Math 538: Asymptotic Methods
List of Theorems

Theorem 1.1 (Order Properties): *The following implications hold:*

(i) $f \prec g\ (z \to z_0\ in\ A) \Rightarrow f \preceq g\ (z \to z_0\ in\ A)$;

(ii) $f \preceq g\ (z \to z_0\ in\ A), \alpha \in \mathbb{R}^+ \Rightarrow |f|^\alpha \preceq |g|^\alpha\ (z \to z_0\ in\ A)$;

(iii) $f \prec g\ (z \to z_0\ in\ A), \alpha \in \mathbb{R}^+ \Rightarrow |f|^\alpha \prec |g|^\alpha\ (z \to z_0\ in\ A)$;

(iv) $f \preceq g \preceq h\ (z \to z_0\ in\ A) \Rightarrow f \preceq h\ (z \to z_0\ in\ A)$;

(v) $f \preceq g \prec h\ (z \to z_0\ in\ A) \Rightarrow f \prec h\ (z \to z_0\ in\ A)$;

(vi) $f \prec g \preceq h\ (z \to z_0\ in\ A) \Rightarrow f \prec h\ (z \to z_0\ in\ A)$;

(vii) $f \preceq \phi,\ g \preceq \psi\ (z \to z_0\ in\ A) \Rightarrow fg \preceq \phi\psi\ (z \to z_0\ in\ A)$;

(viii) $f \preceq \phi,\ g \prec \psi\ (z \to z_0\ in\ A) \Rightarrow fg \prec \phi\psi\ (z \to z_0\ in\ A)$;

(ix) $f \preceq \phi,\ g \preceq \phi\ (z \to z_0\ in\ A) \Rightarrow f + g \preceq \phi\ (z \to z_0\ in\ A)$;

(x) $f \prec \phi,\ g \prec \phi\ (z \to z_0\ in\ A) \Rightarrow f + g \prec \phi\ (z \to z_0\ in\ A)$.

Theorem 1.2 (Asymptotic Functions Have Same Order): $f \sim g\ (z \to z_0\ in\ A) \Rightarrow f = \mathcal{O}(g)$ *and* $g = \mathcal{O}(f)\ (z \to z_0\ in\ A)$.

Lemma 1.1: *If* $\{\phi_n\}$ *and* $\{\psi_n\}$ *are asymptotically equivalent sequences and* $\{\phi_n\}$ *is an asymptotic sequence then* $\{\psi_n\}$ *is an asymptotic sequence.*

Theorem 1.3 (Uniqueness): *Let* $\{\phi_n\}_{n=0}^N$ *be an asymptotic sequence. Then*

$$f(z) \sim \sum_{n=0}^{N} a_n\phi_n(z)\ \ (z \to z_0\ in\ A) \iff$$

$$a_0 = \lim_{\substack{z\to z_0 \\ z\in A}} \frac{f(z)}{\phi_0(z)}, \qquad a_n = \lim_{\substack{z\to z_0 \\ z\in A}} \frac{f(z) - \sum\limits_{j=0}^{n-1} a_j\phi_j(z)}{\phi_n(z)},\ n = 1, 2, \ldots, N.$$

Theorem 1.4 (Nonuniqueness): *If* $f(z) \sim \sum\limits_{n=0}^{\infty} a_n\phi_n(z)$ *and* $f - g = o(\phi_n)\ (z \to z_0\ in\ A)\ \forall n$, *then* $g(z) \sim \sum\limits_{n=0}^{\infty} a_n\phi_n(z)\ (z \to z_0\ in\ A)$.

Theorem 1.5 (Addition): *If* $f(z) \sim \sum_{n=0}^\infty a_n\phi_n(z)$ *and* $g(z) \sim \sum_{n=0}^\infty b_n\phi_n(z)$ $(z \to z_0$ *in* $A)$, *then* $\alpha f(z) + \beta g(z) \sim \sum_{n=0}^\infty(\alpha a_n + \beta b_n)\phi_n(z)\ (z \to z_0\ in\ A)$.

Lemma 1.2: *If* $\{\phi_n\}_{n=0}^{\infty}$ *($z \to z_0$ in A) is an asymptotic sequence that satisfies*

$$\phi_m(z)\phi_n(z) = \alpha(z)\phi_{m+n}(z), \tag{1.1}$$

(where without loss of generality one takes $\alpha(z) = 0$ whenever $\phi_0(z) = 0$), then ϕ_n can be expressed as $\phi_n(z) = \alpha(z)\beta^n(z)$, where $\lim\limits_{\substack{z\to z_0\\ z\in A}} \beta(z) = 0$. Here, we interpret $\beta^0(z) = 1$ for all $\beta(z)$.

