Pseudospectral Simulation of 2D Turbulence

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1 Introduction

In this lab we will learn how to use a *pseudospectral simulation*, a widely used numerical tool for the study of fluid turbulence. The equation we will be solving is the two-dimensional incompressible Navier–Stokes equation. A key step requires solving for the velocity stream function ψ by inverting a Laplace operator: $\nabla^2\psi = \omega$. One of the most efficient methods for doing this is the pseudospectral method, which exploits the Fast Fourier Transform algorithm. Strictly speaking, this method is only appropriate for periodic boundary conditions. However, in practice it can be used even when the physical boundary conditions are nonperiodic, as long as the turbulence *decorrelation length* is smaller than the box size.

In order to set up our simulation, we will first need to determine a consistent set of forcing and dissipation parameters so that a fully developed turbulent state is obtained, in which net forcing balances net dissipation.

2 Theory

We begin with the two-dimensional isotropic incompressible Navier–Stokes equation for the vorticity $\omega = \omega(\boldsymbol{x}, t)$ in the case where the density ρ is constant (so that there is no baroclinic term):

$$\frac{\partial\omega}{\partial t} + (\hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\nabla^{-2}\omega \cdot \boldsymbol{\nabla})\omega = \nu_H(-1)^{p_H}\nabla^{2p_H}\omega + f, \tag{1}$$

where on the right-hand side we have included small-scale dissipation ($p_H \geq 1$) and a random forcing $f = f(\boldsymbol{x}, t)$. Recall that the velocity $\boldsymbol{u}$ is related to the stream function by $\boldsymbol{u} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi$ and the normal component of the vorticity is given by $\omega \doteq \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u} = \nabla^2\psi$. (The notation "$\doteq$" denotes a definition.)

Upon taking the spatial Fourier Transform $\omega_{\boldsymbol{k}}(t) = \int \omega(\boldsymbol{x}, t) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{x}$, Eq. (1) becomes

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \frac{\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}} - \nu_H k^{2p_H} \omega_{\boldsymbol{k}} + f_{\boldsymbol{k}}. \tag{2}$$

We will use a *dealiased* pseudospectral method to solve Eq. (2); this means that the convolution sum in the second term is actually computed in the spatial domain, with the help of the convolution theorem. Note that if we had begun with an infinite spatial domain instead of a periodic domain (as we implicitly assumed), the integral Fourier transform instead of the discrete Fourier transform would be used; the convolution sum would then become a convolution integral.

3 Simulation

In order to determine reasonable parameters for our numerical simulation of Eq. (2), we introduce the *enstrophy*, or *total mean-squared vorticity*, Z, defined as

$$Z \doteq \frac{1}{2} \int k^2 \left|\omega_{\boldsymbol{k}}\right|^2 \, d\boldsymbol{k},$$

where $d\boldsymbol{k} = k\, dk d\theta$ is the volume element. Upon multiplying Eq. (2) by $k^2 \omega^*_{\boldsymbol{k}}$, symmetries in the cross product term cause the nonlinear (advective) contributions to vanish, so that in a steady state, the balance equation for Z becomes just

$$0 = \overline{\frac{\partial Z}{\partial t}} = -2 \int_0^\infty \nu_H k^{2p_H+2} E(k)\, dk + \overline{\int f_{\boldsymbol{k}} \omega^*_{\boldsymbol{k}}\, d\boldsymbol{k}}, \tag{3}$$

where the over-bar denotes a time average and $E(\boldsymbol{k}) \doteq \frac{1}{2}(2\pi k)\overline{|\omega_{\boldsymbol{k}}|^2}/k^2$ is the energy spectrum. The enstrophy dissipation, described by the first integral, occurs mostly at the small scales (high wavenumbers), where the viscous terms dominate. We characterize these scales by the *dissipation wavenumber* k_d. In this lab we restrict the forcing to a narrow band, $k \in [4, 6]$, centered on the *forcing wavenumber* $k_f = 5$. It turns out that it is crucial to include all contributions of the enstrophy dissipation from the *inertial range*, which is the wavenumber interval between the low forcing wavenumber k_f and high dissipation wavenumber k_d. The theory of Kolmogorov [1941], Kraichnan

