Bounds on the Global Attractor of 2D Incompressible Turbulence in the palenstrophy–enstrophy–energy space

John C. Bowman and Pedram Emami
Department of Mathematical and Statistical Sciences
University of Alberta

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Turbulence

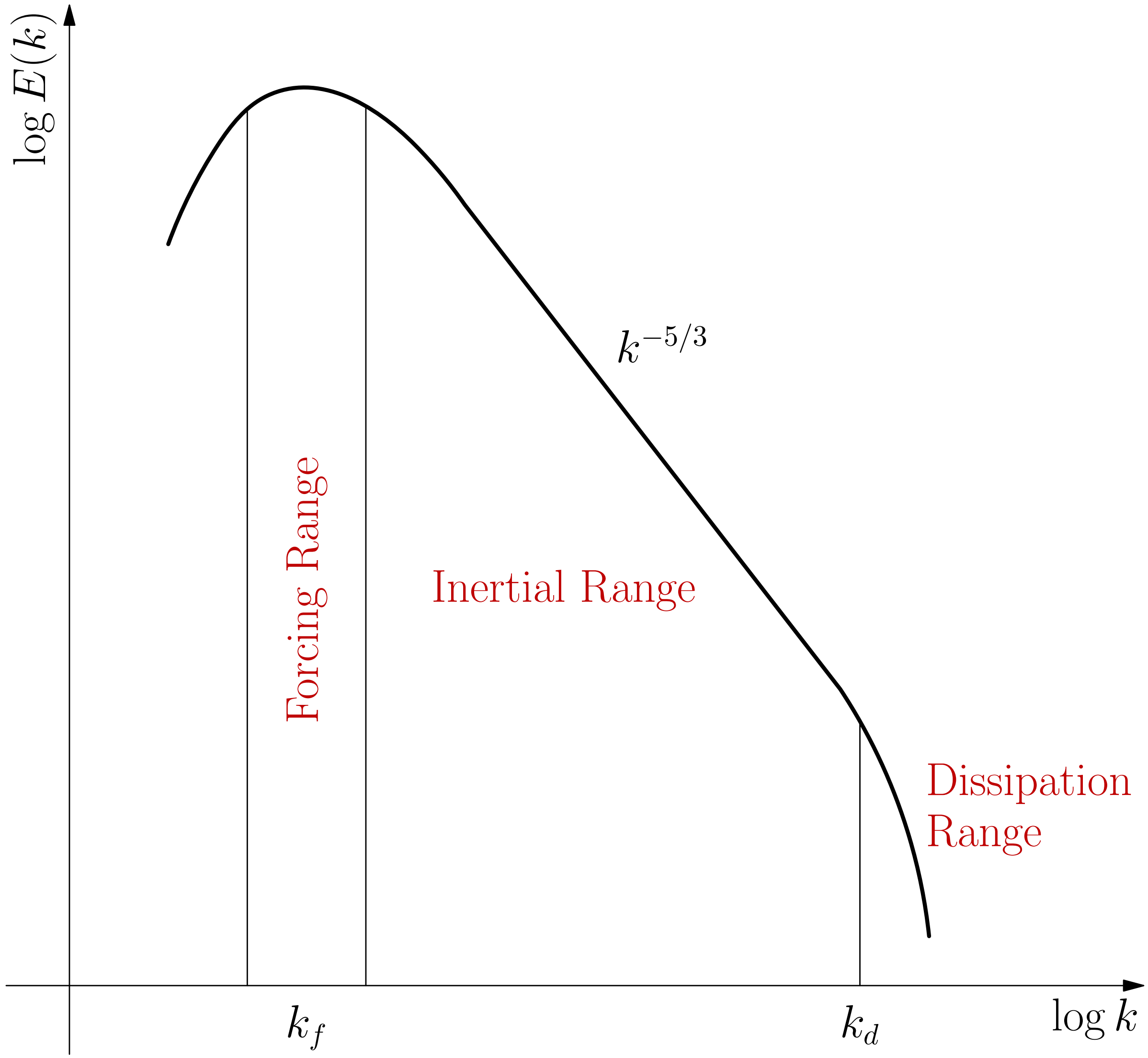
- In 1941, Kolmogorov conjectured that the energy spectrum of 3D incompressible turbulence exhibits a self-similar power-law scaling characterized by a uniform *cascade* of energy to molecular (viscous) scales:

$$E(k) = C\epsilon^{2/3}k^{-5/3}.$$

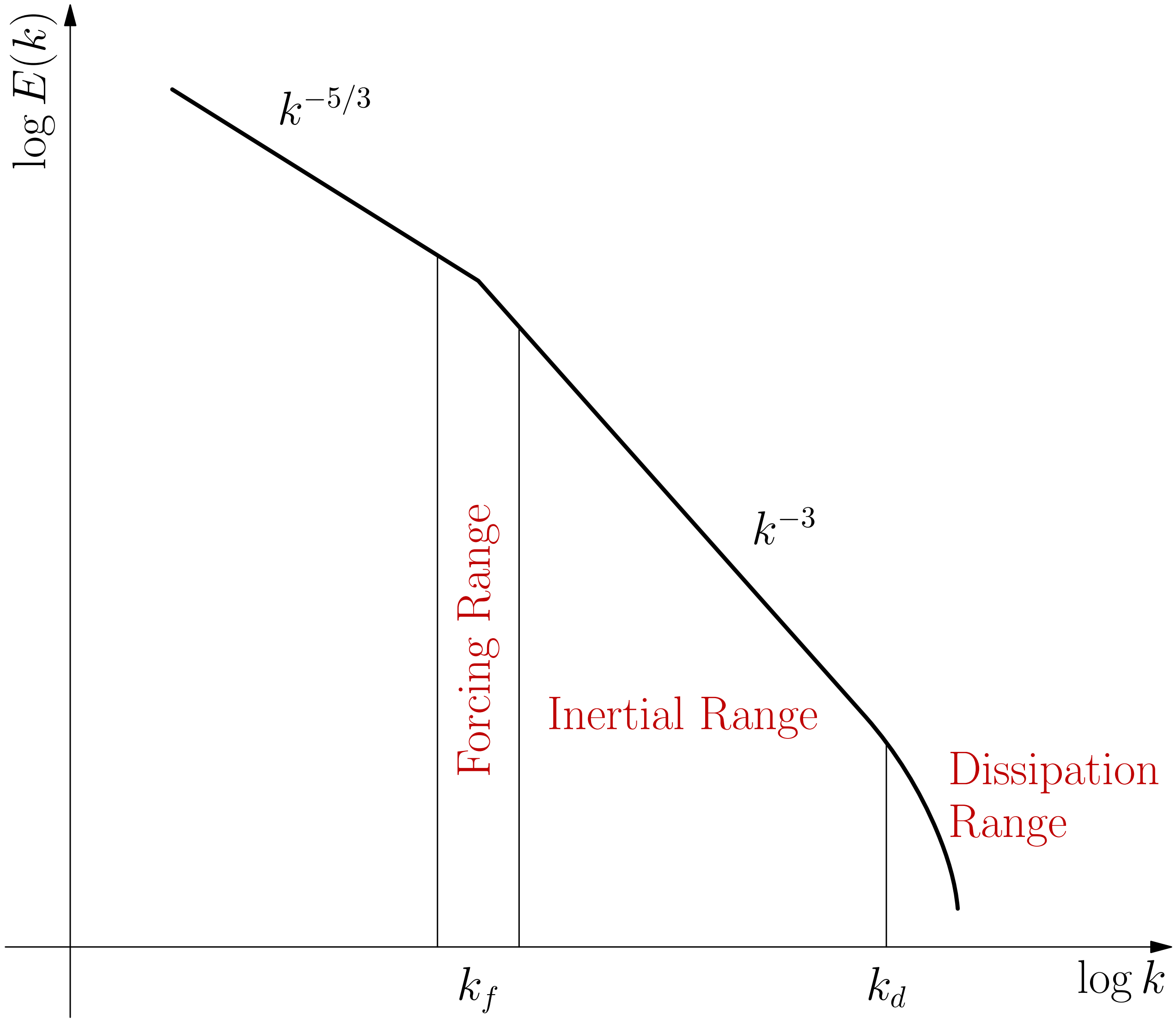
- Here k is the Fourier wavenumber and $E(k)$ is normalized so that $\int E(k)\,dk$ is the total energy.

