Spectral reduction: a statistical description of turbulence

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A method is described for predicting statistical properties of turbulence. Collections of Fourier amplitudes are represented by nonuniformly spaced modes with enhanced coupling coefficients. The statistics of the full dynamics can be recovered from the time-averaged predictions of the reduced model. A Liouville theorem leads to inviscid equipartition solutions. Excellent agreement is obtained with two-dimensional forced-dissipative pseudospectral simulations. For the two-dimensional enstrophy cascade, logarithmic corrections to the high-order structure functions are observed.

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Many practical applications for spectral simulations of turbulence exist where it would be desirable to evolve modes that are distributed nonuniformly in Fourier space, devoting most of the computational resources to the length scales of greatest physical interest. This idea has led to the development of a new reduced statistical description of turbulence, called *spectral reduction* [1], which dramatically reduces the number of spectral modes that must be retained in simulations of turbulent phenomena. It exploits the fact that statistical moments are much smoother functions of wave number than are the underlying stochastic amplitudes.

The concept of wave-number reduction is not new. In the method of constrained decimation [2–4], a stochastic forcing term is added to model the effect of the deleted modes on the retained modes. She and Jackson have proposed a reduction scheme in which the linear (viscous) term is modified [5]. In spectral reduction, a third alternative is chosen: the nonlinear coefficients are enhanced to account for the effect of the discarded modes on the explicitly evolved modes. There have been other more heuristic attempts at wave-number reduction [6–9]; these methods typically neglect nonlocal wave-number triad interactions (which play a particularly important role in two-dimensional turbulence). Unlike the renormalization group [10] method, which retains only large-scale modes and attempts to express the effect of the small-scale modes using a self-similarity *Ansatz*, spectral reduction retains certain modes from all scales, while discarding other modes associated with these same scales. The generality of the formulation allows one to refine the partition wherever the physics dictates.

In this Letter we restrict our attention to homogeneous and isotropic incompressible turbulence in two dimensions. The appropriate spectral transform in this limit is the integral Fourier transform, under which the two-dimensional Navier–Stokes vorticity equation becomes

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int_{\mathcal{D}} d\boldsymbol{p} \int_{\mathcal{D}} d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*. \tag{1}$$

Here $\nu_{\boldsymbol{k}}$ models time-independent linear dissipation or forcing and the interaction coefficient $\epsilon_{\boldsymbol{kpq}} \doteq (\hat{z}\boldsymbol{\cdot}\boldsymbol{p}\boldsymbol{\times}\boldsymbol{q})\, \delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$ is antisymmetric under permutation of any two indices ($*$ denotes complex conjugation, $\doteq$ indicates a definition, and $\hat{z}$ is the unit normal to the plane of motion). We restrict the integration to a bounded wave-number domain $\mathcal{D}$ that excludes a neighborhood of $\boldsymbol{k} = \boldsymbol{0}$. As a consequence of the antisymmetry of $\epsilon_{\boldsymbol{kpq}}$, in the inviscid limit ($\nu_{\boldsymbol{k}} = 0$) Eq. (1) conserves the energy $\frac{1}{2}\int_{\mathcal{D}} d\boldsymbol{k}\, |\omega_{\boldsymbol{k}}|^2/k^2$ and enstrophy $\frac{1}{2}\int_{\mathcal{D}} d\boldsymbol{k}\, |\omega_{\boldsymbol{k}}|^2$. It is believed that the energy and enstrophy play fundamental roles in the dynamics of the turbulent cascade.

We introduce an arbitrary coarse-grained grid that partitions $\mathcal{D}$ into connected regions called bins. The bins are labeled by capital letters to distinguish them from the continuum wave numbers, which we represent by lower-case letters. To this grid, we associate new variables $\Omega_{\boldsymbol{K}} \doteq \Delta_{\boldsymbol{K}}^{-1} \int_{\boldsymbol{K}} \omega_{\boldsymbol{k}}\, d\boldsymbol{k}$, where $\Delta_{\boldsymbol{K}}$ is the area of bin $\boldsymbol{K}$. The exact evolution of $\Omega_{\boldsymbol{K}}$ is given by

