)^{\delta_2} \frac{\epsilon^{1/3}}{\nu}.$$

- Such a balance is possible only if $\ell \sim \eta$.

Convexity

- But the Cauchy–Schwarz inequality implies that the structure functions $S_p = \sum_n \langle |u_n|^p \rangle$ satisfy

$$S_{\frac{p}{2}+\frac{q}{2}} \le S_p^{1/2} S_q^{1/2}.$$

- Let $r = \frac{2\pi}{k}$. Then $\sum_n \langle |u_n|^p \rangle \sim (\epsilon r)^{p/3} \left(\frac{r}{\ell}\right)^{\delta_p} \sim \dfrac{\epsilon^{p/3}}{\ell^{\delta_p}} r^{\zeta_p} \Rightarrow$

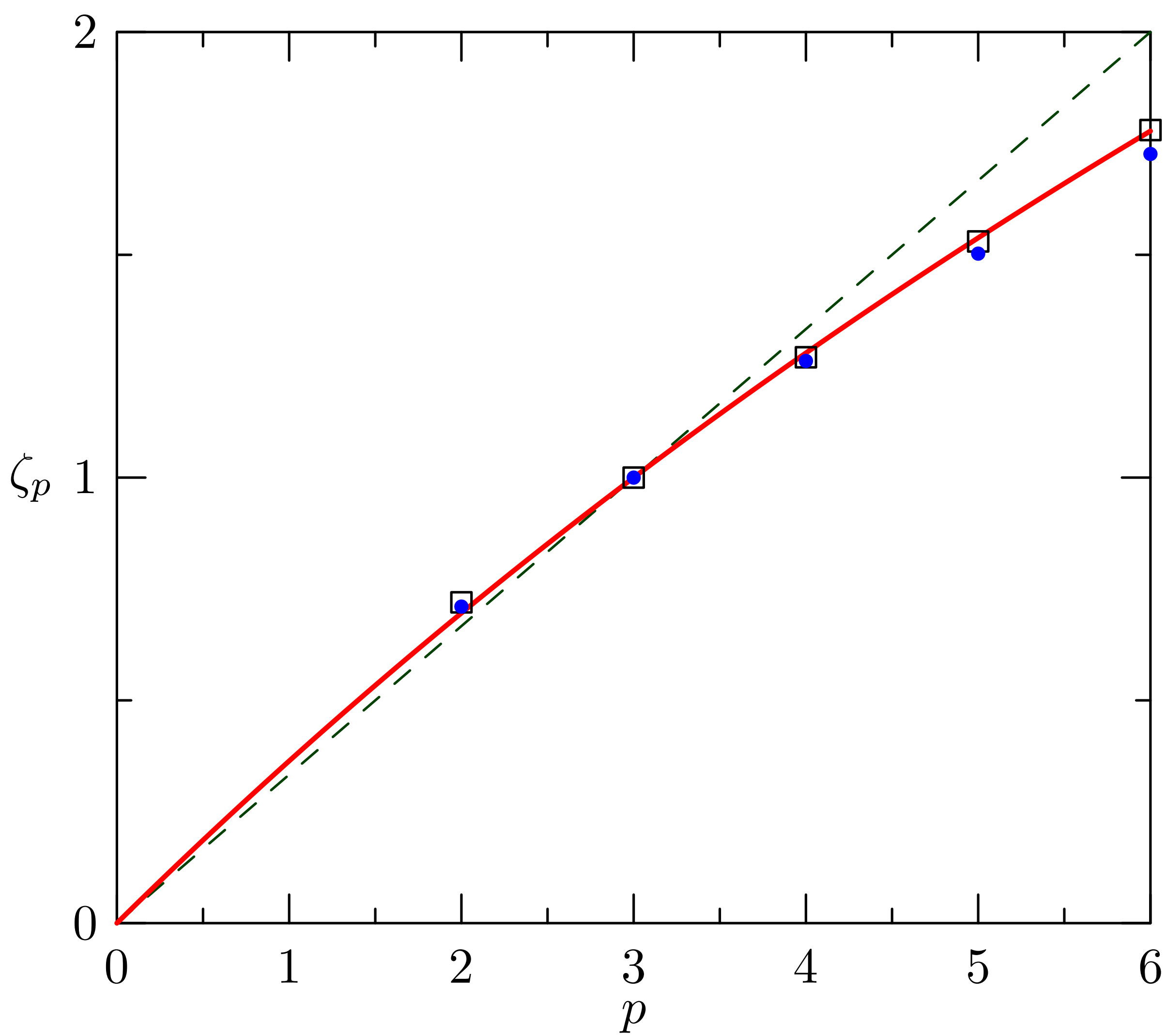
$$\left(\frac{r}{\ell}\right)^{\zeta_{\frac{p+q}{2}}} \le \left(\frac{r}{\ell}\right)^{\frac{\zeta_p}{2}+\frac{\zeta_q}{2}},$$

or

$$1 \le \left(\frac{r}{\ell}\right)^{\frac{\zeta_p+\zeta_q}{2} - \zeta_{\frac{p+q}{2}}}.$$

- If ℓ is the smallest excited length scale (η), then $r/\ell > 1 \; \forall r$ $\Rightarrow \zeta_p$ is convex.
- If ℓ were the largest excited length scale (L), then $r/\ell < 1 \; \forall r$ $\Rightarrow \zeta_p$ is concave.