- Kolmogorov suggested that C might be a universal constant.

3D Energy Cascade



2D Energy Cascade



2D Turbulence

- Consider the Navier–Stokes equations for 2D incompressible homogeneous isotropic turbulence with density $\rho = 1$:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} P = \boldsymbol{F},$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$
$$\int_\Omega \boldsymbol{u}\, d\boldsymbol{x} = \boldsymbol{0}, \qquad \int_\Omega \boldsymbol{F}\, d\boldsymbol{x} = \boldsymbol{0},$$
$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}),$$

with $\Omega = [0, 2\pi] \times [0, 2\pi]$ and periodic boundary conditions on $\partial\Omega$.

- Introduce the Hilbert space

$$H(\Omega) \doteq \mathrm{cl} \left\{ \boldsymbol{u} \in (C^2(\Omega) \cap L^2(\Omega))^2 \mid \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,\ \int_\Omega \boldsymbol{u}\, d\boldsymbol{x} = \boldsymbol{0} \right\}.$$

with inner product $(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t)\, d\boldsymbol{x}$ and L^2 norm $|\boldsymbol{u}| = (\boldsymbol{u}, \boldsymbol{u})^{1/2}$.

- For $\boldsymbol{u} \in H(\Omega)$, the Navier–Stokes equations can be expressed:

$$\frac{d\boldsymbol{u}}{dt} - \nu\nabla^2\boldsymbol{u} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}P = \boldsymbol{F}.$$

- Introduce $A \doteq -\mathcal{P}(\nabla^2)$, $\boldsymbol{f} \doteq \mathcal{P}(\boldsymbol{F})$, and the bilinear map

$$\mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) \doteq \mathcal{P}\left(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}P\right),$$

where $\mathcal{P} : C^2(\Omega) \to H(\Omega)$ is the Helmholtz–Leray projection:

$$\mathcal{P}(\boldsymbol{v}) \doteq \boldsymbol{v} - \boldsymbol{\nabla}\nabla^{-2}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}.$$

- The dynamical system can then be compactly written:

$$\frac{d\boldsymbol{u}}{dt} + \nu A\boldsymbol{u} + \mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}.$$

Stokes Operator A

- The operator $A = \mathcal{P}(-\nabla^2)$ is positive semi-definite and self-adjoint, with a compact inverse.

- On the periodic domain $\Omega = [0, 2\pi] \times [0, 2\pi]$, the eigenvalues of A are

$$\lambda = \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{k}, \qquad \boldsymbol{k} \in \mathbb{Z} \times \mathbb{Z} \backslash \{\boldsymbol{0}\}.$$

- The eigenvalues of A can be arranged as

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \qquad \lambda_0 = 1$$

and its eigenvectors $\boldsymbol{w}_i$, $i \in \mathbb{N}_0$, form an orthonormal basis for the Hilbert space H, upon which we can define any quotient power of A:

$$A^\alpha \boldsymbol{w}_j = \lambda_j^\alpha \boldsymbol{w}_j, \qquad \alpha \in \mathbb{R}, \quad j \in \mathbb{N}_0.$$

Subspace of Finite Enstrophy

- We define the subspace of H consisting of solutions with finite enstrophy:

$$V \doteq \left\{ \boldsymbol{u} \in H \mid \sum_{j=0}^{\infty} \lambda_j (\boldsymbol{u}, \boldsymbol{w}_j)^2 < \infty \right\}.$$

- Another suitable norm for elements $\boldsymbol{u} \in V$ is

$$||\boldsymbol{u}|| = \left| A^{1/2} \boldsymbol{u} \right| = \left(\int_\Omega \sum_{i=1}^{2} \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \frac{\partial \boldsymbol{u}}{\partial x_i} \right)^{1/2} = \left(\sum_{j=0}^{\infty} \lambda_j (\boldsymbol{u}, \boldsymbol{w}_j)^2 \right)^{1/2}.$$

Quadratic Quantities

- For any solution $\boldsymbol{u}$ of the 2D Navier–Stokes equation, the nth-order quadratic quantity is

$$E_n = \frac{1}{2}|A^n \boldsymbol{u}|^2,$$

- E_0, $Z \doteq E_{1/2}$, and $P \doteq E_1$ are called the energy, enstrophy, and palinstrophy.

Properties of the Bilinear Map

- We make use of the antisymmetry

$$(\boldsymbol{\mathcal{B}}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}) = -(\boldsymbol{\mathcal{B}}(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{v}),$$

which implies the conservation of the energy $E_0 = \frac{1}{2}|u|^2$.

- In 2D, we also have orthogonality:

$$(\boldsymbol{\mathcal{B}}(\boldsymbol{u},\boldsymbol{u}),A\boldsymbol{u}) = 0$$

and the strong form of enstrophy invariance:

$$(\boldsymbol{\mathcal{B}}(A\boldsymbol{v},\boldsymbol{v}),\boldsymbol{u}) = (\boldsymbol{\mathcal{B}}(\boldsymbol{u},\boldsymbol{v}),A\boldsymbol{v}).$$

which implies the conservation of the enstrophy $E_{\frac{1}{2}} = \frac{1}{2}|A^{1/2}u|^2$.

- In 2D, the above properties imply the symmetry

$$(\boldsymbol{\mathcal{B}}(\boldsymbol{v},\boldsymbol{v}),A\boldsymbol{u}) + (\boldsymbol{\mathcal{B}}(\boldsymbol{v},\boldsymbol{u}),A\boldsymbol{v}) + (\boldsymbol{\mathcal{B}}(\boldsymbol{u},\boldsymbol{v}),A\boldsymbol{v}) = 0.$$

Dynamical Behaviour

- Our starting point is the incompressible 2D Navier–Stokes equation with periodic boundary conditions:

$$\frac{d\boldsymbol{u}}{dt} + \nu A\boldsymbol{u} + \mathcal{B}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{f}, \qquad \boldsymbol{u} \in H.$$

- Take the inner product with $\boldsymbol{u}$ (respectively $A\boldsymbol{u}$):

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}(t)|^2 + \nu||\boldsymbol{u}(t)||^2 = (\boldsymbol{f}, \boldsymbol{u}(t)),$$
$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{u}(t)||^2 + \nu|A\boldsymbol{u}(t)|^2 = (\boldsymbol{f}, A\boldsymbol{u}(t)).$$

- The Cauchy–Schwarz and Poincaré inequalities yield

$$(\boldsymbol{f}, \boldsymbol{u}(t)) \leq |\boldsymbol{f}||\boldsymbol{u}(t)| \qquad \text{and} \qquad |\boldsymbol{u}(t)| \leq ||\boldsymbol{u}(t)||.$$

- Since the existence and uniqueness for solutions to the 2D Navier–Stokes equation has been proven, a global attractor can be defined [Ladyzhenskaya 1975], [Foias & Temam 1979].