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* \right\rangle_{\boldsymbol{KPQ}}, \tag{2}$$

where $\langle \cdot \rangle_{\boldsymbol{K}}$ denotes a bin average and the operator

$$\langle f \rangle_{\boldsymbol{KPQ}} \doteq \frac{1}{\Delta_{\boldsymbol{K}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}}} \int_{\boldsymbol{K}} d\boldsymbol{k} \int_{\boldsymbol{P}} d\boldsymbol{p} \int_{\boldsymbol{Q}} d\boldsymbol{q}\, f, \tag{3}$$

depends only on the bin geometry. The geometric factors $\langle f \rangle_{\boldsymbol{KPQ}}$ can be efficiently computed using a combination of analytical and numerical methods [11–13]. Since they are independent of both time and initial conditions, they need only be computed once for each new wave-number partition. The reality condition $\Omega_{\boldsymbol{K}} = \Omega^*_{-\boldsymbol{K}}$, where $-\boldsymbol{K}$ denotes the inversion of bin $\boldsymbol{K}$ through the origin, will be respected for partitions that possess inversion symmetry.

Equation (2) is unfortunately not closed. If $\omega_{\boldsymbol{k}}$ were naively approximated by its bin-averaged value $\Omega_{\boldsymbol{K}}$, one would obtain

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*. \tag{4}$$

In the inviscid limit, Eq. (4) conserves the coarse-grained enstrophy $\frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}$ since $\left\langle \epsilon_{\boldsymbol{kpq}}/q^2 \right\rangle_{\boldsymbol{KPQ}}$ is antisymmetric in $\boldsymbol{K} \leftrightarrow \boldsymbol{P}$. However, the coarse-grained energy $\frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}/K^2$ is not conserved since $\left\langle \epsilon_{\boldsymbol{kpq}}/q^2 \right\rangle_{\boldsymbol{KPQ}}/K^2$ is not antisymmetric in $\boldsymbol{K} \leftrightarrow \boldsymbol{Q}$ (here K denotes the magnitude of some characteristic wave number in bin $\boldsymbol{K}$). However, both of these desired symmetries can be reinstated by replacing the factor $\left\langle \epsilon_{\boldsymbol{kpq}}/q^2 \right\rangle_{\boldsymbol{KPQ}}$ in Eq. (4) with the slightly modified coefficient $\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}/Q^2$. The relative error introduced by this modification is negligible in the limit of small bin size, being on the order of the squared relative variation in the wavenumber magnitude over a bin. The result,

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*, \tag{5}$$

which we call the spectrally reduced Navier–Stokes equation, is a more acceptable alternative than Eq. (4) as a closure of Eq. (2): not only does it reduce to the Navier–Stokes equation in this limit, but it also conserves both energy and enstrophy, even when the bins are large. The final modification leading to Eq. (5) partially compensates for the error introduced by the crude approximation $\omega_{\boldsymbol{k}} \approx \Omega_{\boldsymbol{K}}$ and leads to the same general structure and symmetries as Eq. (1); in this sense spectral reduction may be regarded as a renormalization.

If the bins are large, the true vorticity will vary rapidly with wave number within each bin and it is unlikely that Eq. (5) will yield a reasonable description of the instantaneous dynamics. However, the time-averaged (or ensemble-averaged) moments of Eq. (5) satisfy equations that closely approximate the equations governing the exact bin-averaged statistics. For example, a time average (denoted by an over-bar) of the bin-averaged enstrophy equation derived from Eq. (1) leads to

$$\frac{1}{2}\overline{\frac{\partial\left\langle|\omega_{\boldsymbol{k}}|^2\right\rangle_{\boldsymbol{K}}}{\partial t}}+\mathrm{Re}\left\langle\nu_{\boldsymbol{k}}\overline{|\omega_{\boldsymbol{k}}|^2}\right\rangle_{\boldsymbol{K}}=\mathrm{Re}\sum_{\boldsymbol{P},\boldsymbol{Q}}\Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\left\langle\frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\overline{\omega^*_{\boldsymbol{k}}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}}}\right\rangle_{\boldsymbol{KPQ}}. \tag{6}$$