3D Navier-Stokes Structure Exponents



Squares: laboratory measurements (van de Water & Herweijer)

- Apparent contradiction!
  - Energy balance $\Rightarrow \ell = \eta$;
  - But concavity $\Rightarrow \ell = L$.
- Paradox is easily resolved: steeper-than-Kolmogorov spectrum $\Rightarrow$ must integrate to a scale η_d smaller than η to obtain sufficient dissipation [Frisch and Vergassola 1991].
- We see from our result

$$\epsilon^{1/3} = \nu \ell^{-\delta_2} \left(\frac{1}{\eta_d}\right)^{4/3-\delta_2}$$

that in fact $\ell = L$ is consistent with

$$\boxed{\eta_d \sim \nu^{1/(\frac{4}{3}-\delta_2)}}.$$

Dissipation Wavenumber Scaling

$$\boxed{\eta_d \sim \nu^{1/(\frac{4}{3}-\delta_2)}}.$$

- Verify by doing a least-squares fit on the time-averaged pth-order flux

$$\Sigma_{n,p} = \left\langle \left| \mathrm{Im}\left(u_n u_{n+1} u_{n+2} + \frac{1+\beta}{\lambda} u_{n-1} u_n u_{n+1} \right) \right|^{p/3} \right\rangle .$$

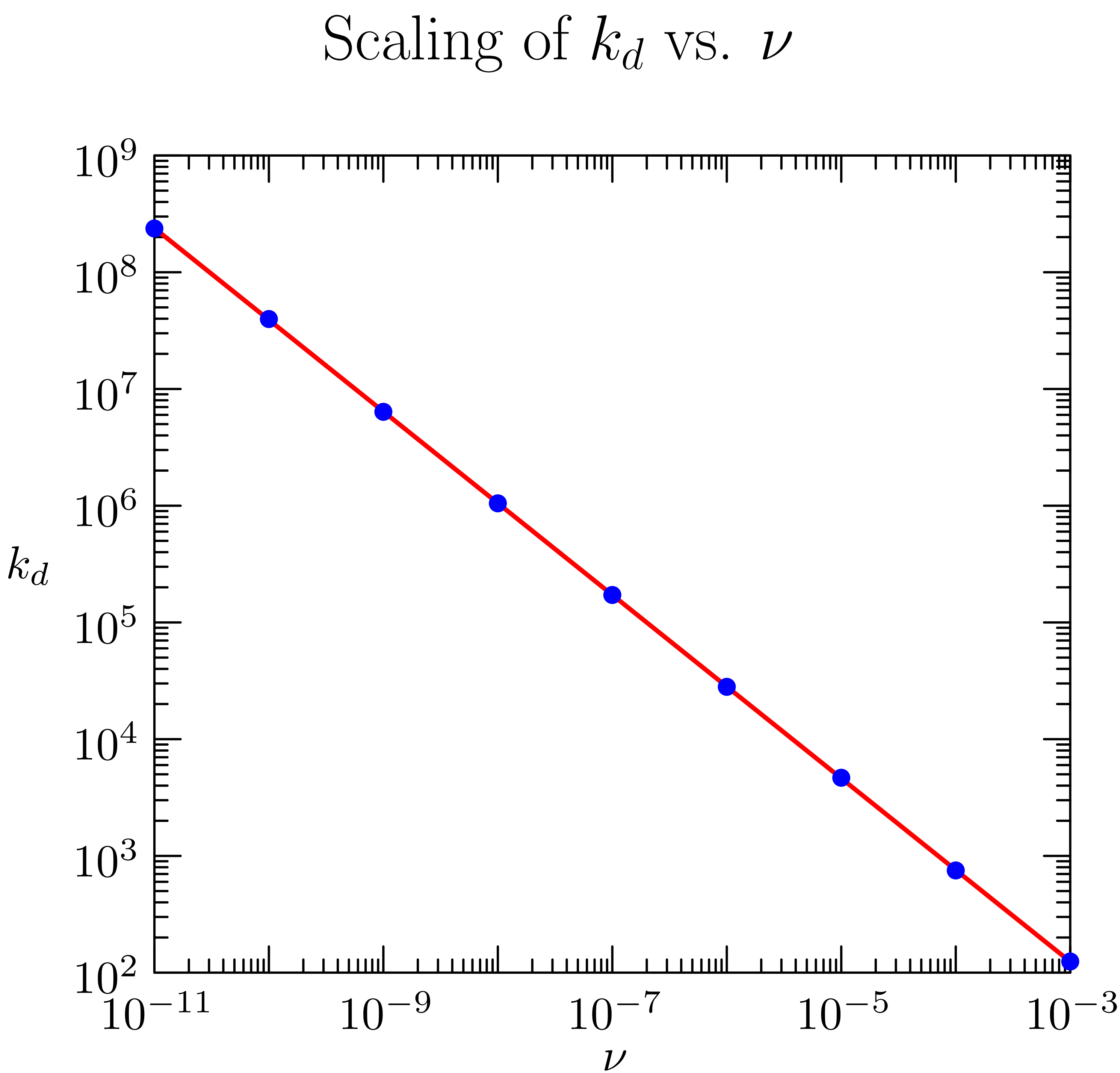
$\Rightarrow \zeta_2 = 0.7105 \pm 0.0005.$

