Math 417:
Honours Real Variables I

John C. Bowman
University of Alberta
Edmonton, Canada

April 16, 2021

Contents

Chapter 1

Measure Theory

[Tao 2011]

The *measure* $m(E)$ of a solid body E is a fundamental concept in Euclidean geometry. In one, two, and three dimensions, we refer to this measure as the length, area, or volume of E respectively.

In classical geometry, the measure of a body is typically determined by partitioning the body into components that can be translated or rotated and then reassembled into a simpler body with the same measure. Alternatively, lower and upper bounds on the measure of a body can be obtained by computing the measure of some inscribed or circumscribed body. Such arguments were justified by viewing the measure of a macroscopic body as the sum of the measures of its microscopic components.

With the advent of analytic geometry, Euclidean geometry was reinterpreted as the study of Cartesian products $\mathbb{R}^d$ of the real line $\mathbb{R}$. Within this analytical framework, it is no longer intuitively obvious how to define the measure $m(E)$ of a general subset E of $\mathbb{R}^d$. This is known as the *problem of measure*.

Remark: If one tries to formalize the physical notion of the measure of a body as the sum of the measure of its component "atoms", one runs into an immediate problem: a typical solid body consists of an uncountably infinite number of points, each of which has zero measure, and the product $\infty \cdot 0$ is indeterminate. Moreover, two bodies with the same number of points need not have the same measure: in one dimension, the intervals $[0, 1]$ and $[0, 2]$ have the same cardinality (using the bijection $x \mapsto 2x$) but different lengths. That is, one can disassemble $[0, 1]$ into an uncountably infinite number of points and reassemble them to form a set of twice the length! Pathological problems of this nature can even occur when one restricts the assembly to a finite number of components. In three or more dimensions, the famous *Banach–Tarski* paradox illustrates that the unit ball B can be disassembled into five pieces that can be translated, rotated, and then reassembled to form two disjoint copies of B!

Remark: The construction of the pathological pieces in the Banach–Tarski paradox requires the controversial *axiom of choice*. While trivial for finite sets (using induction), the axiom of choice for infinite (even countably infinite) sets does not follow from the other axioms of set theory and must be explicitly added to our fundamental list of axioms:

Axiom 1.1 (Axiom of choice): *Let* $\{E_\alpha : \alpha \in A\}$ *be a collection of nonempty sets* E_α*, indexed by elements of* A*. Then one can construct a set* $\{x_\alpha : \alpha \in A\}$ *of elements* x_α *chosen from* E_α*.*

- When $A = \mathbb{N}$, the axiom of choice states that it is possible to select a sequence $x_1, x_2, \ldots$ of elements from a sequence of nonempty sets $E_1, E_2, \ldots$.

We will begin by introducing two important notions of measure that are widely used in mathematics. The *Jordan measure*, which underlies the *Riemann integral*, suffices for many applications, for example, in defining the area under the graph of a continuous function. However, the notion of Jordan measure turns out to be inadequate for certain sets that arise as limits of other sets. The notion of *Lebesgue measure* and the associated *Lebesgue integral* were developed by the French mathematician Henri Lebesgue in 1902 to fill this gap.

Definition: The *indicator function* 1_S for a set S is

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Remark: To see why we might need a new type of integral, let $\{q_1, q_2, \ldots\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and for $n \in \mathbb{N}$ define $f_n = 1_{\{q_1,q_2,\ldots,q_n\}}$. Notice that the sequence of functions $\{f_n\}$ converges pointwise on $[0,1]$ to the *Dirichlet* function $1_{\mathbb{Q}\cap[0,1]}$. However, although the functions f_1, f_2, … are Riemann integrable on $[0,1]$, with $\int_0^1 f_n = 0$, their pointwise limit $1_{\mathbb{Q}\cap[0,1]}$ is not, since every nondegenerate subinterval contains both rational and irrational numbers. Although the interchange of limits and Riemann integration would be guaranteed by uniform convergence, the convergence of f_n is not uniform on $[0,1]$.