Dynamical Behaviour: Constant Forcing

- If the force $\boldsymbol{f}$ is constant with respect to time, a Gronwall inequality can be exploited:

$$|\boldsymbol{u}(t)|^2 \le e^{-\nu t}|\boldsymbol{u}(0)|^2 + (1 - e^{-\nu t})\left(\frac{|\boldsymbol{f}|}{\nu}\right)^2.$$

- Defining a nondimensional Grashof number $G = \dfrac{|\boldsymbol{f}|}{\nu^2}$, the above inequality can be simplified to

$$|\boldsymbol{u}(t)|^2 \le e^{-\nu t}|\boldsymbol{u}(0)|^2 + (1 - e^{-\nu t})\nu^2 G^2.$$

- Similarly,

$$||\boldsymbol{u}(t)||^2 \le e^{-\nu t}||\boldsymbol{u}(0)||^2 + (1 - e^{-\nu t})\nu^2 G^2.$$

- Being on the attractor thus requires

$$|\boldsymbol{u}| \le \nu G \qquad \text{and} \qquad ||\boldsymbol{u}|| \le \nu G.$$

Z–E Bounds: Constant Forcing

- A trivial lower bound is provided by the Poincaré inequality:

$$|\boldsymbol{u}|^2 \le ||\boldsymbol{u}||^2 \quad \Rightarrow \quad E \le Z.$$

- An upper bound is given by

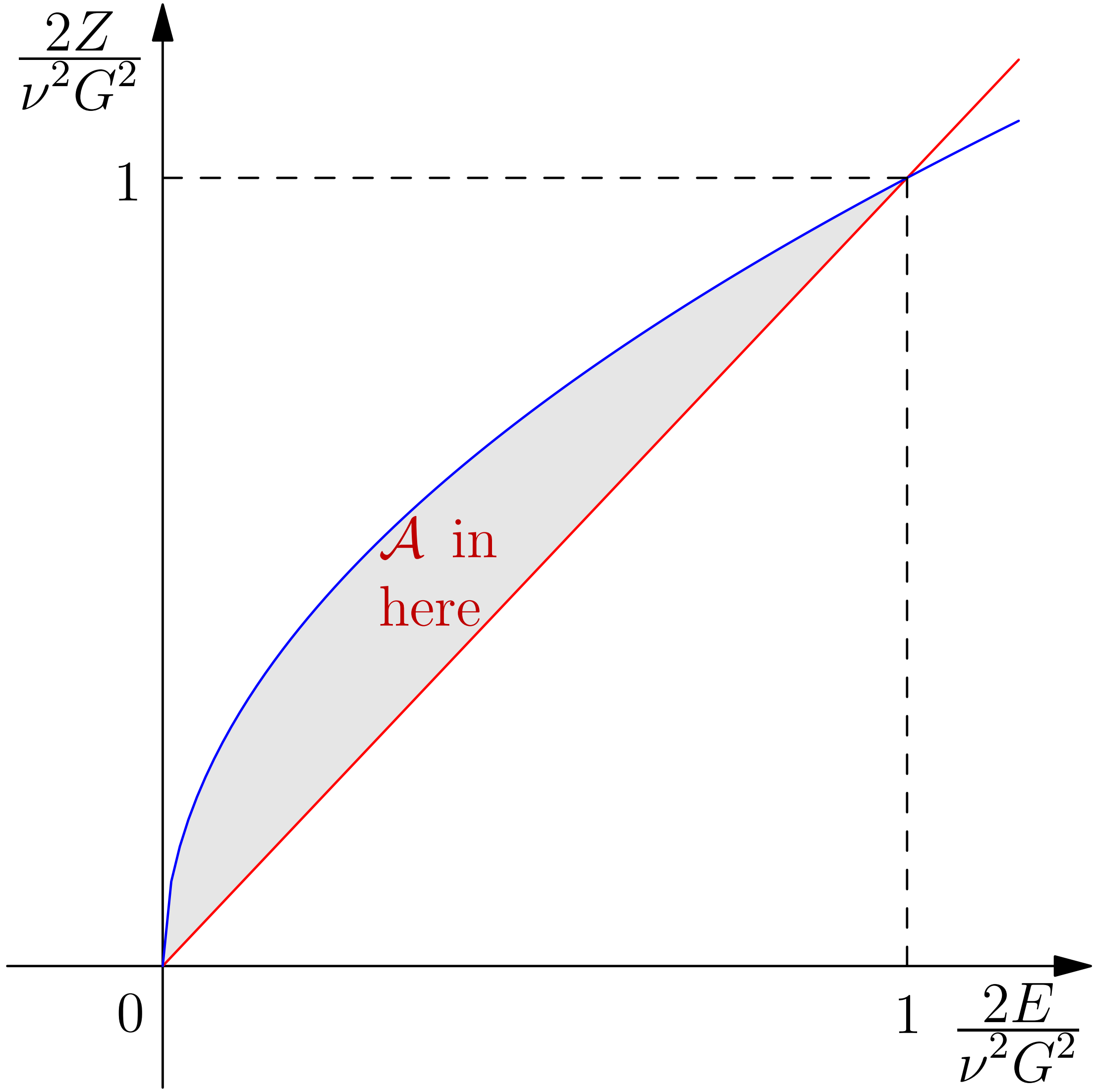
Theorem 1 (Dascaliuc, Foias, and Jolly [2005])
For all $\boldsymbol{u} \in \mathcal{A}$,

$$||\boldsymbol{u}||^2 \le \frac{|\boldsymbol{f}|}{\nu}|\boldsymbol{u}|.$$

- That is,

$$Z \le \nu G \sqrt{E}.$$

Z–E Bounds: Constant Forcing



Extended Norm: Random Forcing

- For a random variable α, with probability density function P, define the ensemble average

$$\langle \alpha \rangle = \int_{-\infty}^{\infty} \alpha \left(\frac{dP}{d\zeta} \right) d\zeta.$$

- The extended inner product is

$$(\boldsymbol{u}, \boldsymbol{v})_{\tilde{\omega}} \doteq \int_{\Omega} \langle \boldsymbol{u} \cdot \boldsymbol{v} \rangle \; d\boldsymbol{x} = \int_{\Omega} \left(\int_{-\infty}^{\infty} \boldsymbol{u} \cdot \boldsymbol{v} \, \frac{dP}{d\zeta} d\zeta \right) d\boldsymbol{x},$$

with norm

$$|\boldsymbol{f}|_{\tilde{\omega}} \doteq \left(\int_{\Omega} \langle |\boldsymbol{f}|^2 \rangle \; d\boldsymbol{x} \right)^{1/2}.$$

- The n-th order injection rate is $\epsilon_n = \big(\boldsymbol{f}, A^{2n} \boldsymbol{u}\big)$.

Dynamical Behaviour: Random Forcing

- Energy balance:

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|^2 + \nu(A\boldsymbol{u}, \boldsymbol{u}) + (\mathcal{B}(\boldsymbol{u}, \boldsymbol{u}), \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u}) \doteq \epsilon,$$

where $\epsilon \doteq \epsilon_0$ is the rate of energy injection.