- On substituting $\delta_2 = \zeta_2 - 2/3 = 0.0438$ we find

$$k_d \sim \nu^{-0.775}.$$

- Numerically determined wavenumber k_d separating regions of equal energy dissipation scales as

$$k_d \sim \nu^{-0.7855}.$$



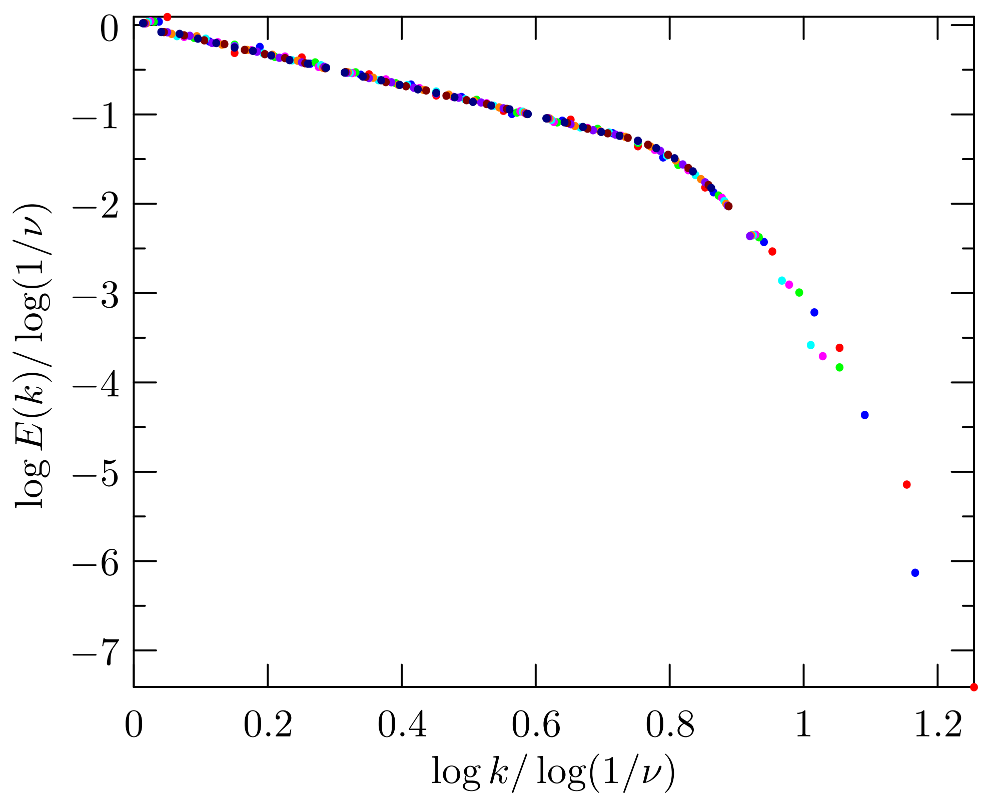
Scaling of k_d vs. ν
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
k_d
10^{-11}
10^{-9}
10^{-7}
10^{-5}
10^{-3}
ν