Remark: Similarly, given

$$f(x) = \int_a^b F(x,t)\, dt,$$

the statement

$$f'(x) = \int_a^b \frac{\partial F(x,t)}{\partial x}\, dt$$

does not hold in general (although it does if $\partial F(x,t)/\partial x$ is continuous in both x and t).

Q. Is it possible to find a new type of integral that generalizes the Riemann integral (so that every Riemann integral is still integrable in the new sense, with the same value) but for which limit processes (limits and integrals, derivatives and integrals) can always be interchanged?

A. No, the following example shows that some restrictions will still be required:

$$f_n(x) = \begin{cases} n^2 x & \text{if } x \in [0, \frac{1}{n}], \\ n^2\left(\frac{2}{n} - x\right) & \text{if } x \in (\frac{1}{n}, \frac{2}{n}], \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\lim_{n\to\infty} f_n(x) = 0$ for $x \in [0,1]$ but

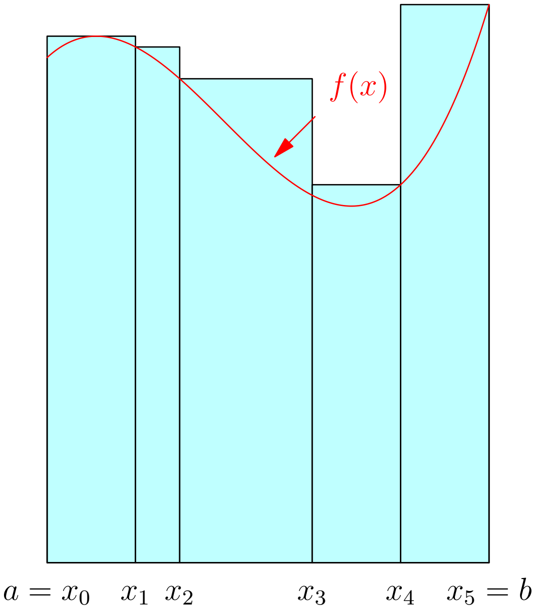
$$\int_0^1 \lim_{n\to\infty} f_n = 0 \neq 1 = \lim_{n\to\infty} \int_0^1 f_n.$$

Nevertheless, the class of functions for which such interchange of limit processes is valid will be much larger with the Lebesgue integral we are about to develop.

Remark: The Riemann integral is computed by approximating the area under a continuous function by an upper sum:

$$\int_a^b f \approx \sum_{k=1}^n f(\xi_k) m(I_k),$$

obtained by partitioning $[a,b]$ into subintervals $I_k = [x_{k-1}, x_k]$ for $k = 1, \ldots, n$, and finding points $\xi_k \in I_k$ at which f achieves its maximum on I_k, as illustrated below:



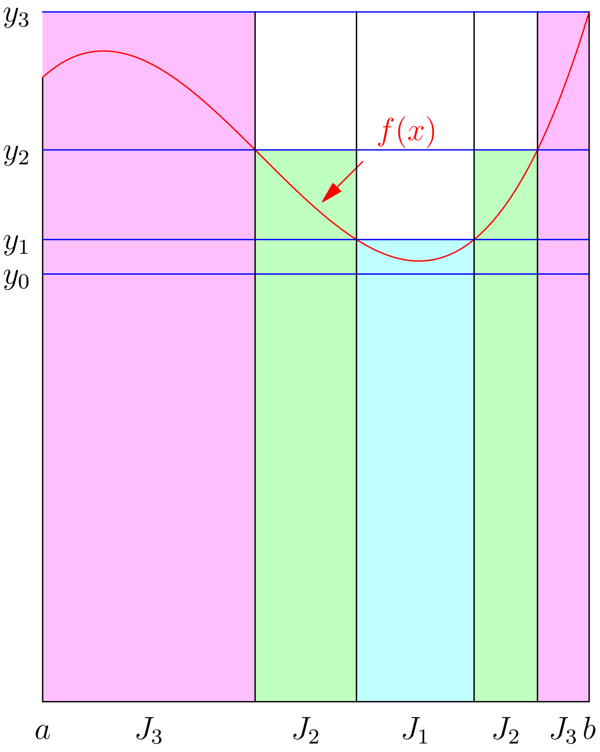
In contrast, to compute the Lebesgue integral one first partitions the y axis into intervals $[y_{k-1}, y_k]$ and computes the preimages

$$J_k = \{x \in [a,b] : f(x) \in [y_{k-1}, y_k]\}.$$

The Lebesgue integral is then approximated as

$$\int_a^b f \approx \sum_{k=1}^n y_k m(J_k),$$

as shown in the following diagram:



Q. In the above example, we notice that J_2 and J_3 are not intervals. What is the "length" of a general subset of $\mathbb{R}$ that is not an interval?

- For example, what is the "length" of the rational numbers within the unit interval $[0,1]$? This is equivalent to determining the integral

$$\int_0^1 1_{\mathbb{Q}},$$

which, as we have already discussed, is not Riemann integrable. However we will soon see that the Lebesgue integral of $1_{\mathbb{Q}}$ does exist and evaluates to 0, as one might expect from Cantor's diagonalization argument (there are vastly more irrationals than rationals within the unit interval). In fact, the Lebesgue measure will be constructed so that the measure of any countable set is zero, precisely so that one can then integrate functions like $1_{\mathbb{Q}}$ that have only a countable number of discontinuities (if the union of the subsets J_k containing the discontinuities has measure zero, they cannot contribute to the integral).

Remark: To attempt to answer the above question, we need to make the notion of "length", or in general, *measure*, more precise.

Definition: Let S be a set. The *power set* $\mathcal{P}(S)$ is the set of all subsets of S:

$$\mathcal{P}(S) = \{s : s \subset S\}.$$

- $\mathcal{P}(\emptyset) = \{\emptyset\}$.
- $\mathcal{P}(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

Problem 1.1: If S is a finite set of n elements, use induction to show that $\mathcal{P}(S)$ has 2^n elements.

Definition: We introduce the *non-negative extended real numbers* $[0,\infty]$, with the convention that $\infty \cdot 0 = 0 \cdot \infty = 0$.

Definition: Given a sequence $\{x_k\}_{k=1}^{\infty}$ of non-negative extended real numbers, we define

$$\sum_{k=1}^{\infty} x_k = \sup_{\substack{F \subset \mathbb{N} \\ F \text{ finite}}} \sum_{n \in F} x_n,$$

which may be finite or infinite.

Definition: Given a set $S = \{x_a\}_{a \in A}$ of non-negative extended real numbers indexed by an arbitrary set A, we define

$$\sum_{\alpha \in A} x_\alpha \doteq \sup_{\substack{F \subset A \\ F \text{ finite}}} \sum_{\alpha \in F} x_\alpha.$$

(We use the symbol $\doteq$ to emphasize a definition, although the notation := is more common.)

Remark: One can relabel the set S in an arbitrary manner without affecting the sum. In particular, we have:

Theorem 1.1 (Tonelli's theorem for series): *Let* $\{x_{n,k}\}_{n,k\in\mathbb{N}}$ *be a doubly infinite sequence of extended non-negative real numbers. Then*

$$\sum_{(n,k)\in\mathbb{N}^2} x_{n,k} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} x_{n,k} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} x_{n,k}.$$

Remark: Note that this rearrangement does not hold when dealing with signed summands (cf. *Riemann rearrangement theorem*).

If we wish to generalize our notion of measure from classical geometry to arbitrary subsets of $\mathbb{R}^d$, it seems reasonable to seek a function $m(\mathcal{P}(\mathbb{R}^d)) \mapsto [0,\infty]$ that satisfies the properties:

1. $m(\emptyset) = 0$; *nullity*
2. $m([a_1,b_1]\times\ldots\times[a_d,b_d]) = \prod_{k=1}^{d}(b_k - a_k), \qquad a_k \le b_k \in \mathbb{R};$ *measure of a box*
3. $m(S+x) = m(S) \quad \forall x \in \mathbb{R}^d, S \subset \mathbb{R}^d;$ *translational invariance*
4. $m\left(\bigcup_{k=1}^{\infty} S_k\right) = \sum_{k=1}^{\infty} m(S_k)$ for *disjoint* subsets S_k of $\mathbb{R}^d$. *countable disjoint additivity*

