Spectral Reduction of the GOY Shell Turbulence Model

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Outline

Shell Models

- Shell models are reduced models of turbulence formulated in Fourier space.
- Typical velocity Fourier amplitudes on each shell n in wavenumber space are represented by a single quantity u_n.
- The shell wavenumbers $k_n = \lambda^n$ scale geometrically.
- General form:

$$(d_t + \nu k_n^2) u_n = i k_n \sum_{l,m} A_{l,m} u^*_{n+l} u^*_{n+m} + F.$$

- We take the forcing F to be a white-noise random process.
- This allows one to control the mean rate of energy injection [Novikov 1964]: $\epsilon = F^2/2$.

DN Model

Restrict shell model:

- Only nearest neighbour couplings.
- Enforce conservation of energy $\frac{1}{2}\sum_n |u_n|^2$ by nonlinearity.

  $\Rightarrow$ generalized Desnyansky and Novikov [1974] model (DN):

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i k_n (a_n u_{n-1}^{*2} - \lambda a_{n+1} u_n^* u_{n+1}^*)$$
$$+ i k_n (b_n u_{n-1}^* u_n^* - \lambda b_{n+1} u_{n+1}^{*2}),$$

- For constant coefficients a_n and b_n of opposite sign, Bell and Nelkin [1977] showed that when $\nu = 0$ this model has a (linearly) stable fixed point, corresponding to the Kolmogorov scaling

$$u_n = A k_n^{-1/3}.$$

- Since the fixed point is stable, the DN system does not exhibit intermittency.

GOY Model

- Complex version of the Gledzer [1973] model proposed by Yamada and Ohkitani [1987]:

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i k_n \left(\bar{\alpha} u_{n+1}^* u_{n+2}^* + \bar{\beta} u_{n-1}^* u_{n+1}^* + \bar{\gamma} u_{n-1}^* u_{n-2}^*\right)$$
$$+ F\delta_{n,0},$$

where

$$\bar{\alpha} \doteq \alpha, \qquad \bar{\beta} \doteq \frac{\beta}{\lambda}, \qquad \bar{\gamma} \doteq \frac{\gamma}{\lambda^2}.$$

- When $\nu = F = 0$, the Goy model has an unstable fixed point, again corresponding to the Kolmogorov power law

$$u_n = A k_n^{-1/3}.$$

- Given periodic or zero Dirichlet boundary conditions in wavenumber space on u_n, nonlinearity conserves the energy

$$E \doteq \frac{1}{2}\sum_n |u_n|^2,$$

provided $\alpha + \beta + \gamma = 0$.

- Set $\alpha = 1$ by rescaling time $\Rightarrow$ one free parameter δ:

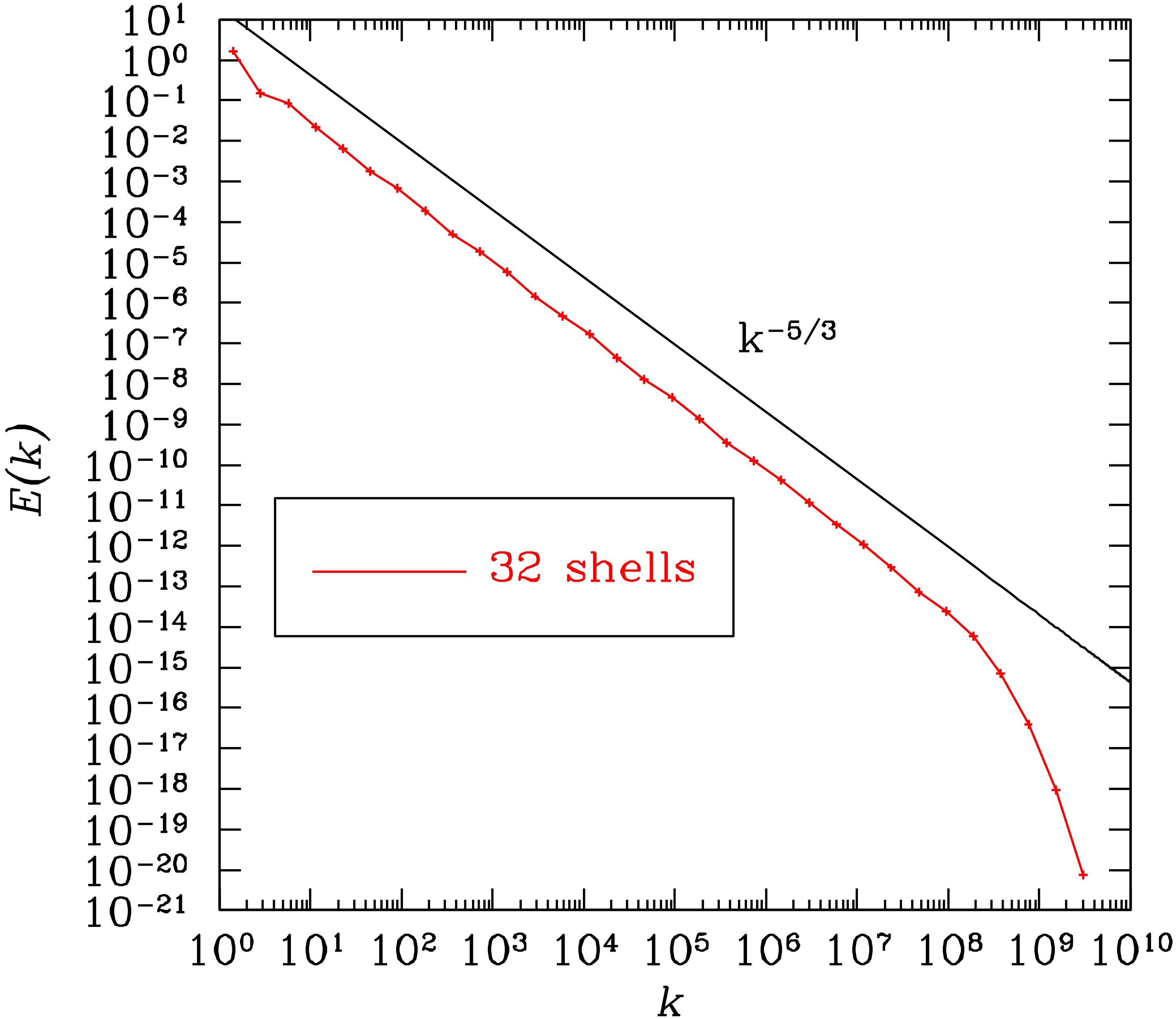
$$\alpha = 1 \qquad \beta = -\delta \qquad \gamma = \delta - 1.$$