If the true vorticity is a continuous function of wave number, the mean value theorem for integrals guarantees the existence of a wave number $\boldsymbol{\kappa}$ in bin $\boldsymbol{K}$ such that $\Omega_{\boldsymbol{K}}=\omega_{\boldsymbol{\kappa}}$. Furthermore, time-averaged quantities such as $\overline{|\omega_{\boldsymbol{k}}|^2}$ are generally smooth functions of the wave number $\boldsymbol{k}$. We thus deduce that $\overline{|\Omega_{\boldsymbol{K}}|^2}=\overline{|\omega_{\boldsymbol{\kappa}}|^2}\approx\overline{|\omega_{\boldsymbol{k}}|^2}$ for all $\boldsymbol{k}$ in bin $\boldsymbol{K}$. Similarly, the triplet correlation $\overline{\omega^*_{\boldsymbol{k}}\omega^*_{\boldsymbol{p}}\omega^*_{\boldsymbol{q}}}$ is a smooth function of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ when restricted to the surface defined by the triad condition $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}$.

To good accuracy the statistical averages in Eq. (6) may therefore be evaluated at the characteristic wave numbers $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ of each bin, yielding

$$\frac{1}{2}\overline{\frac{\partial|\Omega_{\boldsymbol{K}}|^2}{\partial t}}+\mathrm{Re}\left\langle\nu_{\boldsymbol{k}}\right\rangle_{\boldsymbol{K}}\overline{|\Omega_{\boldsymbol{K}}|^2}=\mathrm{Re}\sum_{\boldsymbol{P},\boldsymbol{Q}}\Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\left\langle\frac{\epsilon_{\boldsymbol{kpq}}}{q^2}\right\rangle_{\boldsymbol{KPQ}}\overline{\Omega^*_{\boldsymbol{K}}\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}}. \tag{7}$$

Moreover, to the extent that the wave-number magnitudes vary slowly over a bin, Eq. (6) may equally well be reduced to the (nonlinearly conservative) approximation

$$\frac{1}{2}\overline{\frac{\partial|\Omega_{\boldsymbol{K}}|^2}{\partial t}}+\mathrm{Re}\left\langle\nu_{\boldsymbol{k}}\right\rangle_{\boldsymbol{K}}\overline{|\Omega_{\boldsymbol{K}}|^2}=\mathrm{Re}\sum_{\boldsymbol{P},\boldsymbol{Q}}\Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\frac{\left\langle\epsilon_{\boldsymbol{kpq}}\right\rangle_{\boldsymbol{KPQ}}}{Q^2}\overline{\Omega^*_{\boldsymbol{K}}\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}}, \tag{8}$$

which is precisely the evolution equation for the time-averaged enstrophy obtained from Eq. (5). Similar arguments for the higher-order statistical moments can also be made, suggesting that spectral reduction can indeed provide an accurate statistical description of turbulence, even when each bin contains many statistically independent modes. As the partition is refined, one expects the solutions of Eq. (8) to converge to the those of Eq. (6).

In the absence of forcing and dissipation, the (untruncated) two-dimensional Euler equations can be written in a noncanonical Hamiltonian framework [14] as $\dot{\omega}_{\boldsymbol{k}}=\int d\boldsymbol{q}\,J_{\boldsymbol{kq}}\delta H/\delta\omega_{\boldsymbol{q}}$, where $H\doteq\frac{1}{2}\int d\boldsymbol{k}\,|\omega_{\boldsymbol{k}}|^2/k^2$ and $J_{\boldsymbol{kq}}\doteq\int d\boldsymbol{p}\,\epsilon_{\boldsymbol{kpq}}\omega^*_{\boldsymbol{p}}$. The Liouville theorem

$$\int d\boldsymbol{k}\frac{\delta\dot{\omega}_{\boldsymbol{k}}}{\delta\omega_{\boldsymbol{k}}}=\int d\boldsymbol{k}\int d\boldsymbol{q}\left[\epsilon_{\boldsymbol{k}(-\boldsymbol{k})\boldsymbol{q}}\frac{\delta H}{\delta\omega_{\boldsymbol{q}}}+J_{\boldsymbol{kq}}\frac{\delta^2H}{\delta\omega_{\boldsymbol{k}}\delta\omega_{\boldsymbol{q}}}\right]=0$$

then follows immediately from the properties of $\epsilon_{\boldsymbol{kpq}}$.