Remark: The above properties imply the *monotonicity* property

$$S \subset T \Rightarrow m(S) \le m(T).$$

To see this set $S_1 = S$, $S_2 = T \setminus S$, and $S_k = \emptyset$ for $k \ge 3$. Then

$$m(S) \le m(S) + m(T\setminus S) = \sum_{k=1}^{\infty} m(S_k) = m\left(\bigcup_{k=1}^{\infty} S_k\right) = m(T).$$

Q. Would it make sense to consider a stronger version of property 4:

$$m\left(\bigcup_{\alpha\in A} S_a\right) = \sum_{\alpha\in A} m(S_\alpha)$$

for disjoint subsets S_α of $\mathbb{R}^d$?

A. No; this would imply that

$$m([0,1]) = m\left(\bigcup_{a\in[0,1]} \{a\}\right) = \sum_{a\in[0,1]} m(\{a\}) = \sum_{a\in[0,1]} 0 = 0,$$

which would contradict property 2.

Remark: Unfortunately, the following counterexample shows that even in one dimension ($d = 1$), there exists no measure m that satisfies all four properties for arbitrary subsets of $\mathbb{R}$.

- Define an equivalence relation $\sim$ on $[0,1]$:

$$x \sim y \iff x - y \in \mathbb{Q}.$$

For $x \in [0,1]$, let $[x] \doteq \{a \in [0,1] : a \sim x\}$.

For example, $[0] = \mathbb{Q} \cap [0,1]$.

Claim: If $[x] \neq [y]$, then $[x] \cap [y] = \emptyset$.
Proof: Given $a \in [x]$ and $b \in [x] \cap [y]$, we see that $a \sim b$ and also $b \sim y$, which implies that $a \in [y]$. Thus $[x] \subset [y]$. Similarly, $[y] \subset [x]$. □

We can then construct a set $S \subset [0,1]$ consisting of a representative element chosen from each distinct set $[x]$ such that for each $x \in [0,1]$, $S \cap [x]$ consists of precisely one element. This requires the axiom of choice.

Claim:

$$[0,1] \subset \bigcup_{q\in\mathbb{Q}\cap[-1,1]} q + S \subset [-1,2].$$

Proof: The second inclusion is straightforward. To see the first, let $x \in [0,1]$ and a be the common element of S and $[x]$. Then $q \doteq x - a \in \mathbb{Q} \cap [-1,1]$. Hence $x \in q + S$. □

Let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [-1,1]$ such that $q_n \neq q_m$ for $n \neq m$.

Claim: $(q_n + S) \cap (q_m + S) = \emptyset$ for $n \neq m$.
Proof: Let $x \in (q_n + S) \cap (q_m + S)$. Then $\exists s_n, s_m \in S \ni x = q_n + s_n = q_m + s_m$. Hence $[s_n] = [x] = [s_m]$, which, by the choice of S, implies that $s_n = s_m$. Thus $q_n = q_m$ and hence $n = m$. □

Now if $m(S) = 0$, then

$$1 = m([0,1]) \leq m\left(\bigcup_{k=1}^{\infty} q_k + S\right) = \sum_{k=1}^{\infty} m(q_k + S) = \sum_{k=1}^{\infty} m(S) = 0. \ \#$$

Alternatively, if $m(S) > 0$, then

$$\infty = \sum_{k=1}^{\infty} m(S) = \sum_{k=1}^{\infty} m(q_k + S) = m\left(\bigcup_{k=1}^{\infty} q_k + S\right) \leq m([-1, 2]) = 3. \ \#$$

We thus see that it is impossible to find a measure m for arbitrary subsets of $\mathbb{R}$ that satisfies the given four properties. The nonmeasurable set S constructed here is called a *Vitali set*.