- From the energy conservation identity $(\mathcal{B}(\boldsymbol{u}, \boldsymbol{u}), \boldsymbol{u}) = 0$,

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|^2 + \nu||\boldsymbol{u}||^2 = \epsilon.$$

- The Poincaré inequality $||\boldsymbol{u}|| \geq |\boldsymbol{u}|$ leads to

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|^2 \leq \epsilon - \nu|\boldsymbol{u}|^2,$$

which implies that $|\boldsymbol{u}(t)|^2 \leq e^{-2\nu t}|\boldsymbol{u}(0)|^2 + \left(\frac{1 - e^{-2\nu t}}{\nu}\right)\epsilon.$

- So for every $\boldsymbol{u} \in \mathcal{A}$, we expect $|\boldsymbol{u}(t)|^2 \leq \epsilon/\nu$.

- From $|\boldsymbol{u}(t)| \le \sqrt{\epsilon/\nu}$ we then obtain a lower bound for $|\boldsymbol{f}|$:

$$\sqrt{\nu\epsilon} \le \frac{\epsilon}{|\boldsymbol{u}|} = \frac{(\boldsymbol{f}, \boldsymbol{u})}{|\boldsymbol{u}|} \le \frac{|\boldsymbol{f}||\boldsymbol{u}|}{|\boldsymbol{u}|} = |\boldsymbol{f}|.$$

- It is convenient to use this lower bound for $|\boldsymbol{f}|$ to define a lower bound for the Grashof number $G = |\boldsymbol{f}|/\nu^2$, which we use as the normalization $\tilde{G}$ for random forcing:

$$\tilde{G} = \sqrt{\frac{\epsilon}{\nu^3}}.$$

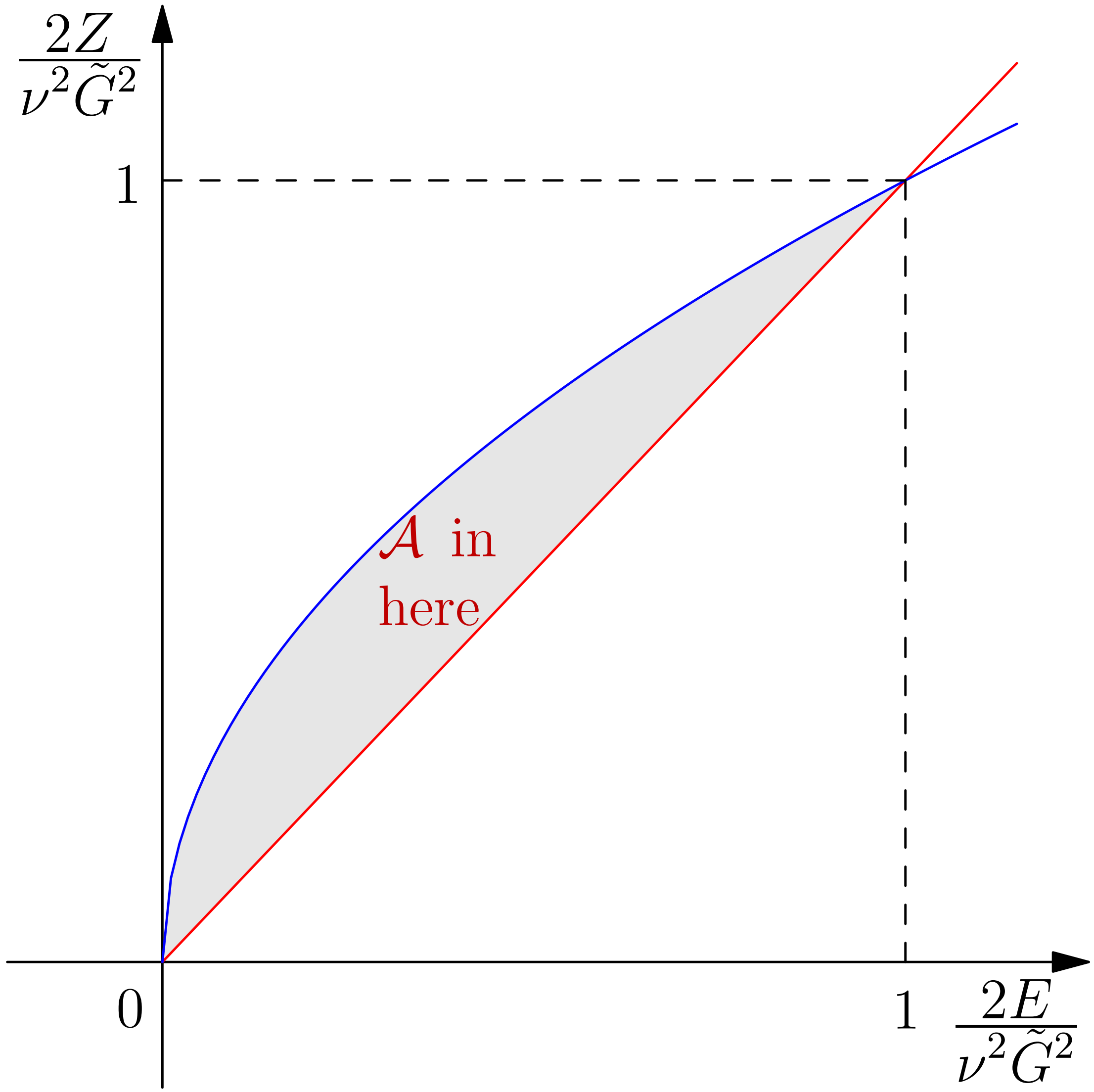
- We proved the following theorem (JDE 2018):

Theorem 2 (Emami & Bowman [2018]) *For all* $\boldsymbol{u} \in \mathcal{A}$ *with energy injection rate* ϵ,

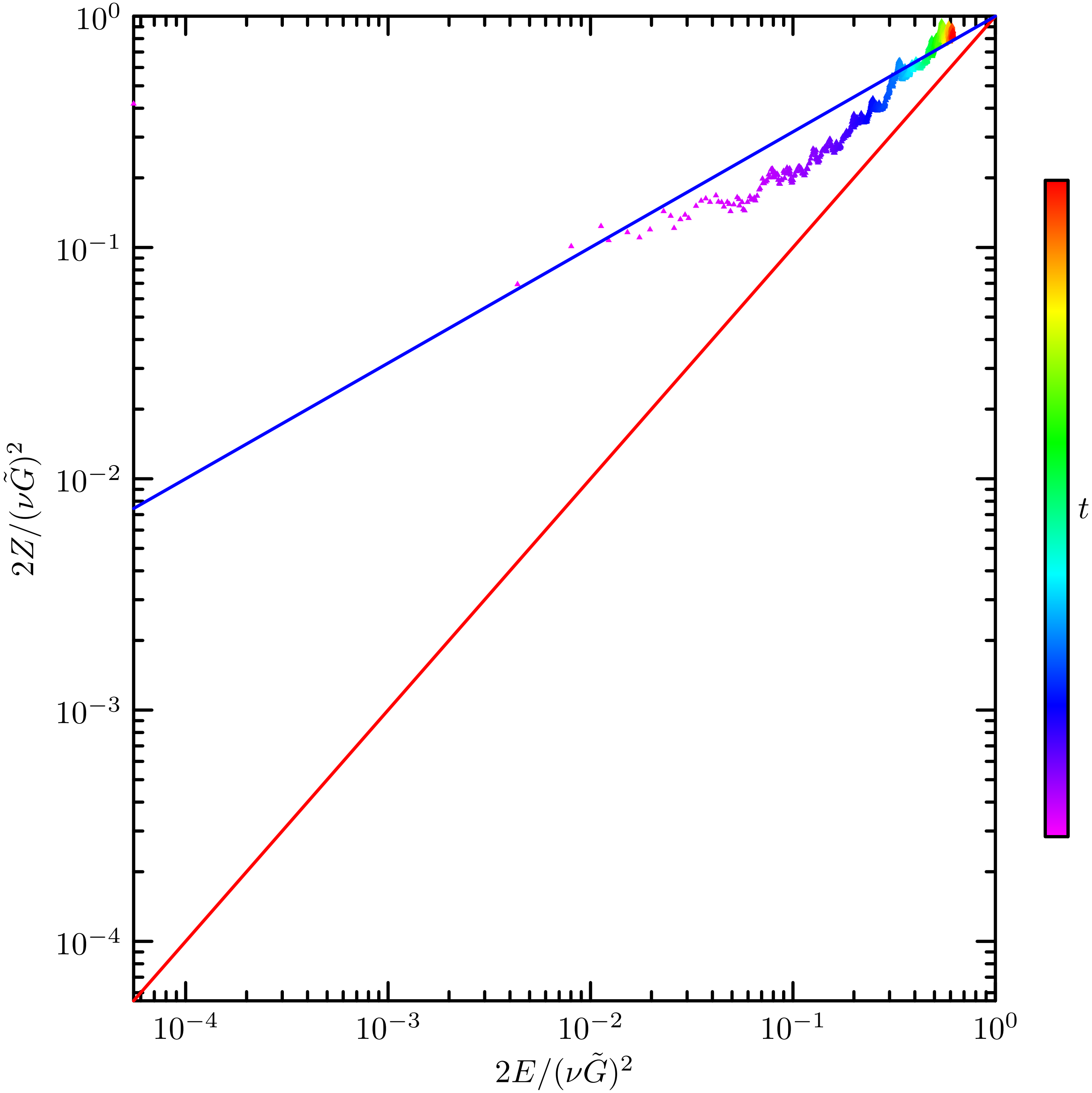
$$||\boldsymbol{u}||^2 \le \sqrt{\frac{\epsilon}{\nu}}\,|\boldsymbol{u}|.$$