- A second invariant $E \doteq \frac{1}{2}\sum_n k_n^p |u_n|^2$ is also conserved, where

$$p = -\log_\lambda(\delta - 1).$$

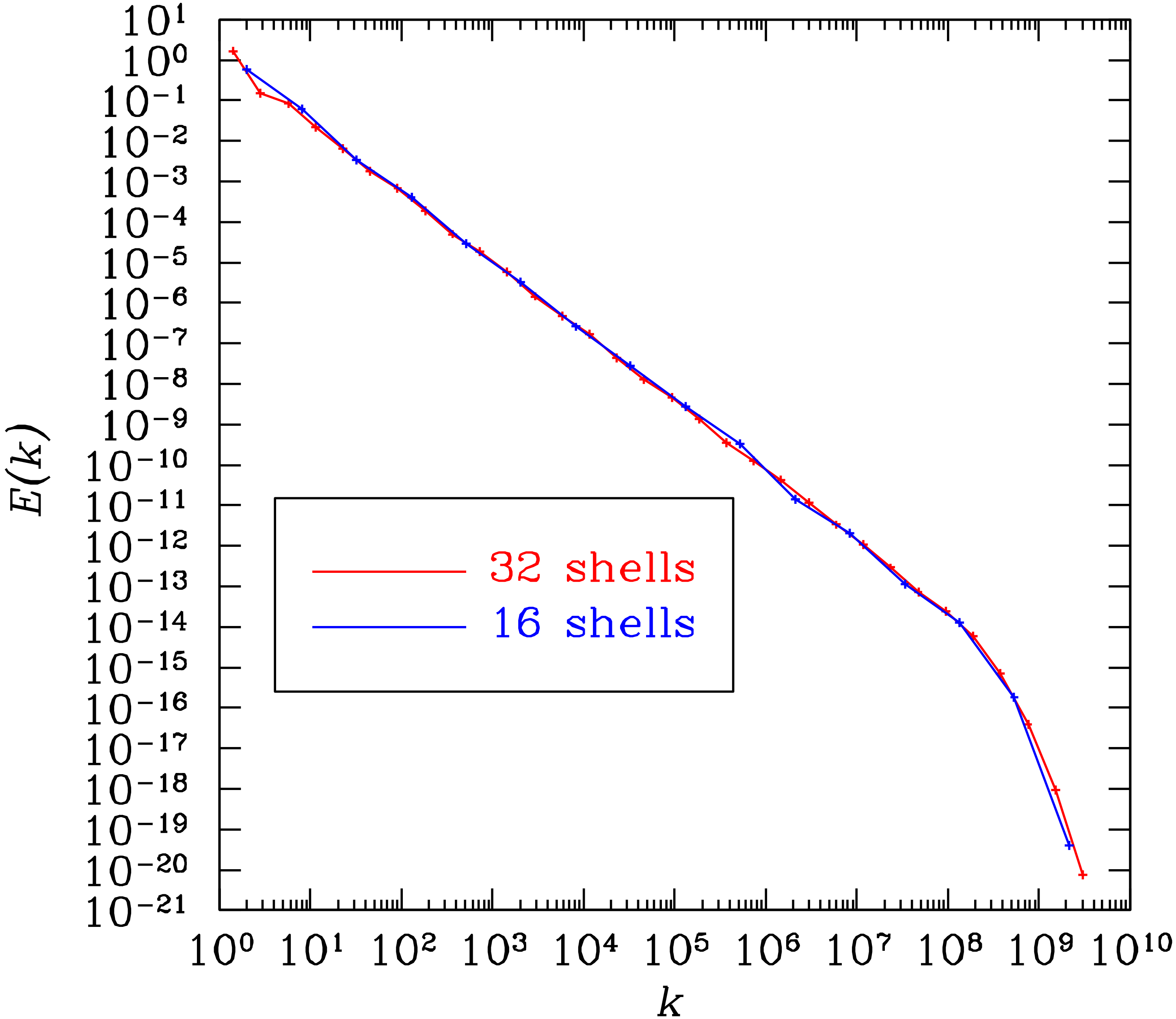
- Consider the case $\lambda = 2$.
- [2D Turbulence:] For $\delta = 5/4$, the second invariant $\frac{1}{2}\sum_n k_n^2 |u_n|^2$ has the dimensions of enstrophy.
- [3D Turbulence:] For $\delta = 1/2$, the second invariant $\frac{1}{2}\sum_n (-1)^n k_n |u_n|^2$ has the dimensions of helicity.