[1967], and Batchelor [1969] predict that between k_f and k_d, Eq. (2) supports power law solutions of the form

$$E(k) = C\eta^{\frac{2}{3}}k^{-3},$$

where

$$\eta \doteq 2\int_{k_f}^{k_d} \nu_H k^{2p_H+2} E(k)\, dk,$$

the rate of enstrophy dissipation in the inertial range, is constant. In other words,

$$\eta = 2\nu_H C\eta^{\frac{2}{3}} \int_{k_f}^{k_d} k^{2p_H-1}\, dk, \tag{4}$$

which we may solve for η.

We assume that Eq. (1) is ergodic, so that ensemble averages are equivalent to time averages. According to Novikov's theorem [1964], if we prescribe that $f_{\boldsymbol{k}}$ is an isotropic white-noise random process satisfying $\langle f_{\boldsymbol{k}}(t) f^*_{\boldsymbol{k'}}(t')\rangle = F_k^2 \delta_{\boldsymbol{k}\boldsymbol{k'}}\delta(t-t')$, with F_k nonzero only for $k \in [4,6]$, the second integral in Eq. (3) may be expressed in terms of the Green's function for Eq. (2). Equation (3) simplifies to the balance

$$\eta = 2\pi \int_4^6 kF_k^2\, dk.$$

We are now ready to describe the procedure to determine the parameters for our pseudospectral run.

4 Procedure

1. Choose a resolution $N \times N$ where N is an odd number. See the graphical user interface `xtriad` to the program `triad` for allowable values. The maximum wavenumber $k_{\max}$ will then be $(N-1)/2$.

2. Choose k_d to be slightly less than (say 90% of) $k_{\max}$.

3. Pick a value for the vorticity injection rate η, which is denoted `eta` in the simulation. Letting $p_H = 3$, $k_f = 5$, and the forcing width $\delta_f = 2$, solve Eq. (4) for the high-wavenumber viscosity ν_H in terms of the enstrophy dissipation rate η. An approximate value for the constant C is 3.

4. Now try running the simulation with these parameters by typing `xtriad` at the command line prompt. Random initial conditions corresponding to a statistical-mechanical equipartition will be used by default. If you select the dynamic time-stepping option, you will not have to worry about calculating an optimal value for the time step. Just pick some small value like 10^{-6} as a starting time step; the dynamic time-stepping mechanism will then quickly determine the best value. The various graphs and movies may be viewed even while `triad` is running [the numbers in square brackets periodically displayed by `triad` correspond to 1000 time steps or the movie sampling interval (if nonzero), whichever is less]. In the graph of the evolution of $E = \sum_{\boldsymbol{k}} E_{\boldsymbol{k}}$ and $Z = \sum_{\boldsymbol{k}} k^2 E_{\boldsymbol{k}}$, we also depict the palinstrophy $P = \sum_{\boldsymbol{k}} k^4 E_{\boldsymbol{k}}$, which is proportional to the rate of enstrophy dissipation. A similar calculation can be used to determine a large-scale damping $-\nu_L k^{2p_L}$, where $p_L < 0$, that can be added to the right-hand side of Eq. (2) so that a true statistically stationary state will develop. However, in this lab we simply set $\nu_L = 0$.

5. What happens when you pick k_d bigger than $k_{\max}$ (say $k_d = 10\,k_{\max}$)? Try it!

6. For a low-resolution inviscid run with $\nu_H = \nu_L = F_k = 0$, compare the long-time energy spectrum predicted by the Fifth-Order Runge–Kutta (RK5) and the Conservative Fifth-Order Runge–Kutta (C–RK5) integrators. Which result do you think is closer to the true solution?

7. Another revealing numerical experiment is that of *decaying turbulence*, where we set the forcing amplitude `eta` to zero. Starting from randomized initial conditions, you will eventually observe the formation of a large-scale dipole vortex pair, which will continue to grow in amplitude until the box size is reached. This upscale energy transfer involves the same nonlinear mechanism as the *inverse cascade* that we talk about in the lectures.