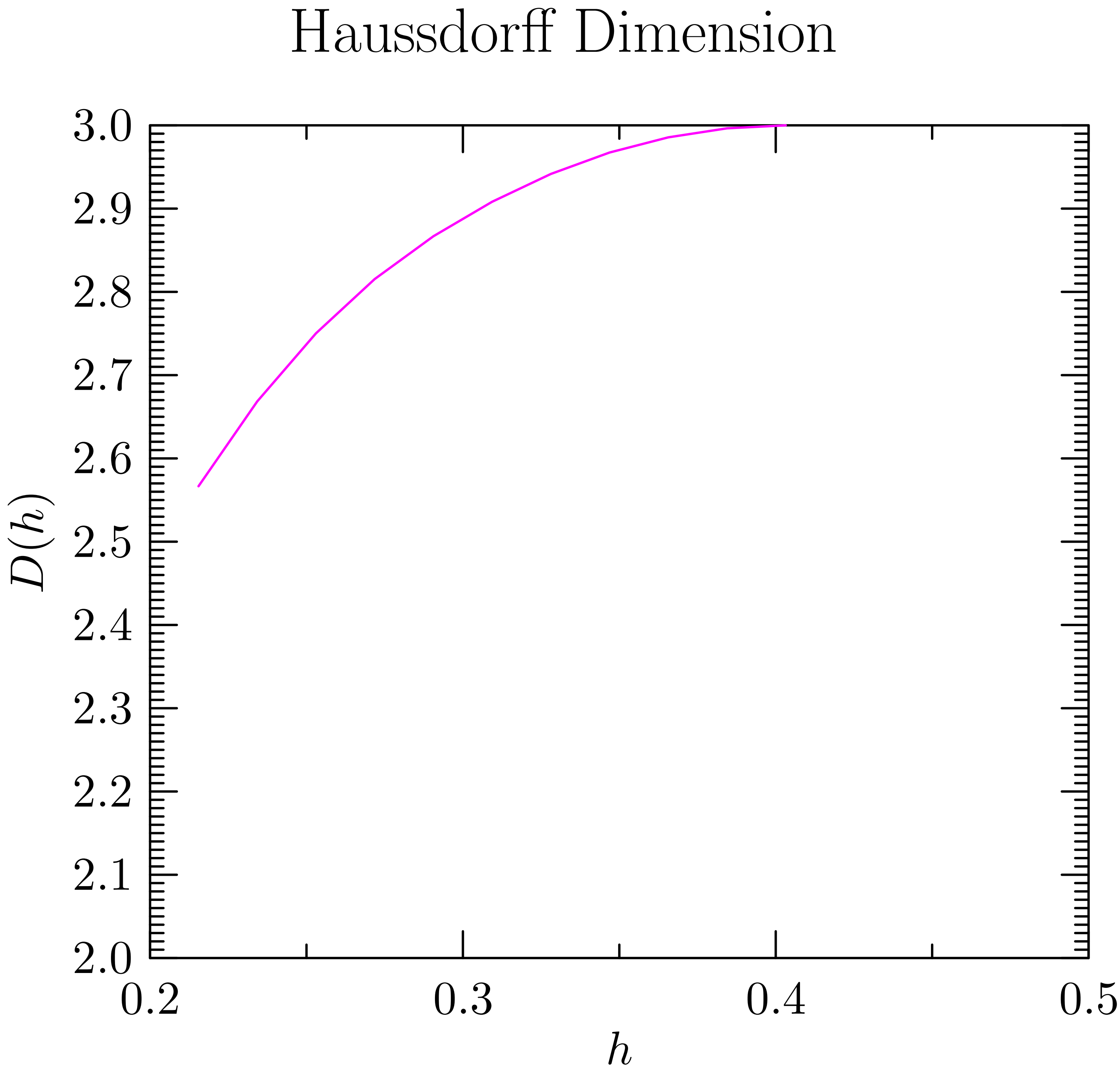
Intermediate dissipation range

- Frisch and Vergassola [1991]:

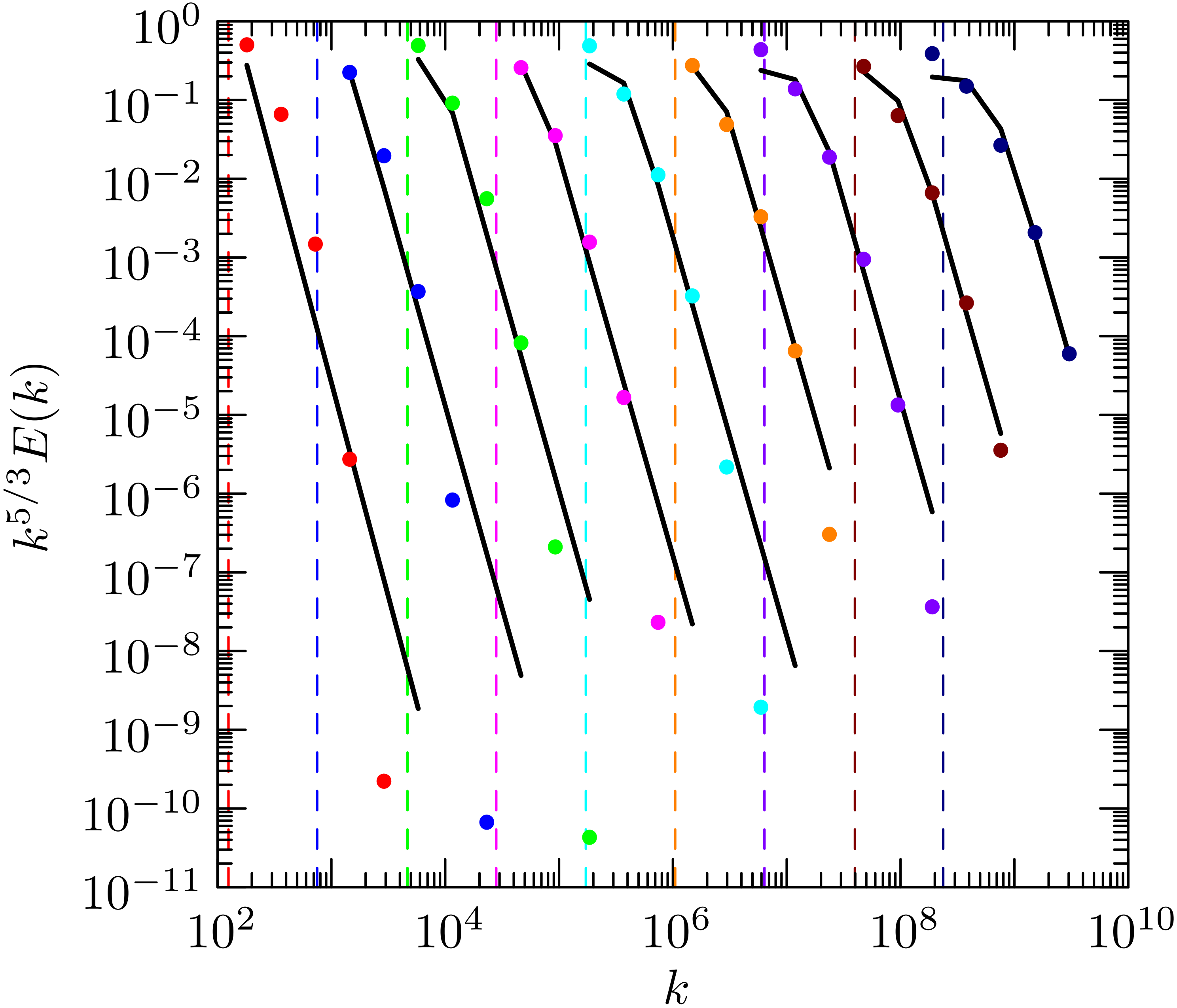
$$E(k) \sim k^{-4-2h(k)+D(h(k))} \qquad (k \geq 1/\eta),$$

where $D(h) = \inf_p(ph + 3 - \zeta_p)$ is the effective Hausdorff dimension and $h(k) = -1 - \log\nu/\log k$.