Kolmogorov Law



Energy spectrum for 3D GOY model ($\lambda = 2$)

Spectral Reduction



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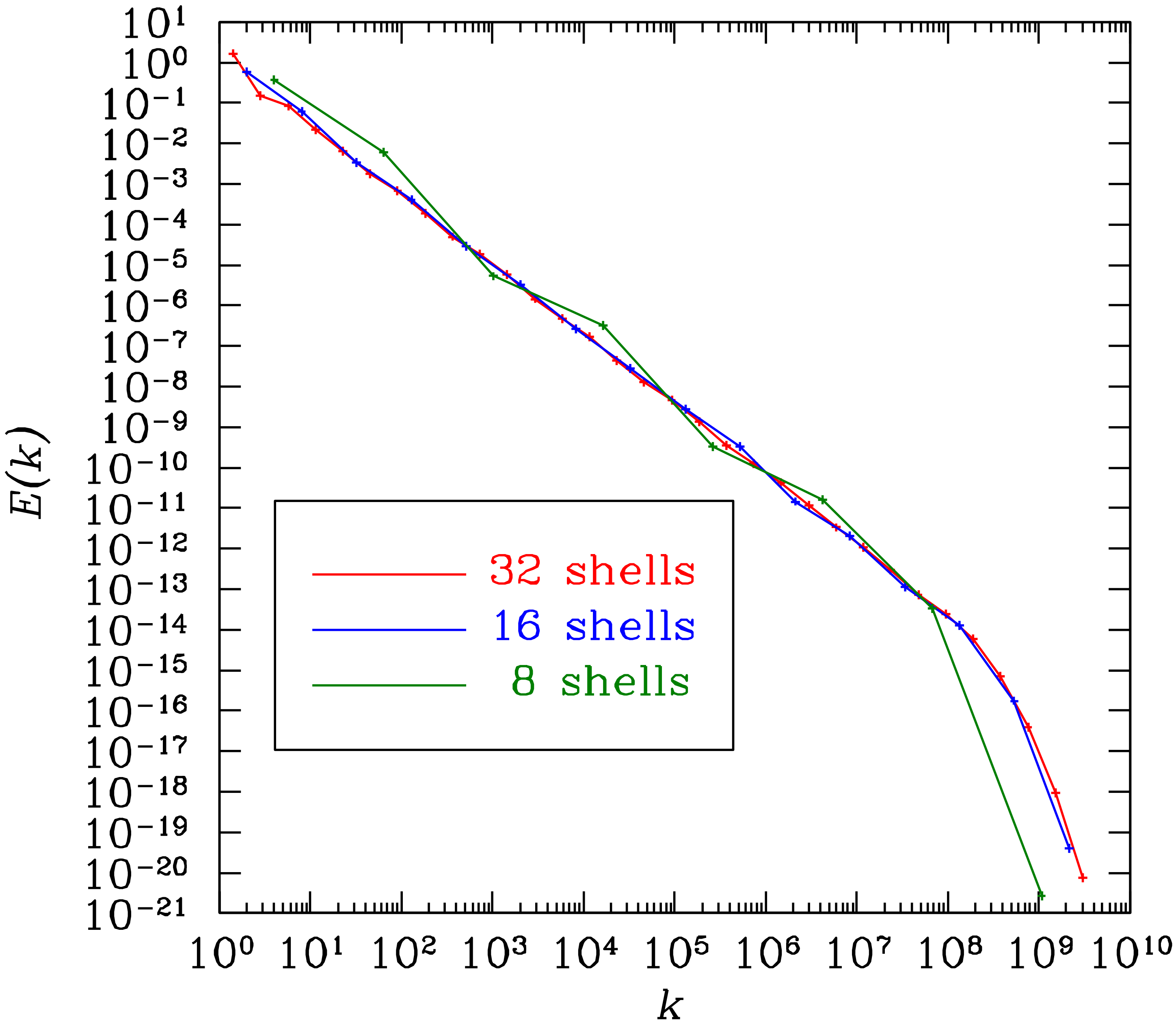