Theorem 1.6 (Multiplication/Division): *If*

(i) the sequence $\{\phi_n\}_{n=0}^{\infty}$ satisfies Eq. (1.1);

(ii) $f(z) \sim \sum_{n=0}^{\infty} a_n\phi_n(z) \quad (z \to z_0 \text{ in } A)$;

(iii) $g(z) \sim \sum_{n=0}^{\infty} b_n\phi_n(z) \quad (z \to z_0 \text{ in } A)$;

with $b_0 \neq 0$, then the product fg and the quotient f/g satisfy

$$f(z)g(z) \sim \alpha(z)\sum_{n=0}^{\infty} c_n\phi_n(z) \quad (z \to z_0 \text{ in } A)$$

$$\frac{f(z)}{g(z)} \sim \frac{1}{\alpha(z)}\sum_{n=0}^{\infty} d_n\phi_n(z) \quad (z \to z_0 \text{ in } A),$$

where

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}$$

and

$$d_0 = \frac{a_0}{b_0}, \qquad d_n = \frac{1}{b_0}\left(a_n - \sum_{j=0}^{n-1} d_j b_{n-j}\right) \quad n \geq 1.$$

Theorem 1.7 (Termwise Integration): *If*

(i) $\{\phi_n\}_{n=0}^{\infty}$ *($z \to z_0$ in A) is an asymptotic sequence;*

(ii) $\forall n$, ϕ_n is analytic in A, with antiderivative Φ_n satisfying $\lim\limits_{\substack{z\to z_0\\ z\in A}} \Phi_n(z) = 0$;

(iii) f is analytic in A;

(iv) $f(z) \sim \sum_{n=0}^{\infty} a_n\phi_n(z)$ $(z \to z_0 \text{ in } A)$;

Then the antiderivatives $\Phi_n(z)$ *form an asymptotic sequence and*

$$\int_{z_0}^{z} f(\zeta)\, d\zeta \sim \sum_{n=0}^{\infty} a_n \Phi_n(z) \quad (z \to z_0 \text{ in } A),$$

provided the path of integration (except possibly for z_0*) lies in* A*.*

Theorem 1.8 (Termwise Differentiation): *Let* $z_0 \in \mathbb{C}$*. If*

(i) ϕ_n *is holomorphic in an open set* A *such that* $z_0 \in \overline{A}$*;*

(ii) $\{\phi_n\}_{n=0}^{\infty}$*, and* $\{\phi'_n\}_{n=0}^{\infty}$ *are asymptotic sequences as* $(z \to z_0 \text{ in } A)$

(iii) $\dfrac{\phi_n}{\phi'_n} = \begin{cases} \mathcal{O}(z - z_0) & (z \to z_0 \text{ in } A), \text{ if } z_0 \neq \infty, \\ \mathcal{O}(z) & (z \to \infty \text{ in } A), \text{ if } z_0 = \infty; \end{cases}$

(iv) f *is analytic in* A*;*

(v) $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \quad (z \to z_0 \text{ in } A),$

then

$$f'(z) \sim \sum_{n=0}^{\infty} a_n \phi'_n(z) \quad (z \to z_0 \text{ in } A).$$

Lemma 2.1 (Small Laplace Tail): *Let* $\delta > 0$*. Given a function* $r(t)$ *and some* $\bar{x} \in \mathbb{R}$ *such that*

$$\int_0^{\infty} e^{-\bar{x}t} r(t)\, dt$$

converges, the function

$$J(x) \doteq \int_{\delta}^{\infty} e^{-xt} r(t)\, dt = o(x^{-\mu}) \qquad (x \to \infty)$$

for all $\mu \in \mathbb{R}$*.*

Theorem 2.1 (Watson's Lemma): *If* $f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n}$ $(t \to 0^+)$*, where* $\alpha > -1$ *and* $\beta > 0$*, then*

$$\int_0^{\infty} e^{-xt} f(t)\, dt \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} \quad (x \to \infty),$$

provided the integral converges for all sufficiently large x*.*

Corollary 2.1.1 (Generalized Watson's Lemma): If $f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n}$ $(t \to 0^+)$, where $\alpha > -1$ and $\beta > 0$, and $\int_0^\infty e^{-xt} f(t)\,dt$ converges for all sufficiently large x, then for any $a > 0$,

$$I_a(x) \doteq \int_0^a e^{-xt} f(t)\,dt \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \quad (x \to \infty).$$

Lemma 2.2: *If $f \sim g$ and g is bounded as $x \to x_0$ then $e^f \sim e^g$ as $x \to x_0$.*

Corollary 2.1.2 (Laplace's Method): Suppose f and h are real-valued functions on $[a, b]$, such that $f \in C$, $h \in C^1$, and $h' < 0$ on some subinterval (a, c). Suppose also that $h(t) \le M < h(a)$ for $t \in (c, b)$, so that the maximum of h is approached only at a. Define $H(t) \doteq h(a) - h(t)$ for $t \in (a, c)$ and $F(u) \doteq f(H^{-1}(u))(H^{-1})'(u)$ and suppose

$$F(u) \sim u^\alpha \sum_{n=0}^{\infty} \gamma_n u^{\beta n} \qquad u \to 0^+,$$

with $\alpha > -1$ and $\beta > 0$. Then

$$I(x) = \int_a^b e^{xh(t)} f(t)\,dt \sim e^{xh(a)} \sum_{n=0}^{\infty} \frac{\gamma_n \Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \quad (x \to \infty),$$

provided the integral converges absolutely for all $x \ge X$.