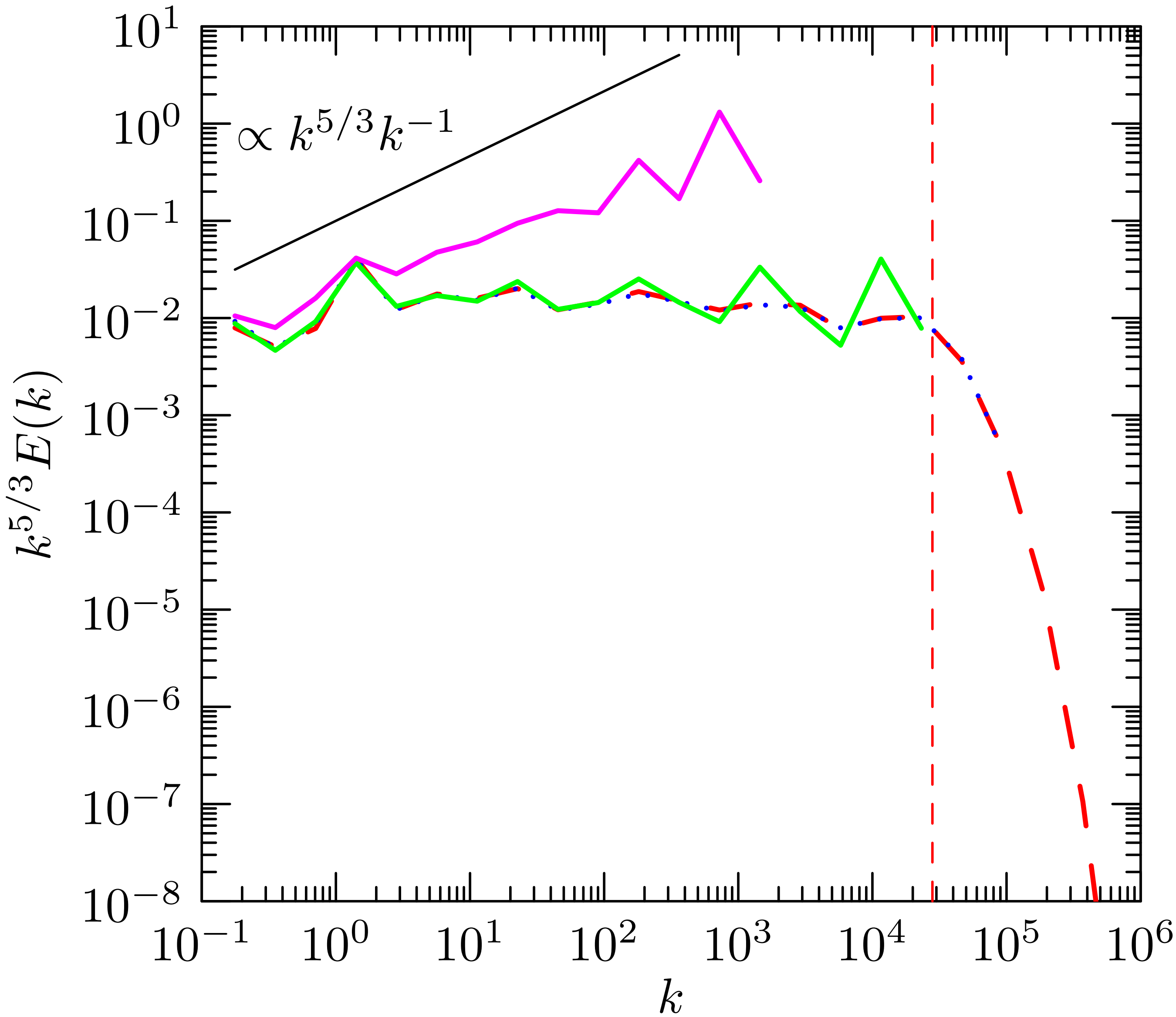



Haussdorff Dimension
3.0
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.0
D(h)
0.2
0.3
0.4
0.5
h