- This leads to the same form as for a constant force: $Z \le \nu\tilde{G}\sqrt{E}$.

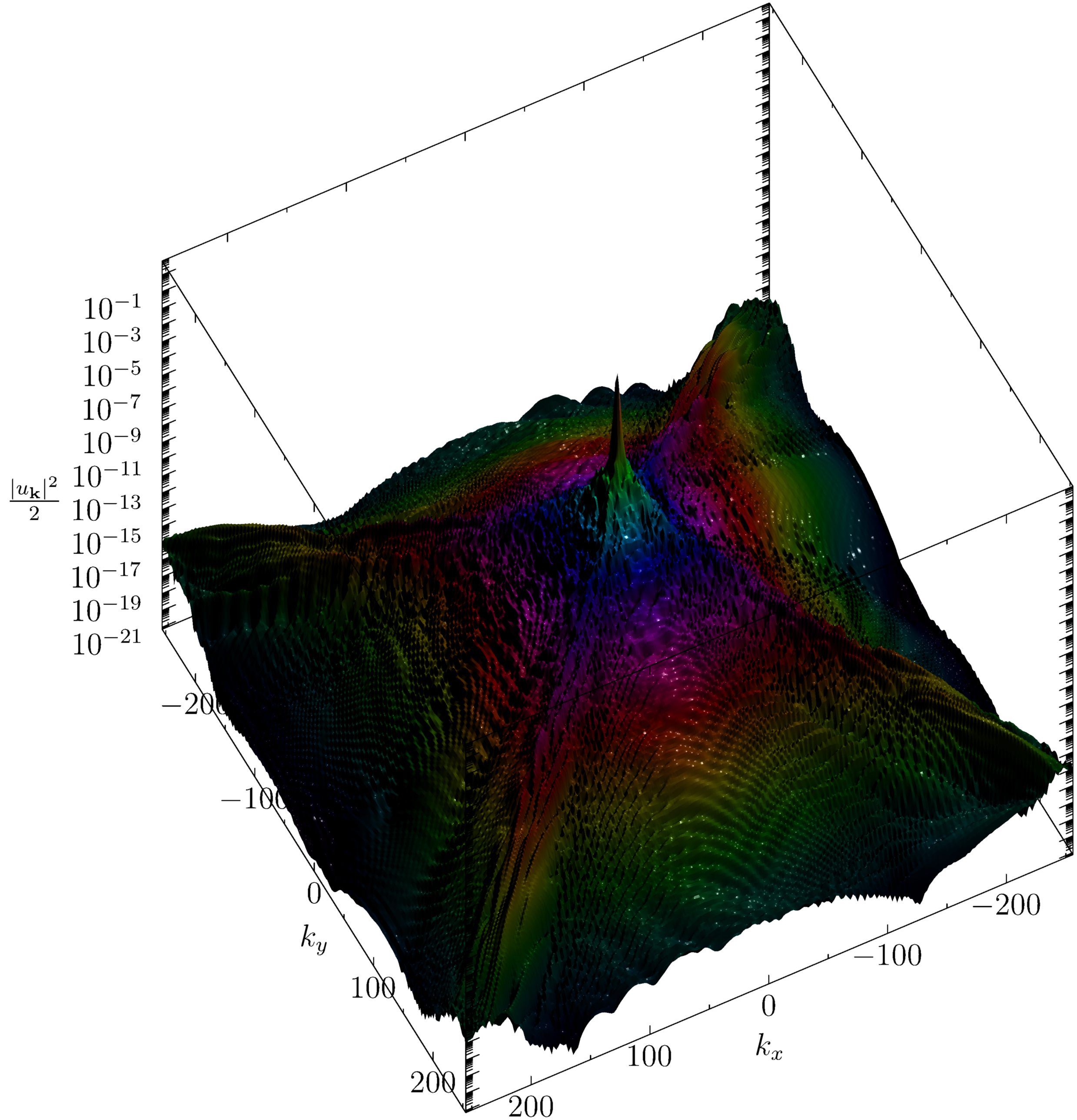
Z–E Bounds: Random Forcing



Z–E Bounds: Random Forcing



3D Energy Spectrum



Large-Scale friction

- In the random-forcing case, we have recently extended the analysis to include a large-scale friction term:

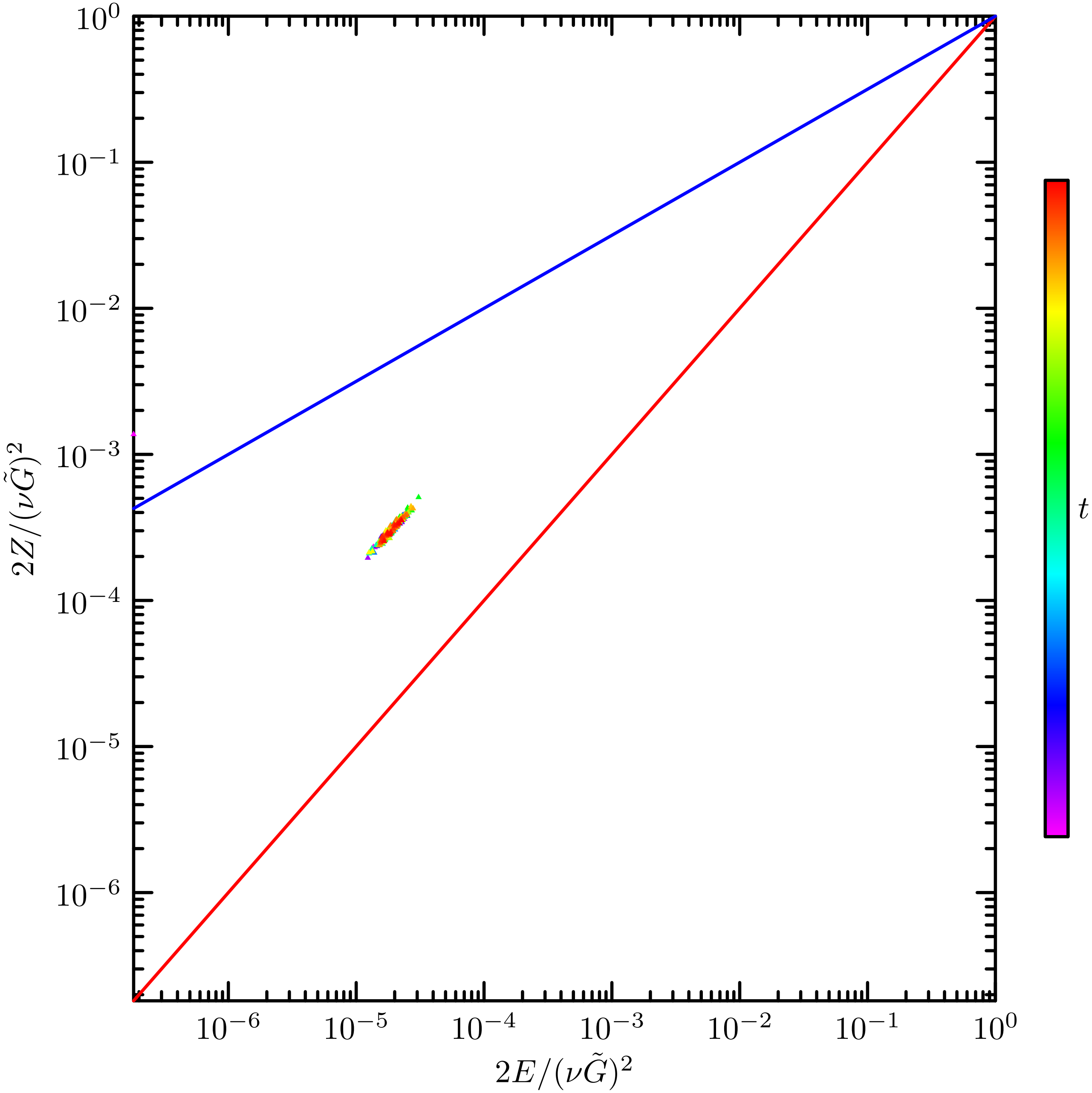
$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = -\nu_0 w + \nu \nabla^2 \omega + f.$$

- If we generalize our definition of the Grashof number to account for ν_0:

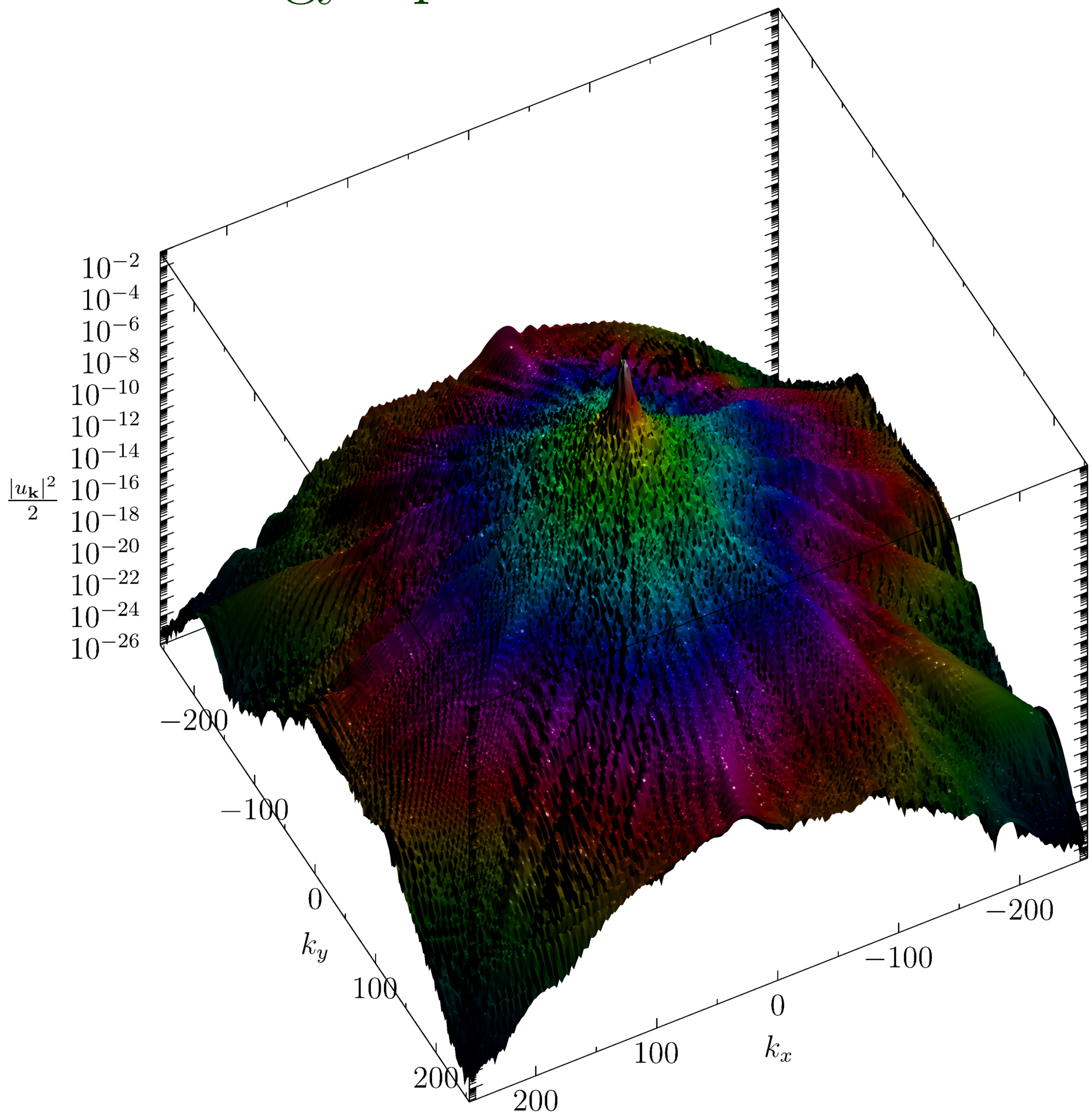
$$\tilde{G} = \frac{\sqrt{\epsilon(\nu + \nu_0)}}{\nu^2},$$

the resulting analytic bounds retain the same form!

Z–E Bounds: Random Forcing+Friction



3D Energy Spectrum with Friction



P–Z Bounds

- Just as the rate of energy dissipation is $2\nu Z$, the rate of enstrophy dissipation is $2\nu P$ where P is the palenstrophy.

- Dascaliuc, Foias, and Jolly also obtained bounds for the palenstrophy–enstrophy plane.

- A critical step in their argument is the application of the Cauchy–Schwarz inequality to estimate the bilinear triplet

$$(\mathcal{B}(\boldsymbol{u},\boldsymbol{u}), A^n\boldsymbol{u}) \text{ for } n = 2.$$

- For this bound to be sharp: $\mathcal{B}(\boldsymbol{u},\boldsymbol{u}) = \alpha A^n\boldsymbol{u}$ a.e. for some $\alpha \in \mathbb{R}$.