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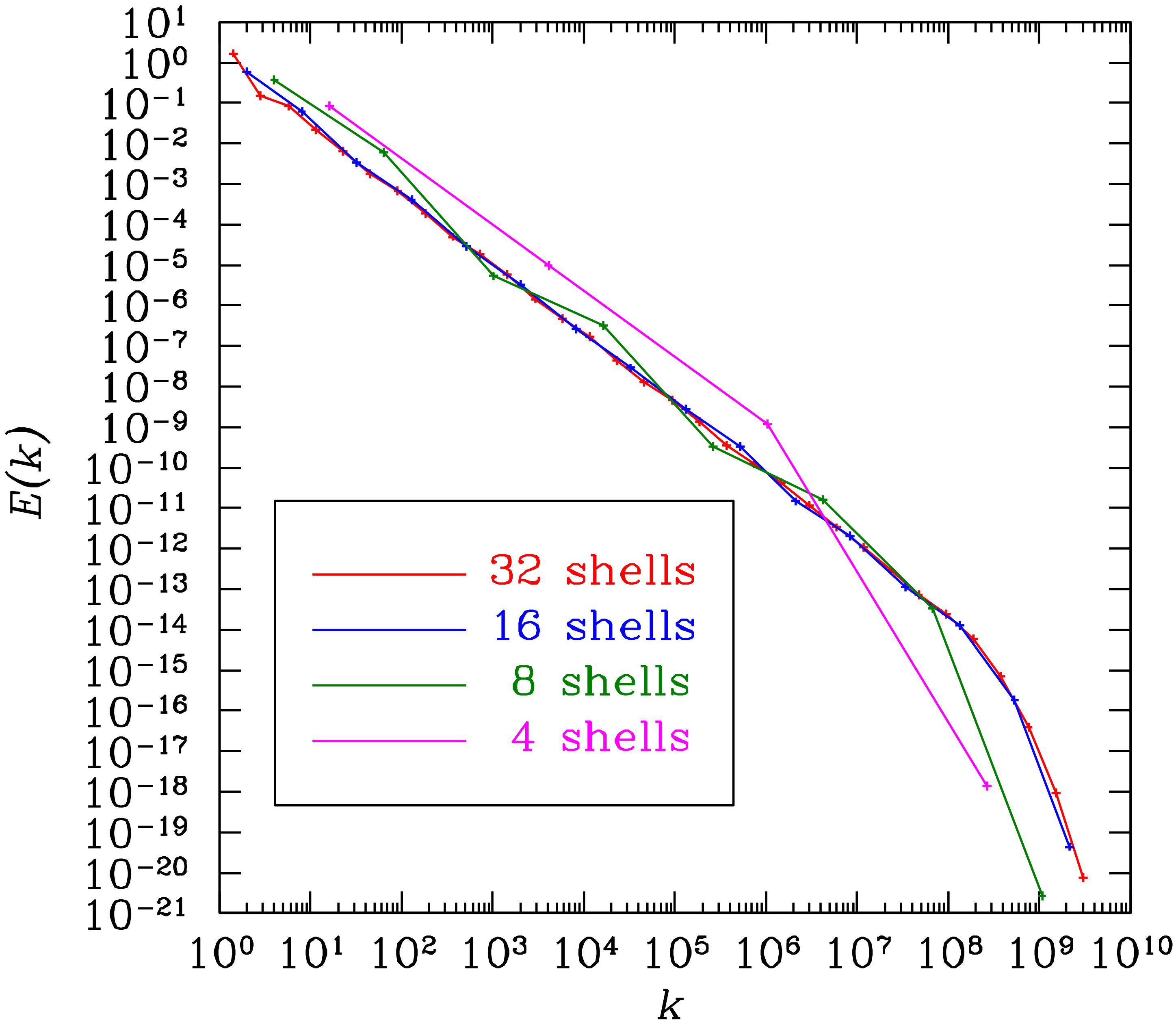
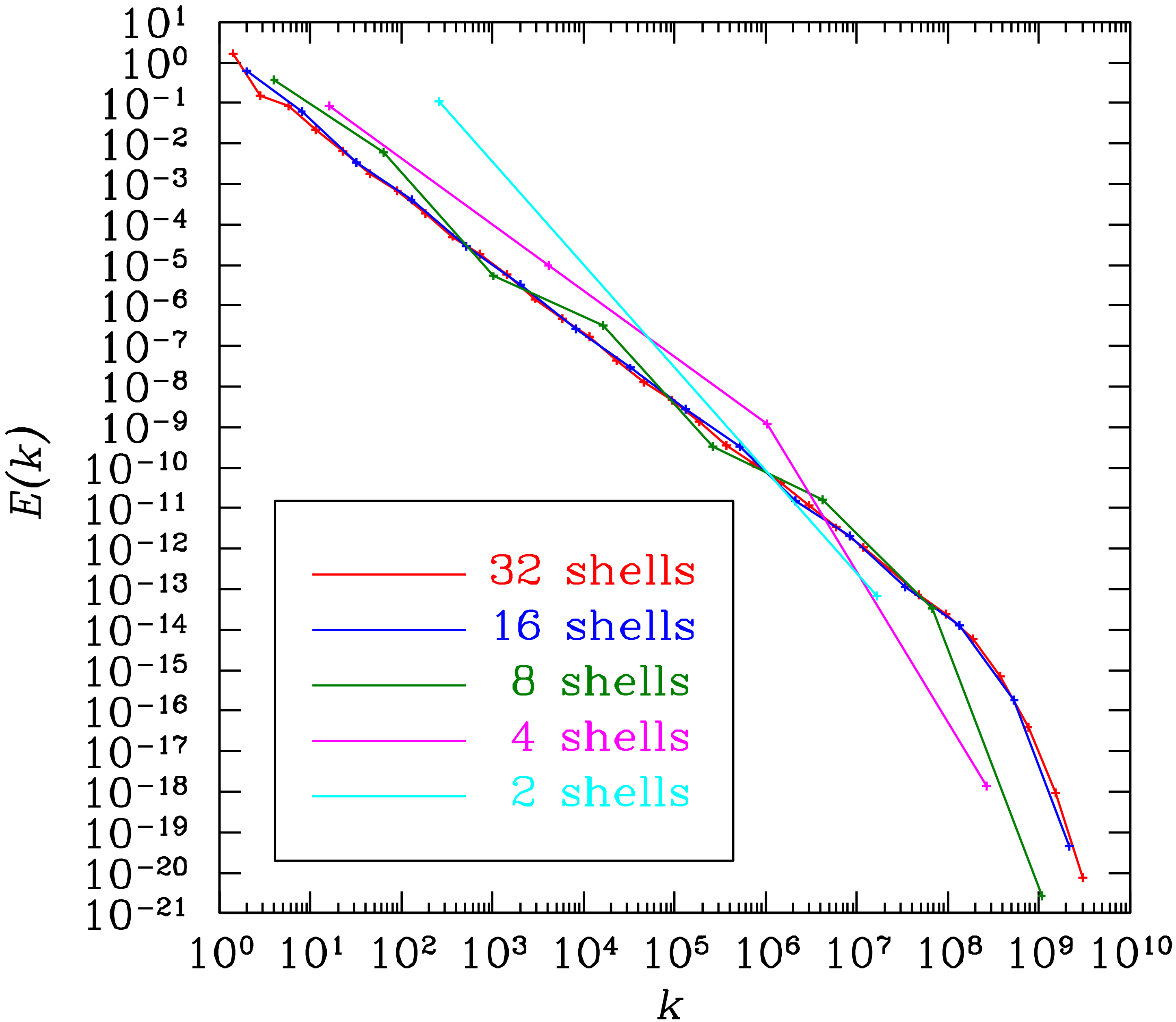