When $\nu_{\boldsymbol{k}}=0$, Eq. (5) can be written in a similar form as $\dot{\Omega}_{\boldsymbol{K}}=\sum_{\boldsymbol{Q}}J_{\boldsymbol{KQ}}\,\partial H/\partial\Omega_{\boldsymbol{Q}}$, where $H\doteq\frac{1}{2}\sum_{\boldsymbol{K}}|\Omega_{\boldsymbol{K}}|^2\Delta_{\boldsymbol{K}}/K^2$ and

$$J_{\boldsymbol{KQ}}\doteq\sum_{\boldsymbol{P}}\Delta_{\boldsymbol{P}}\left\langle\epsilon_{\boldsymbol{kpq}}\right\rangle_{\boldsymbol{KPQ}}\Omega^*_{\boldsymbol{P}}. \tag{9}$$

It is an open question whether (an untruncated version of) Eq. (9) satisfies the Jacobi identity, which would make spectral reduction a Hamiltonian approximation. What is certain is that the respective Liouville theorem

$$\sum_{\boldsymbol{K}}\frac{\partial\dot{\Omega}_{\boldsymbol{K}}}{\partial\Omega_{\boldsymbol{K}}}=\sum_{\boldsymbol{K},\boldsymbol{Q}}\left(\frac{\partial J_{\boldsymbol{KQ}}}{\partial\Omega_{\boldsymbol{K}}}\frac{\partial H}{\partial\Omega_{\boldsymbol{Q}}}+J_{\boldsymbol{KQ}}\frac{\partial^2H}{\partial\Omega_{\boldsymbol{K}}\partial\Omega_{\boldsymbol{Q}}}\right)=0$$

is obeyed, as a consequence of the antisymmetry of $\left\langle\epsilon_{\boldsymbol{kpq}}\right\rangle_{\boldsymbol{KPQ}}$. If the dynamics is *mixing*, the inviscid system will then evolve toward equipartition [15,16]; this was verified for spectral reduction numerically, using a fifth-order conservative Runga–Kutta integration algorithm that conserves quadratic invariants to all orders in the time step [17,18]. When using nonuniform bins it is necessary to rescale the time derivative $\partial/\partial t$ in Eq. (5) to $(\Delta_0/\Delta_{\boldsymbol{K}})\partial/\partial t$, where Δ_0 is the minimum bin area, to obtain an equipartition of modal (instead of bin-averaged) energies. This modification to the transient evolution will be discussed further in a future paper.

Upon adding to Eq. (5) a random stirring force for $k\in[5,7]$ and adopting the dissipation function $\nu_{\boldsymbol{k}}=\nu_L\theta(7-k)+\nu_Hk^2$ (θ is the Heaviside function), we graph in Fig. 1 the time-averaged saturated energy spectra for four wave-number partitions to test how rapidly spectral reduction converges. The excellent agreement demonstrated between the predictions of spectral reduction and a (computationally more expensive) full dealiased pseudospectral simulation is obtained without fitting or the introduction of adjustable parameters. We also show the predictions of the realizable test field model (RTFM) [13], using wavenumber binning and setting the eddy-damping multiplier in this heuristic statistical closure to one.

High-order moments are also accurately described by spectral reduction. A quantity of interest is the angular average $S_n(r)$ of the n-th (time-averaged) moment of velocity increments $\overline{|\boldsymbol{v}(\boldsymbol{r})-\boldsymbol{v}(\boldsymbol{0})|^n}$, or *structure function*. In Fig. 2 we illustrate the scaling with distance r of a typical high-order structure function, $S_{10}(r)$, for the runs depicted in Fig. 1. Slight variations in the predicted large-scale velocities are evident as overall vertical offsets. Note that the (discrete) pseudospectral calculation is an approximation to Eq. (1) at the large scales.