Corollary 2.1.3 (Maximum with $N-1$ Zero Derivatives): Let f and h be infinitely differentiable real-valued functions on $[a, b]$. Suppose $f(a) \ne 0$ and h has an exterior maximum at a, with $h^{(n)}(a) = 0$ for $n = 1, 2, \ldots, N-1$, $h^{(N)}(a) < 0$, and $\sup_{[c,b]} h(t) < h(a)$ for all $c \in (a, b)$. Then the leading-order asymptotic expansion as $x \to \infty$ of $I(x) = \int_a^b e^{xh(t)} f(t)\,dt$ is

$$I(x) \sim \frac{1}{N}\Gamma\left(\frac{1}{N}\right) e^{xh(a)} f(a) \left(\frac{N!}{-h^{(N)}(a)x}\right)^{1/N} \quad (x \to \infty).$$

Theorem 2.2 (Riemann–Lebesgue Lemma):

(i) If f is piecewise continuous on a bounded interval $[a, b]$ then

$$\int_a^b e^{ixt} f(t)\,dt = o(1) \qquad (x \to \infty).$$

(ii) If f is continuous on (a possibly unbounded interval) (a,b) except perhaps at a finite number of points then

$$\int_a^b e^{ixt} f(t)\,dt = o(1) \qquad (x \to \infty),$$

provided for sufficiently large x the integral converges uniformly.

Lemma 2.3: *For $0 < \alpha < 1$ and $x > 0$,*

$$\int_0^\infty e^{ixt} t^{\alpha-1}\,dt = \frac{i^\alpha \Gamma(\alpha)}{x^\alpha}.$$

Lemma 2.4 (Steepest Descent): *If*

(i) $h(x+iy) = u(x,y) + iv(x,y)$ is analytic at $z_0 \doteq x_0 + iy_0$,

(ii) $h'(z_0) \neq 0$,

(iii) C is the curve through z_0 defined by $v(x,y) = v_0$,

then $\boldsymbol{\nabla} u$ is tangent to C at z_0.

Lemma 2.5 (Saddle Points): *If*

(i) h is analytic at z_0,

(ii) $h^{(n)}(a) = 0$ for $n = 1, 2, \ldots, N-1$ and $h^{(N)}(a) = ae^{i\alpha}$, with $a > 0$,

then there are N paths of steepest descent (ascent) through z_0, with direction $\frac{(2n+1)\pi-\alpha}{N}$ $(\frac{2n\pi-\alpha}{N})$ for $n = 0, 1, 2, \ldots, N-1$.

Theorem 3.1: *Suppose that*

$$y'' + p(x)y' + q(x)y = 0$$

has a regular singular point at x_0 and that the corresponding indicial equation $P(r) = 0$ has roots at r_1 and r_2.

1. *If $r_1, r_2 \in \mathbb{R}$ with $r_1 - r_2 \notin \mathbb{Z}$, then there exist two linearly independent solutions of the form*

$$y_1(x) = \sum_{n=0}^\infty a_n (x-x_0)^{n+r_1}, \qquad a_0 \neq 0,$$

$$y_2(x) = \sum_{n=0}^\infty b_n (x-x_0)^{n+r_2}, \qquad b_0 \neq 0.$$

2. *If $r_1, r_2 \in \mathbb{R}$ with $r_1 = r_2$, then there exist two linearly independent solutions of the form*

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \qquad a_0 \neq 0,$$

$$y_2(x) = y_1(x) \log(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_1},$$

   where the constants b_n may be zero.

3. *If $r_1, r_2 \in \mathbb{R}$ with $r_1 - r_2 \in \mathbb{N}$, then there exist two linearly independent solutions of the form*

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \qquad a_0 \neq 0,$$

$$y_2(x) = A y_1(x) \log(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \qquad b_0 \neq 0,$$

   where the constant A may be zero.

4. *If $r_1, r_2 = \alpha \pm i\beta$, then there exist two linearly independent solutions of the form*

$$y_1(x) = \cos(\beta \log(x - x_0)) \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\alpha}, \qquad a_0 \neq 0,$$

$$y_2(x) = \sin(\beta \log(x - x_0)) \sum_{n=0}^{\infty} b_n (x - x_0)^{n+\alpha}, \qquad b_0 \neq 0.$$