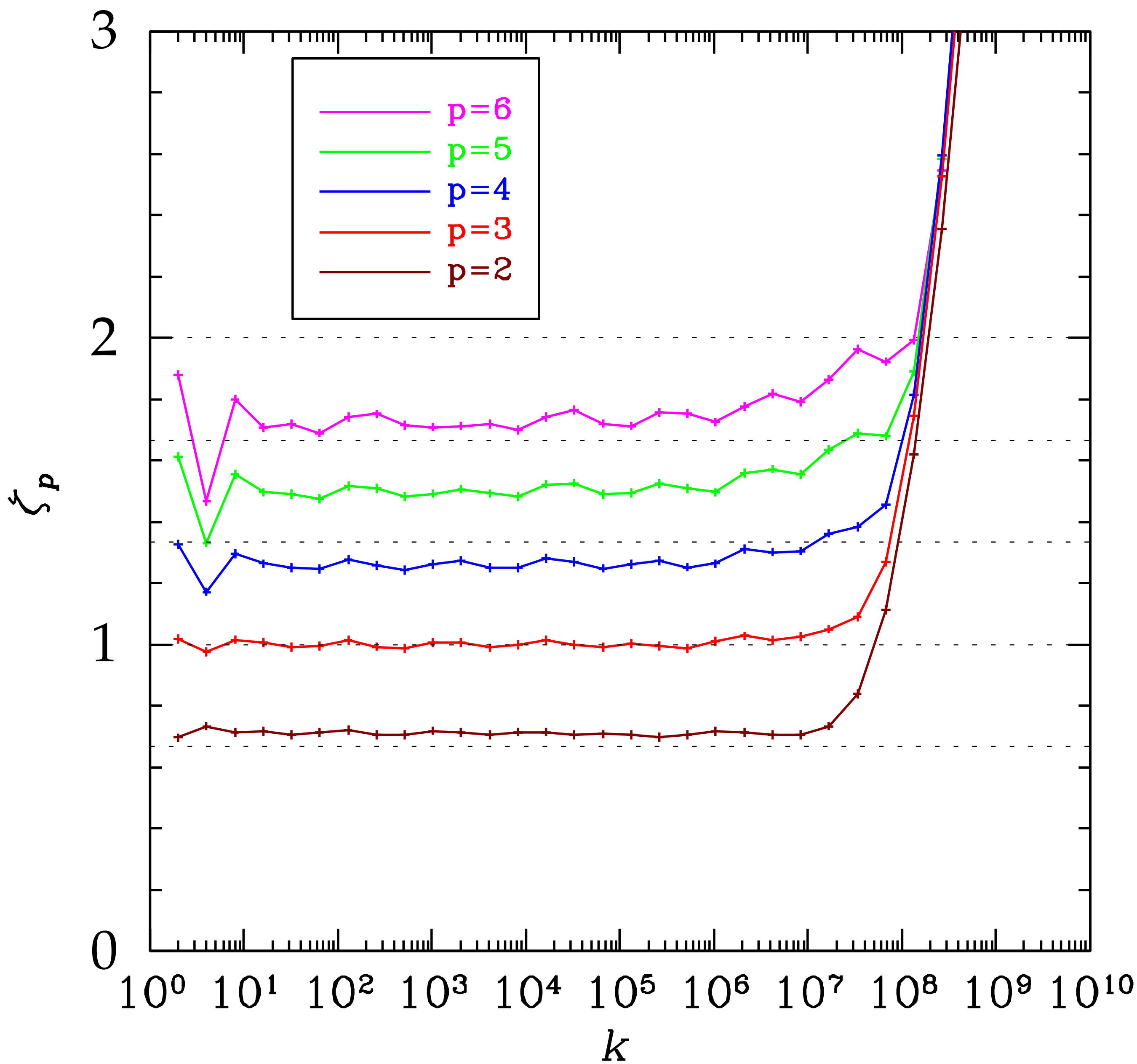
Energy spectrum for 3D GOY model ($\lambda = 2$)