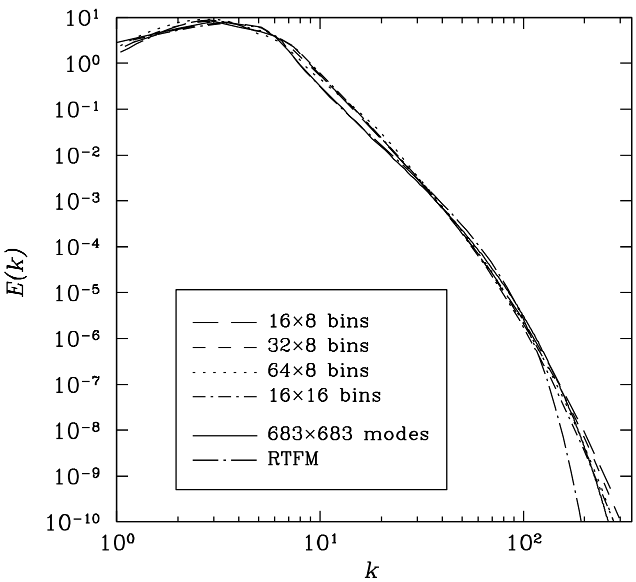

FIG. 1. Comparison of the turbulent energy spectra obtained with 16×8, 32×8, 64×8, and 16×16 (logarithmically spaced radial × uniformly spaced angular) wave-number partitions, the RTFM, and a full 683×683 dealiased pseudospectral simulation (1024×1024 total modes).

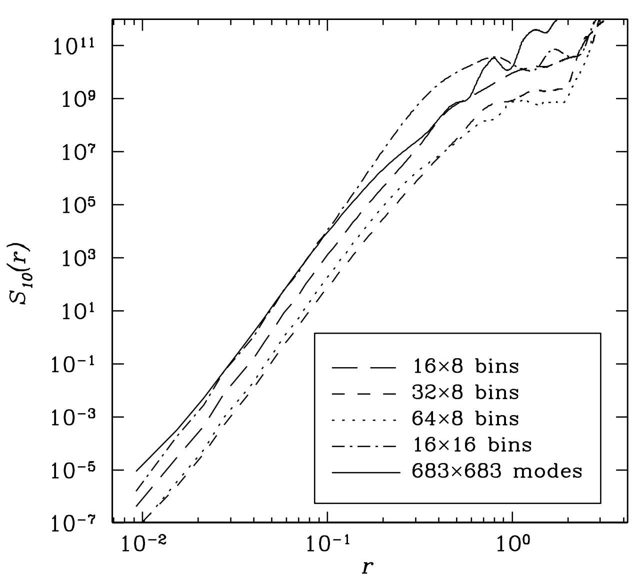

FIG. 2. Angle-averaged structure function $S_{10}(r)$.

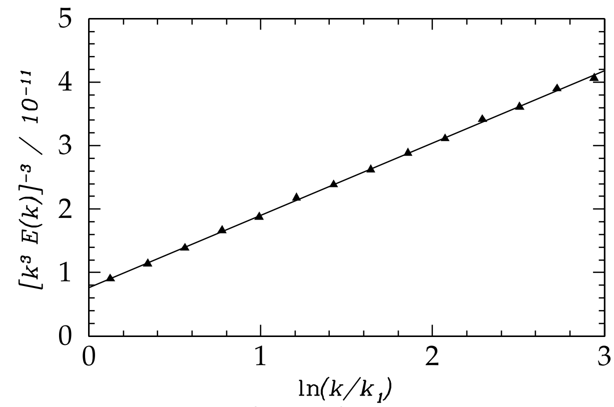

FIG. 3. Linearity of $[k^3E(k)]^{-3}$ with respect to $\ln(k/k_1)$ for an enstrophy inertial range between $k_1 = 16.4$ and $k = 330$. The solid triangles are the predictions of spectral reduction.