Remark: Since pathological sets like those encountered in the above example and in the Banach–Tarski paradox rarely occur in practical applications of mathematics, the standard approach to the problem of measure is to abandon the goal of assigning a measure to **every** subset of $\mathbb{R}^d$, focusing instead on a certain subclass of "nonpathological" subsets of $\mathbb{R}^d$ known as *measurable sets*.

1.A Elementary measure

Before we introduce Lebesgue measure and the associated Lebesgue integral, we will first review the more elementary concept of Jordan measure. To formally define Jordan measure, it is convenient to introduce the concept of *elementary measure*, which allows one to assign a notion of measure to *elementary sets*:

Definition: An *interval* I is a subset of $\mathbb{R}$ of the form $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, where $a \leq b$ are real numbers.

Definition: The *length* of an interval I is $|I| \doteq b - a$.

Definition: A *box* in $\mathbb{R}^d$ is a Cartesian product $I_1 \times \ldots \times I_d$ of intervals $I_1, \ldots, I_d$.

Definition: The *measure* of a box B is $|B| = |I_1| \cdot \ldots \cdot |I_d|$.

Definition: An *elementary set* is a finite union of boxes.

Problem 1.2 (Boolean closure): . If $E, F \subset \mathbb{R}^d$ are elementary sets, show that the union $E \cup F$, the intersection $E \cap F$, the set theoretic difference $E \setminus F \doteq \{x \in E : x \notin F\}$, and the symmetric difference $E \bigtriangleup F \doteq (E \setminus F) \cup (F \setminus E)$ are also elementary. If $x \in \mathbb{R}^d$, show that the translation $E + x \doteq \{y + x : y \in E\}$ is also an elementary set.

Lemma 1.1 (Measure of an elementary set): *Let $E \subset \mathbb{R}^d$ be an elementary set. Then*

(i) *E can be expressed as a finite union of* **disjoint** *boxes;*

(ii) *if E is partitioned as a finite union $B_1 \cup \ldots \cup B_n$ of disjoint boxes, then the quantity $m(E) \doteq |B_1|+ \ldots + |B_n|$ is independent of the partition. In other words, given any other partition $B'_1 \cup \ldots \cup B'_{n'}$ of E, one has $|B'_1|+ \ldots + |B'_{n'}|= |B_1|+ \ldots + |B_n|$.*

Proof:

(i) First consider the one-dimensional case $d = 1$. We can sort the $2n$ endpoints of any finite set of intervals $I_1, \ldots, I_n$ in ascending order, discarding repetitions. Denote the open intervals between successive endpoints, together with the endpoints themselves (treated as degenerate intervals), by $J_1, \ldots, J_{n'}$. Each interval I_i can be expressed as a union of a finite subset of the disjoint intervals $J_1, \ldots, J_{n'}$. The union $\cup_{i=1}^n I_i$ can thus be expressed as a finite union of disjoint intervals. For $d > 1$, express $E = \cup_{i=1}^n B_i$ where $B_i = I_{i,1} \times \ldots \times I_{i,d}$. For each $j = 1, \ldots, d$ we can decompose $I_{1,j}, \ldots, I_{n,j}$ as the union of disjoint intervals. On taking the Cartesian product over $j = 1, \ldots, d$ we can then express E as a finite union of disjoint boxes.

(ii) Let $\#A$ denote the cardinality of a finite set A and define $\frac{1}{N}\mathbb{Z} \doteq \{\frac{n}{N} : n \in \mathbb{Z}\}$. The length $|I|$ of any interval I can then be computed as

$$|I|= \lim_{N\to\infty} \frac{1}{N} \# \left(I \cap \frac{1}{N}\mathbb{Z} \right).$$

On taking the Cartesian product, we find that the measure of a box B can be expressed as

$$|B|= \lim_{N\to\infty} \frac{1}{N^d} \# \left(B \cap \left(\frac{1}{N}\mathbb{Z}\right)^d \right).$$

The measure of an elementary set E can thus be expressed as

$$m(E) = \lim_{N\to\infty} \frac{1}{N^d} \# \left(E \cap \left(\frac{1}{N}\mathbb{Z}\right)^d \right),$$

independent of its decomposition into disjoint boxes.