Spectral Reduction



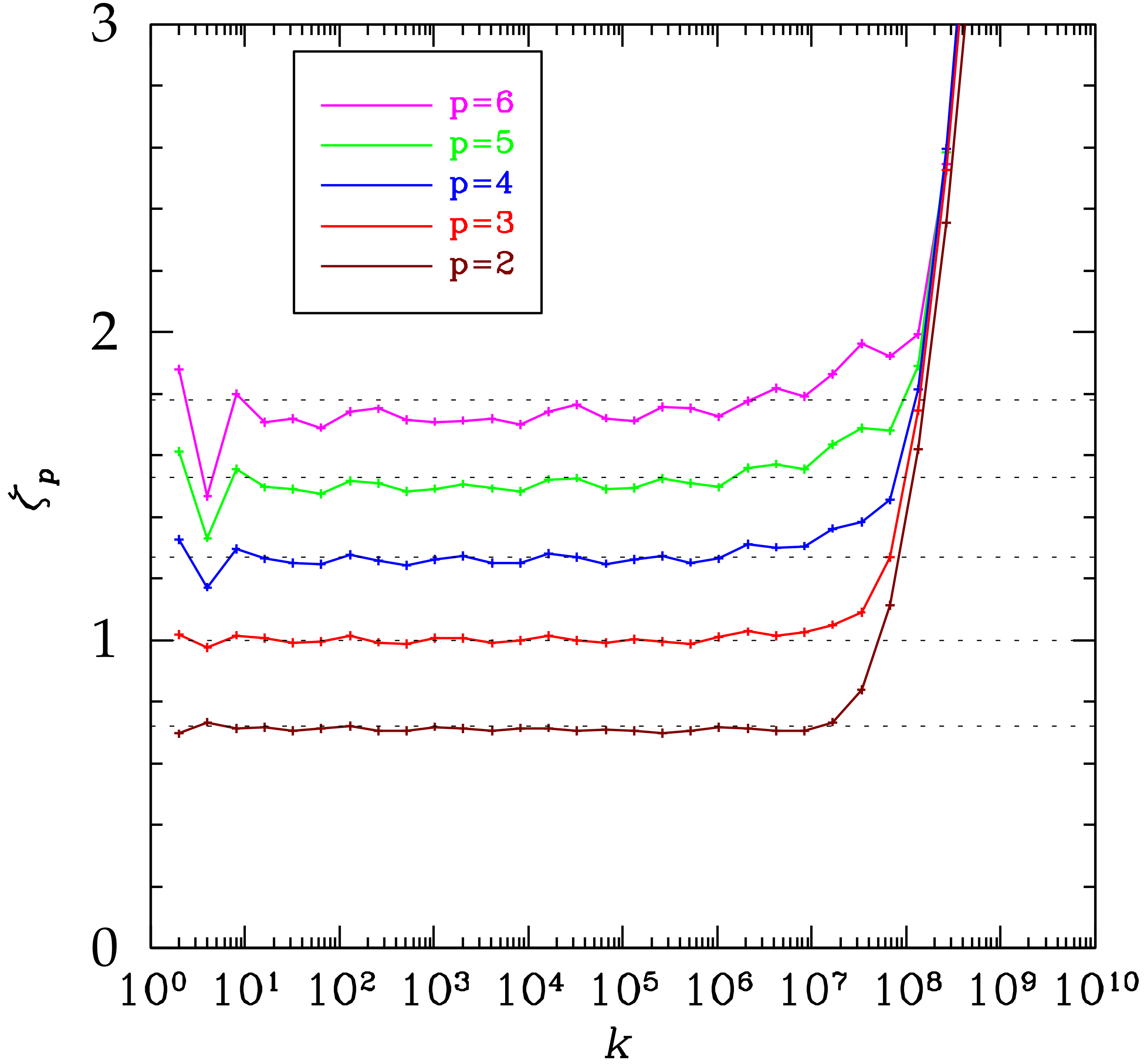
Energy spectrum for 3D GOY model ($\lambda = 2$)