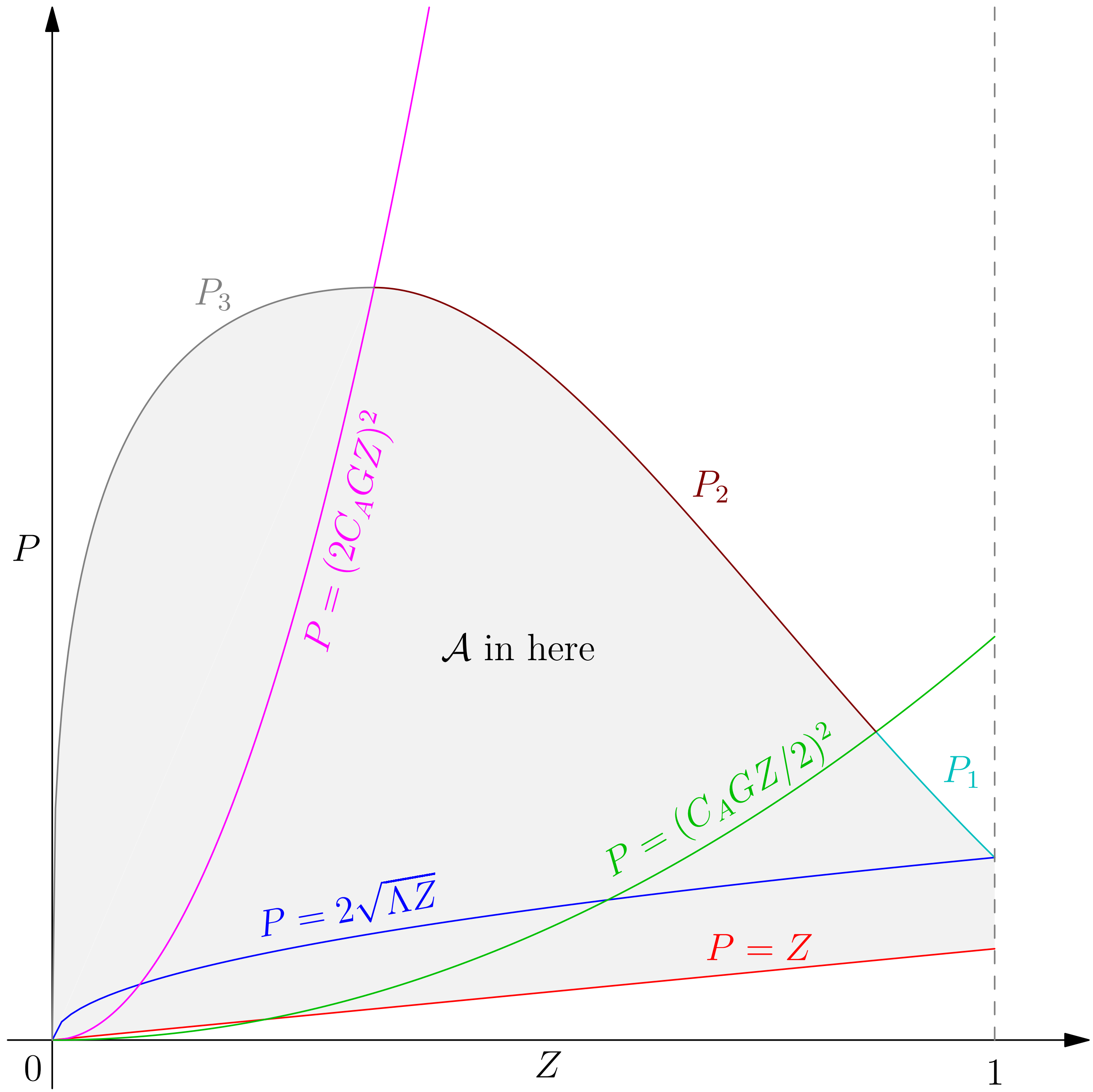
- From the self-adjointness of A, such an alignment would require

$$\begin{aligned} 0 = (\mathcal{B}(\boldsymbol{u},\boldsymbol{u}),\boldsymbol{u}) &= (\alpha \boldsymbol{A}^n\boldsymbol{u},\boldsymbol{u}) = (\alpha \boldsymbol{A}^{n/2}\boldsymbol{u}, \boldsymbol{A}^{n/2}\boldsymbol{u}) \\ &= \alpha|A^{n/2}u|^2 \quad \Rightarrow \quad \mathcal{B}(\boldsymbol{u},\boldsymbol{u}) = 0 \text{ a.e.}, \end{aligned}$$

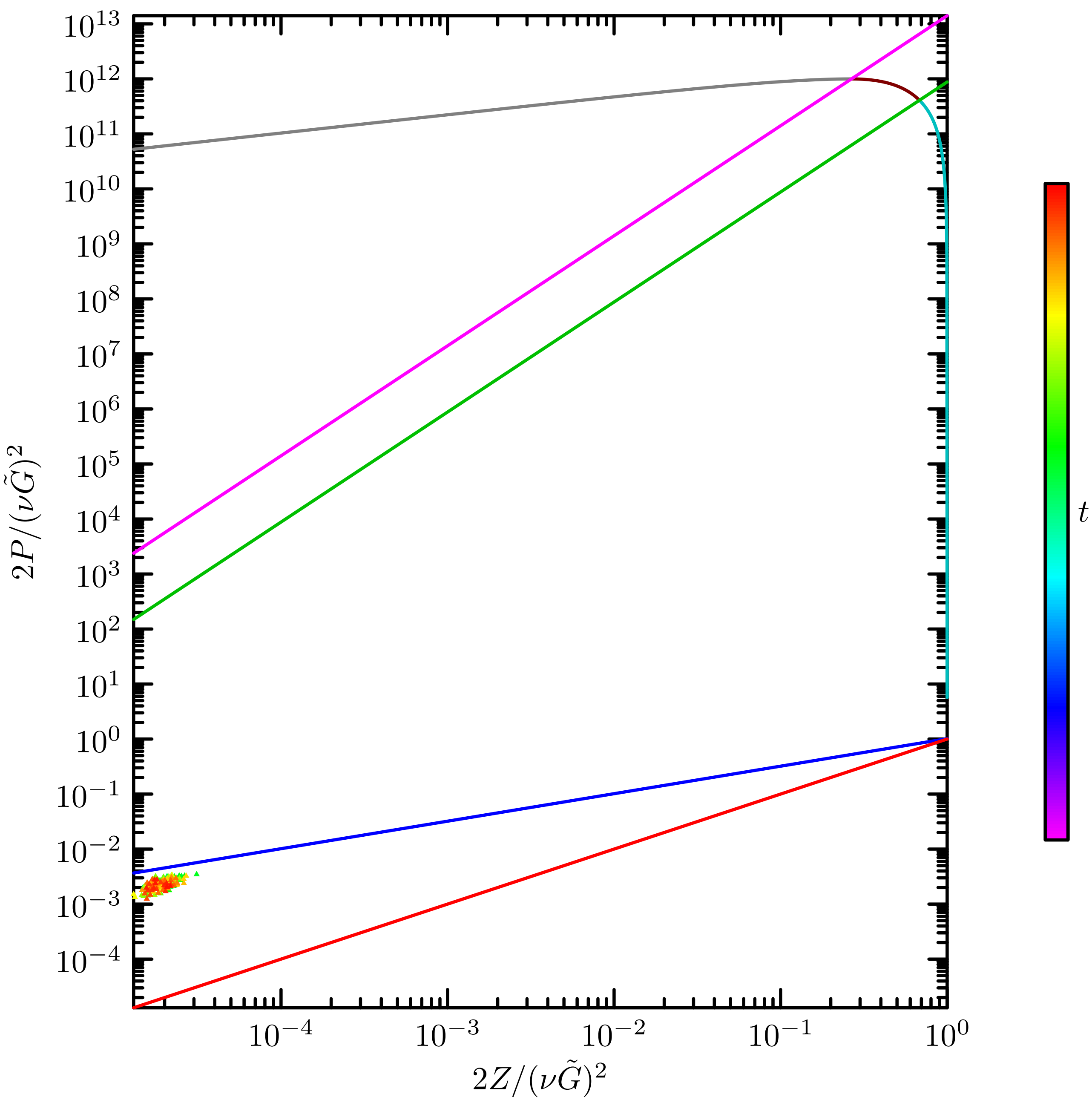
which would imply no cascade!