One can readily investigate high-Reynolds number turbulence with spectral reduction, using a polar partition in which the bins are logarithmically spaced in the radial wave number. A saturated turbulent state can be evolved for thousands of eddy turnover times to obtain statistically meaningful moments for comparison with theoretical predictions. For example, Kolmogorov's idea of self-similar energy transfer in the inertial range [19] led Kraichnan [20] to propose a logarithmically corrected asymptotic form for the energy spectrum $E(k)$ of the enstrophy cascade. In a simulation with viscous dissipation active only at the smallest scales (to yield a pristine inertial range) and forcing via a linear instability (negative $\nu_{\boldsymbol{k}}$), we apply spectral reduction to demonstrate the recent extension $E(k) \sim k^{-3}\chi^{-1/3}(k)$ of Kraichnan's result to the entire inertial range, where k_1 is the smallest inertial-range wave number, $\chi(k) \doteq \ln(k/k_1) + \chi_1$, and the positive constant χ_1 is set by the large-scale dynamics [21]. We verify in Fig. 3 the linear behavior of $[k^3E(k)]^{-3}$ with respect to $\ln(k/k_1)$, using the values $k_1 = 16.4$ and $\chi_1 = 0.67$ determined by a least-squares fit. The inertial-range energy spectrum is thus well described by Kraichnan's logarithmically corrected k^{-3} law.

For the second simulation, we demonstrate in Fig. 4 the linear behavior of $[r^{-n}S_n(r)]^{3/n}$ with $\ln(r_1/r)$ on the interval $0.043 \le r \le r_1 = 0.26$ for various values of n. The implied scaling $S_n(r) \sim r^n[\ln(r_1/r) + \chi'_n]^{n/3}$, where χ'_n is a constant, is in agreement with both the asymptotic theory of Falkovich and Lebedev [22] and the recent experimental results of Paret *et al.* [23], lending support to the claim that there are no high-order intermittency corrections in two-dimensional turbulence. The universality of this result will be investigated in a future paper.

In this Letter, we propose a new technique that dramatically decreases the number of degrees of freedom required to simulate homogeneous turbulence. The statistically stationary state described by Fig. 3, which would require 2048×2048 dealiased (3071×3071 total) pseudospectral modes, can be successfully modeled using only

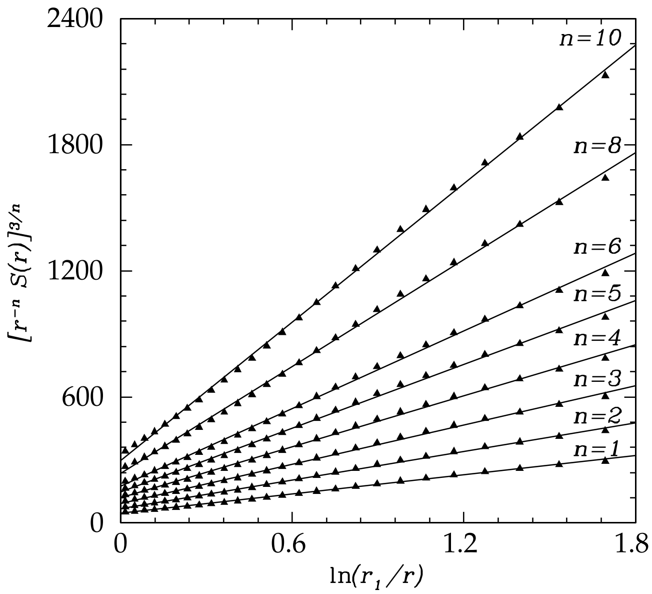

FIG. 4. Linearity of $[r^{-n}S(r)]^{3/n}$ with respect to $\ln(r_1/r)$ for $0.043 \le r \le r_1 = 0.26$.

32×8 bins. A notable feature of spectral reduction that distinguishes it from other statistical theories of turbulence is the existence of a control parameter (bin size) that can be varied to increase the accuracy of a solution. Moreover, spectral reduction does not make a closure assumption on the triplet correlation $\overline{\Omega^*_{\boldsymbol{K}}\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}}$ appearing in Eq. (8); it circumvents the closure problem entirely by reducing the number of triplet correlations to a tractable number, instead of eliminating them in favor of lower-order statistical variables. Unlike statistical closures, spectral reduction thus does not destroy the phase information embodied in the triplet correlation.

Spectral reduction appears to be a promising candidate as a statistical description of turbulence. We propose that it could be used to assess the effect of various dissipation mechanisms in large-eddy simulations, as a subgrid model, or even as a substitute for full simulation of high-Reynolds number turbulence. However, as it does not provide explicit insight into underlying dynamical processes, spectral reduction should be considered more as a computational tool than as a true analytical theory of turbulence. The latter challenge still awaits us.

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