Intermittency



Zeta exponents: 3D GOY model vs. NS (Kolmogorov)

Intermittency



Zeta exponents: 3D GOY model vs. NS (experimental)

Spectral Reduction of GOY Model

- Exploit the continuity of velocity moments in wavenumber.
- Goal: replace neighbouring shells by a reduced number of representative shells with enhanced couplings.
- Define sum and difference variables v_n and d_n:

$$v_n = \frac{u_{2n} + \sigma_n u_{2n+1}}{\sqrt{1+\sigma_n^2}},$$

$$d_n = \frac{-\sigma_n u_{2n} + u_{2n+1}}{\sqrt{1+\sigma_n^2}},$$

  for some real numbers σ_n.
- Neglect the contribution of d_n to nonlinearity: whenever a term u_{2n+1} appears, replace it by $\sigma_n u_{2n}$.
- If the u_n are real and independent of time, the σ_n factors can be chosen to make this approximation exact.

Rescaled sum variables

- Even-index shell variables in the nonlinearity can then be related directly to the sum variables v_n:

$$v_n \approx \frac{u_{2n}(1+\sigma_n^2)}{\sqrt{1+\sigma_n^2}} = u_{2n}\sqrt{1+\sigma_n^2}.$$

- Introduce

$$s_n \doteq \frac{v_n}{\sqrt{1+\sigma_n^2}} = \frac{u_{2n}+\sigma_n u_{2n+1}}{1+\sigma_n^2} \approx u_{2n}.$$

- The even-index shell velocities appearing in the nonlinear term may now be replaced simply by s_n.

- Evolution of rescaled sum variables s_n:

$$\frac{d}{dt}s_n = \frac{1}{1+\sigma_n^2}\frac{d}{dt}(u_{2n} + \sigma_n u_{2n+1})$$
$$= -\mu\Lambda^{2n}s_n + \frac{i\Lambda^n}{1+\sigma_n^2}$$
$$\times\left(a_n s_{n-1}^{*2} - \Lambda a_{n+1}s_n^* s_{n+1}^* + b_n s_{n-1}^* s_n^* - \Lambda b_{n+1}s_{n+1}^{*2}\right),$$

where

$$\mu = \nu\left(\frac{1+\sigma_n^2\lambda^2}{1+\sigma_n^2}\right), \qquad \Lambda = \lambda^2,$$
$$a_n = \bar{\gamma}\sigma_{n-1}, \qquad b_n = -\frac{\alpha}{\lambda}\sigma_{n-1}\sigma_n.$$

- Course-grained energy $\frac{1}{2}\sum_n |s_n|^2(1+\sigma_n^2)$ is conserved.

- In the case where $\sigma_n = \sigma$, we can repeat the renormalization procedure to compute the evolution of

$$S_n \doteq \frac{s_{2n} + \Sigma_n s_{2n+1}}{1 + \Sigma_n^2} \approx s_{2n}.$$

- All of the nonlinear terms containing S_n^{*2} cancel, leaving

$$\frac{d}{dt} S_n = -\bar{\mu}\bar{\Lambda}^{2n} S_n + \frac{i\bar{\Lambda}^n}{(1 + \Sigma_n^2)(1 + \sigma^2)}$$
$$\times \quad \left[A_n S_{n-1}^{*2} - \bar{\Lambda} A_{n+1} S_n^* S_{n+1}^* + B_n S_{n-1}^* S_n^* - \bar{\Lambda} B_{n+1} S_{n+1}^{*2}\right],$$