Evidence for Frisch–Vergarssola Spectrum

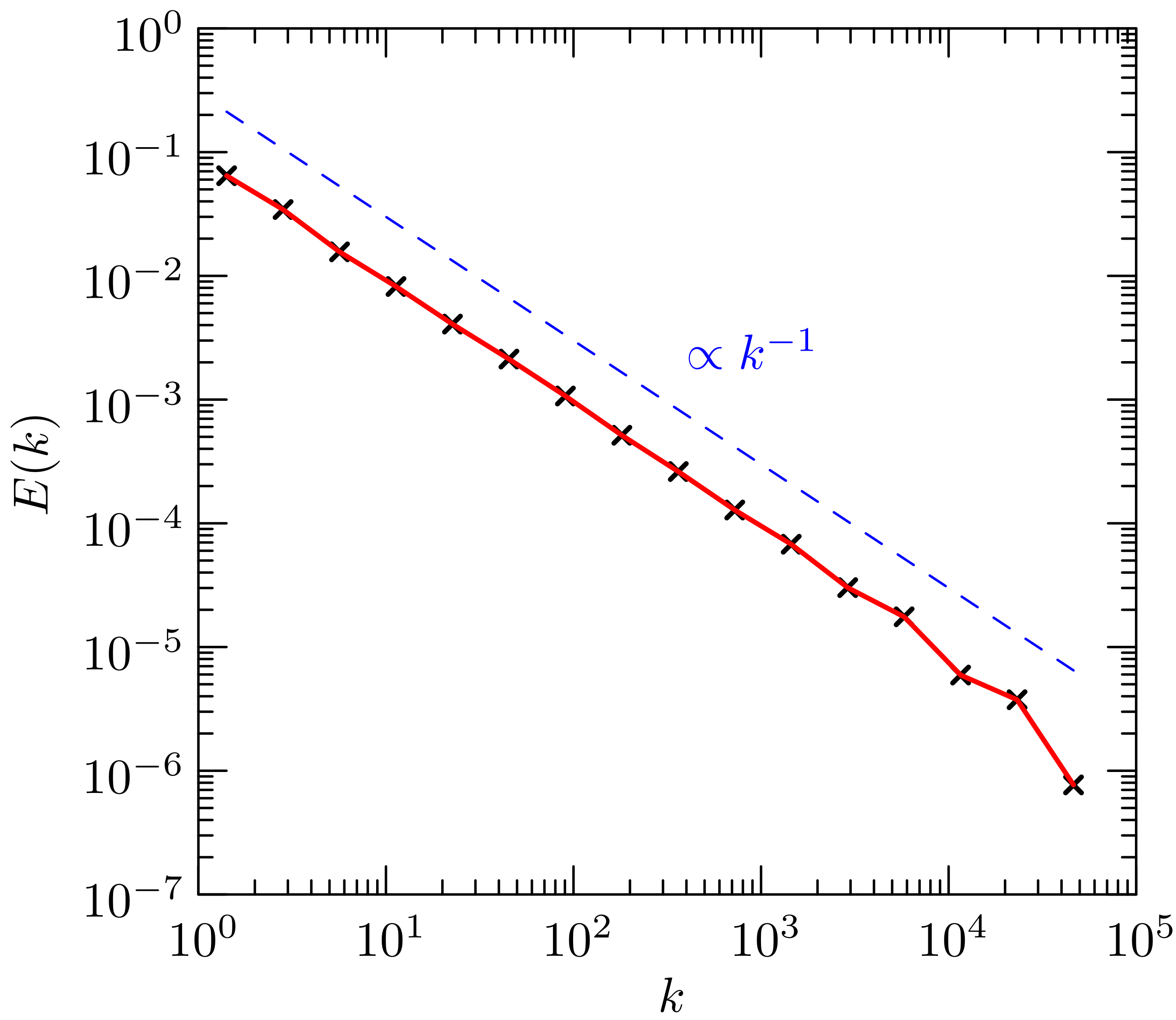


Modal Truncation

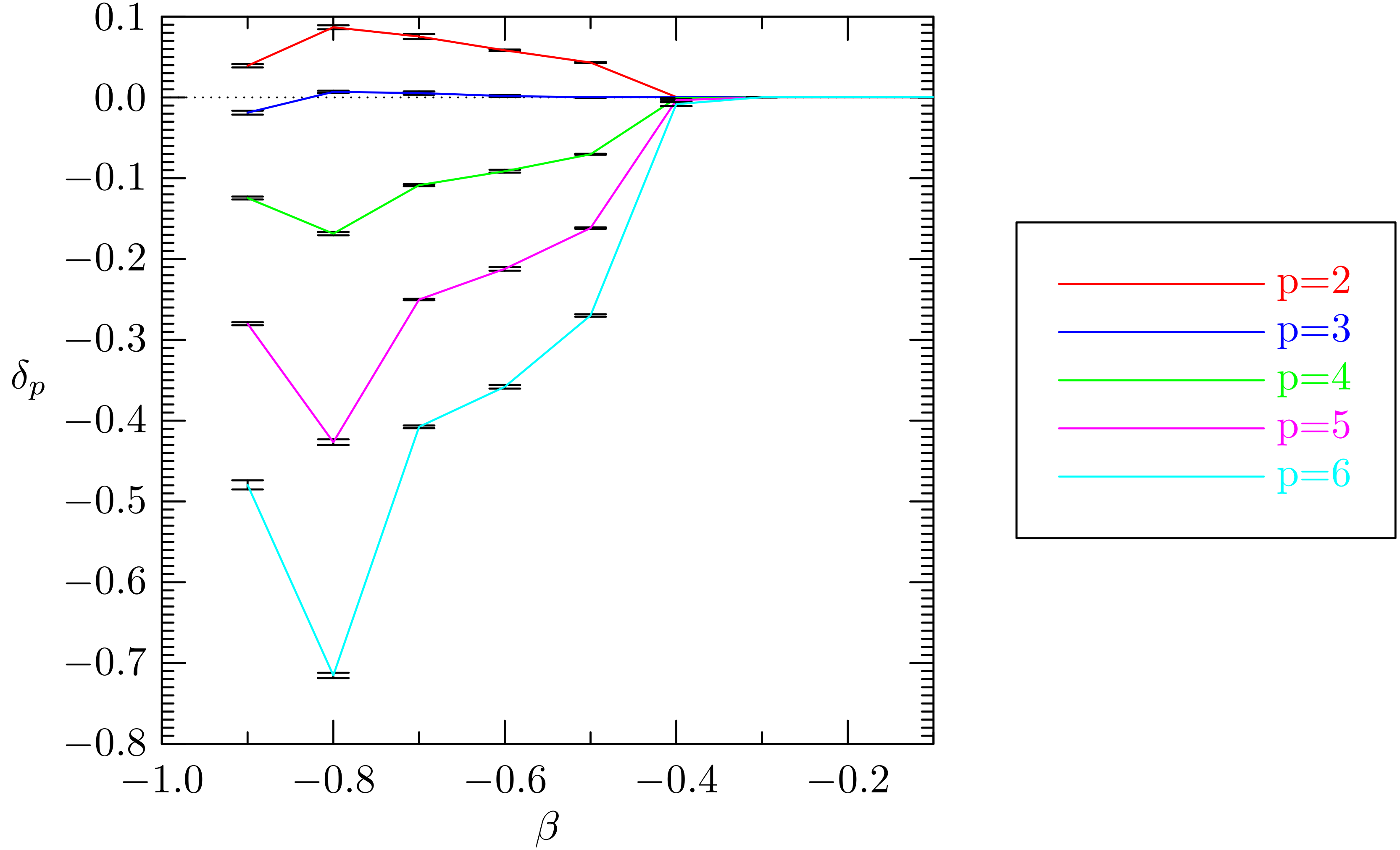


Effect of a finite number of shells on the energy spectrum.

Equipartition Spectrum ($\epsilon = \nu = 0$).