- Numerical simulations show that these quantities are far from being aligned; in fact they are extremely close to being perpendicular!
- Consequently, the observed palenstrophy values are much lower than the predicted bounds.

P–Z Upper Bounds



P–Z Bounds: Random Forcing+Friction



Isotropic turbulence

- For statistically isotropic turbulence, the expected value of the bilinear triplet $(\boldsymbol{\mathcal{B}}(\boldsymbol{u},\boldsymbol{u}), A^n\boldsymbol{u})$ is zero:

Theorem 3 (Emami & Bowman [2020]) *In incompressible statistically isotropic 2D turbulence,*

$$\left\langle \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\cdot A^n\boldsymbol{u}\, d\boldsymbol{x} \right\rangle = 0, \qquad \forall n \in \mathbb{R}.$$

- Proof: Express $\boldsymbol{u} = (u, v) = (-\psi_y, \psi_x)$, where ψ is the stream function and define:

$$\alpha \doteq -u_x = \psi_{yx} = v_y, \quad \beta \doteq -u_y = \psi_{yy}, \quad \gamma \doteq v_x = \psi_{xx}.$$

- Statistical isotropy then implies

$$\begin{aligned}\left\langle \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\cdot A^n \boldsymbol{u}\, d\boldsymbol{x} \right\rangle &= \left\langle \int_\Omega (uu_x + vu_y)A^n u + (uv_x + vv_y)A^n v\, d\boldsymbol{x} \right\rangle \\ &= \left\langle \int_\Omega (-\alpha u - \beta v)A^n u + (\gamma u + \alpha v)A^n v\, d\boldsymbol{x} \right\rangle \\ &= \left\langle \int_\Omega \alpha(vA^n v - uA^n u) + (\gamma u A^n v - \beta v A^n u)\, d\boldsymbol{x} \right\rangle \\ &= 0.\end{aligned}$$

- We then find, by normalizing to $\tilde{G} = \sqrt{\epsilon_{\frac{1}{2}}(\nu + \nu_0)}/\nu^2$, that

$$\frac{2P}{(\nu\tilde{G})^2} \le \sqrt{\epsilon_{\frac{1}{2}} \frac{2Z}{(\nu\tilde{G})^2}}.$$

General upper bound

- For every $\sigma \in \mathbb{R}$ and for all $\boldsymbol{u} \in \mathcal{A}$ driven by a random forcing having injection rate equal to ϵ_σ,

Theorem 4 (Emami & Bowman [2020])

$$\left| A^{\sigma+1/2}\boldsymbol{u} \right|^2 \leq \sqrt{\frac{\epsilon_\sigma}{\nu}}\, |A^\sigma \boldsymbol{u}|.$$

DNS code

- We have released a highly optimized 2D pseudospectral code in C++: https://github.com/dealias/dns.
- It uses our FFTW++ library to implicitly dealias the advective convolution, while exploiting Hermitian symmetry [Bowman & Roberts 2011], [Roberts & Bowman 2018].
- Advanced computer memory management, such as implicit padding, memory alignment, and dynamic moment averaging allow DNS to attain its extreme performance.
- The formulation proposed by Basdevant [1983] is used to reduce the number of FFTs required for 2D (3D) incompressible turbulence to 4 (8).
- We also include simplified 2D (146 lines) and 3D (287 lines) versions called ProtoDNS for educational purposes: https://github.com/dealias/dns/tree/master/protodns.

Implicit Dealiasing

- Let $N = 2m$. For $j = 0, \ldots, 2m-1$ we want to compute

$$f_j = \sum_{k=0}^{2m-1} \zeta_{2m}^{jk} F_k.$$

- If $F_k = 0$ for $k \geq m$, one can easily avoid looping over the unwanted zero Fourier modes by decimating in wavenumber:

$$f_{2\ell} = \sum_{k=0}^{m-1} \zeta_{2m}^{2\ell k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} F_k,$$

$$f_{2\ell+1} = \sum_{k=0}^{m-1} \zeta_{2m}^{(2\ell+1)k} F_k = \sum_{k=0}^{m-1} \zeta_m^{\ell k} \zeta_{2m}^{k} F_k, \qquad \ell = 0, 1, \ldots m-1.$$

- This requires computing two subtransforms, each of size m, for an overall computational scaling of order $2m \log_2 m = N \log_2 m$.

- Parallelized multidimensional implicit dealiasing routines have been implemented as a software layer **FFTW++** (v2.06) on top of the **FFTW** library under the Lesser GNU Public License:

  http://fftwpp.sourceforge.net/

Conclusions

- The upper bound in the Z–E plane obtained previously for constant forcing also works for white-noise forcing and large-scale friction (hypoviscosity).

- Previous bounds in the P–Z plane vastly overestimate the values obtained from numerical simulations.

- These bounds can be greatly tightened by exploiting isotropy.

- Analytical bounds for random forcing provide a means to evaluate various heuristic turbulent subgrid (and supergrid!) models by characterizing the behaviour of the global attractor under these models.

References

[Basdevant 1983] C. Basdevant, Journal of Computational Physics, **50**:209, 1983.

[Bowman & Roberts 2010] J. C. Bowman & M. Roberts, `FFTW++`: A fast Fourier transform C^{++} header class for the `FFTW3` library, http://fftwpp.sourceforge.net, May 6, 2010.

[Bowman & Roberts 2011] J. C. Bowman & M. Roberts, SIAM J. Sci. Comput., **33**:386, 2011.

[Dascaliuc *et al.* 2005] R. Dascaliuc, M. Jolly, & C. Foias, Journal of Dynamics and Differential Equations, **17**:643, 2005.

[Dascaliuc *et al.* 2007] R. Dascaliuc, C. Foias, & M. Jolly, Journal of mathematical physics, **48**:065201, 2007.

[Dascaliuc *et al.* 2010] R. Dascaliuc, C. Foias, & M. Jolly, Journal of Differential Equations, **248**:792, 2010.

[Emami & Bowman 2018] P. Emami & J. C. Bowman, Journal of Differential Equations, **264**:4036, 2018.

[Emami & Bowman 2020] P. Emami & J. C. Bowman, Journal of Differential Equations, 2020, to be submitted.

[Foias & Temam 1979] C. Foias & R. Temam, Journal de mathematiques pures et appliquees, **58**:339, 1979.

[Foias *et al.* 2013] C. Foias, M. S. Jolly, & M. Yang, Journal of Dynamics and Differential Equations, **25**:393, 2013.

[Ladyzhenskaya 1975] O. Ladyzhenskaya, Journal of Soviet Mathematics, **3**:458, 1975.

[Roberts & Bowman 2018] M. Roberts & J. C. Bowman, J. Comput. Phys., **356**:98, 2018.