where

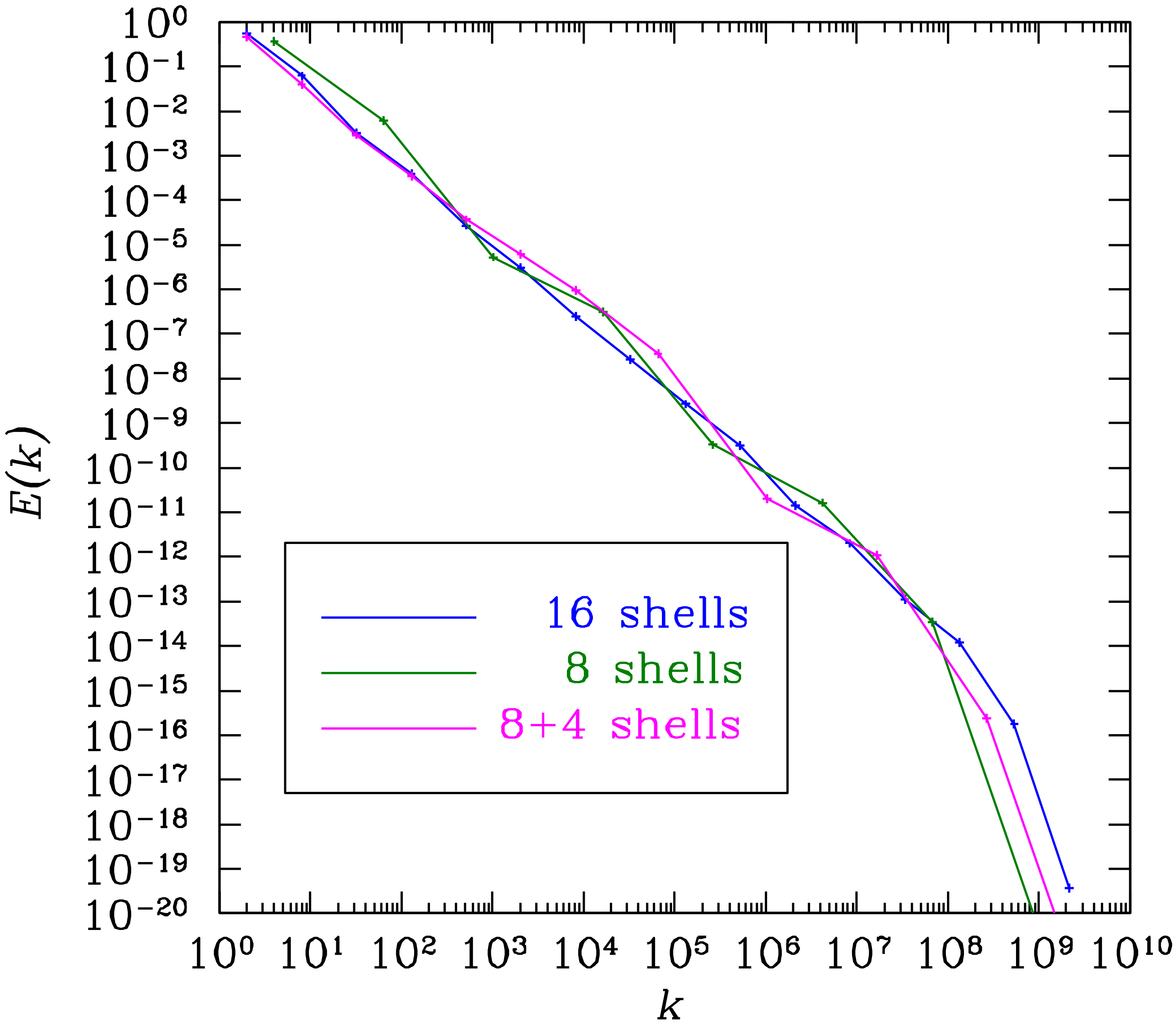
$$\bar{\mu} = \mu \left(\frac{1 + \Sigma_n^2 \Lambda^2}{1 + \Sigma_n^2} \right), \qquad \bar{\Lambda} = \Lambda^2,$$

$$A_n = \Sigma_{n-1}^2 a_{2n}, \qquad B_n = \Sigma_{n-1} b_{2n}.$$

Properties

- Energy $\frac{1}{2}\sum_n |S_n|^2(1+\Sigma_n^2)(1+\sigma^2)$ is again conserved.
- After the first renormalization, the form of the equations remains invariant under susbequent renormalization.
- The form is identical to that of the DN model but with coefficients of the same sign (for the 3D Goy model).

Subgrid Model



Conclusions

- The Goy model is an interesting dynamical system that mirrors many properties (scaling, intermittency) of real turbulence.
- It provides an excellent testbed for new ideas and methods for two- and three-dimensional turbulence, for example, the method of Spectral Reduction.
- These ideas can be used to develop reliable dynamical subgrid models.