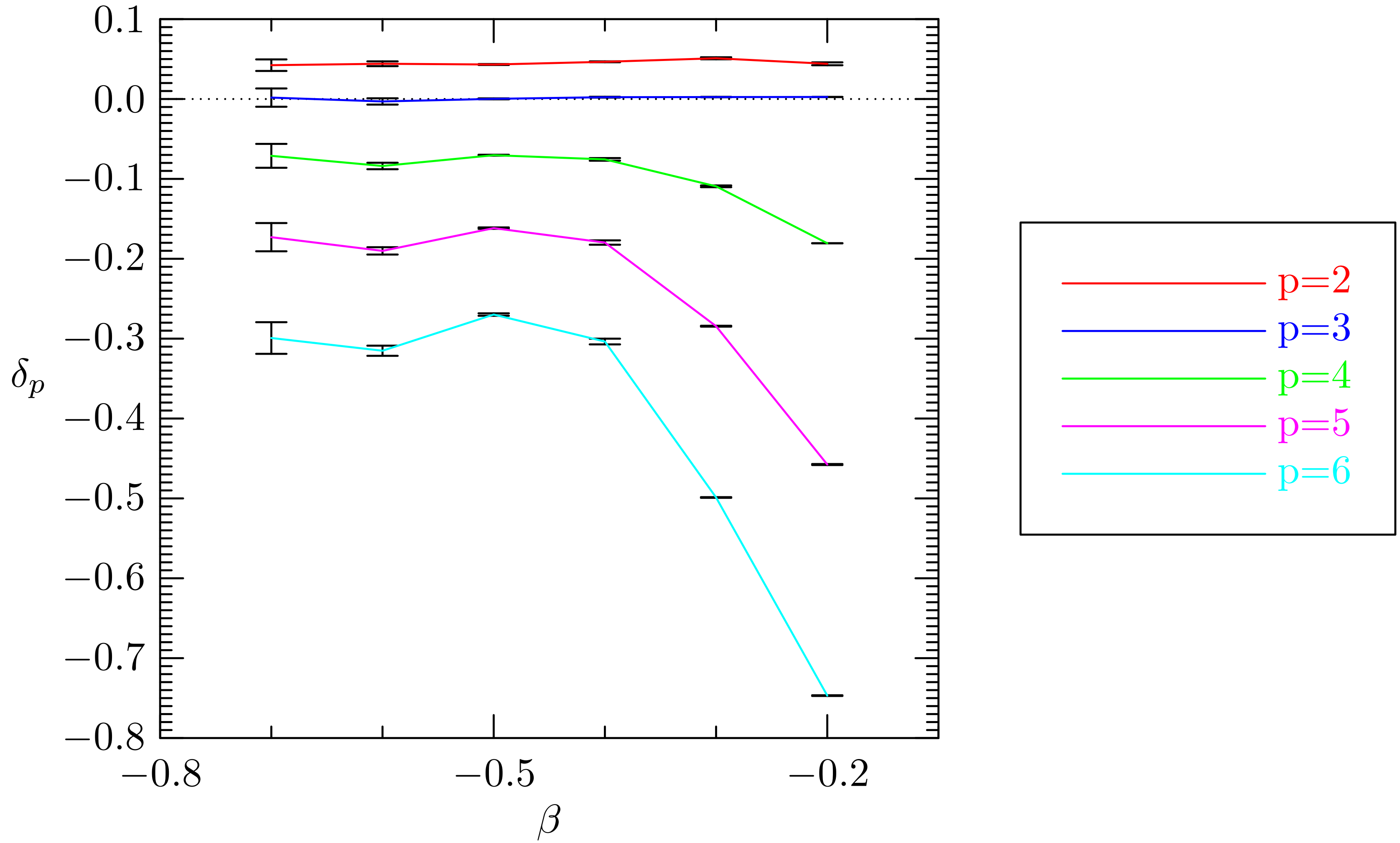


Intermittency vs. β



Kadanoff: no variation on helicity-preserving curve $\lambda = \dfrac{1}{1+\beta}$.

Helicity Preserving Cases



Evidence against conjecture of Kadanoff *et al.*

Conclusions

- The GOY model mirrors many properties (scaling, intermittency) of real turbulence.
- The typical dissipation scale is much smaller than the Kolmogorov length, due to the inherent steepening of the spectrum by intermittency corrections.
- Derived and confirmed a relation between the second-order intermittency correction and the dissipation wavenumber k_d.
- Need to fully resolve the small scales: insufficient resolution of the viscous subrange can affect inertial-range scaling exponents.
- For Kadanoff's standard case, truncation wavenumber must typically be three times higher than k_d.
- Strong variations in the intermittency exponent occur even along the helicity preserving curve, disproving the conjecture of Kadanoff *et al* [1995].

Asymptote: The Vector Graphics Language

`http://asymptote.sf.net`

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