Definition: We refer to $m(E)$ in Lemma 1.1 as the *elementary measure* of E.

- The elementary measure of $[1, 2] \cup (3, 5)$ is 3.

Remark: The elementary measure $m(E)$ of an elementary set E is non-negative and satisfies the properties:

1. $m(\emptyset) = 0$; *nullity*
2. $m([a_1, b_1]\times \ldots \times[a_d, b_d]) = \prod_{k=1}^{d}(b_k - a_k)$; *measure of a box*
3. $m(E + x) = m(E) \quad \forall x \in \mathbb{R}^d$; *translational invariance*
4. $m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$ for disjoint elementary sets E_k. *finite disjoint additivity*

Remark: As previously shown, these properties imply the *monotonicity* property

$$E \subset F \Rightarrow m(E) \leq m(F)$$

for nested elementary sets E and F.

Problem 1.3: Show that the above properties imply the *finite subadditivity* property

$$m(E \cup F) \leq m(E) + m(F),$$

for arbitrary (not necessarily disjoint) elementary sets E and F. By induction, deduce that

$$m\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} m(E_k)$$

for arbitrary elementary sets E_k. Hint: $E \cup F = E \cup (F/E)$.

Problem 1.4 (Uniqueness of elementary measure): Let $d = 1$. Let $m' : \mathcal{E}(\mathbb{R}^d) \mapsto [0, \infty)$ be a map from the collection $\mathcal{E}(\mathbb{R}^d)$ of elementary subsets of $\mathbb{R}^d$ to $[0, \infty)$ that satisfies non-negativity, finite disjoint additivity, and translational invariance. Prove that there exists a constant $c \in \mathbb{R}$ such that $m'(E) = cm(E)$ for all elementary sets E. Moreover, if we enforce the normalization $m'([0, 1)^d) = 1$, show that $m' = m$. Hint: Set $c \doteq m'([0, 1)^d)$ and compute $m'([0, 1/n)^d)$ for any positive integer n.

Problem 1.5: Let $d_1, d_2 \geq 1$, and $E_1 \subset \mathbb{R}^{d_1}, E_2 \subset \mathbb{R}^{d_2}$ be elementary sets. Show that $E_1 \times E_2 \subset \mathbb{R}^{d_1+d_2}$ is elementary, with measure $m^{d_1+d_2}(E_1 \times E_2) = m^{d_1}(E_1)m^{d_2}(E_2)$, where $m^d(E)$ denotes elementary measure in dimension d.

1.B Jordan measure

Definition: Let $S \subset \mathbb{R}^d$ be a bounded set. The *Jordan inner measure* $m_{*J}(S)$ of S is

$$m_{*J}(S) \doteq \sup_{\substack{E \subset S \\ E \text{ elementary}}} m(E).$$

The *Jordan outer measure* $m^{*J}(S)$ of S is

$$m^{*J}(S) \doteq \inf_{\substack{E \supset S \\ E \text{ elementary}}} m(E).$$

Problem 1.6: If E is an elementary set, prove that $m_{*J}(E) = m^{*J}(E) = m(E)$.

Definition: If $m_{*J}(S) = m^{*J}(S)$, we say that S is Jordan measurable, and call $m(S) \doteq m_{*J}(S) = m^{*J}(S)$ the *Jordan measure* of S. To emphasize the dimension d, we sometimes write $m(S)$ as $m^d(S)$.

Remark: The finite disjoint additivity and subadditivity properties of elementary measure allow us to rewrite the Jordan outer measure $m^{*J}(S)$ of S as

$$m^{*J}(S) = \inf_{\substack{\bigcup_{k=1}^{n} B_k \supset S \\ B_k \text{ boxes}}} \sum_{k=1}^{n} |B_k|.$$

Remark: Unbounded sets are not Jordan measurable (we say they have *infinite Jordan outer measure* but leave the inner Jordan measure undefined).

Remark: The following lemma shows that Jordan measurable sets are those sets that are "almost elementary" with respect to Jordan outer measure.

Lemma 1.2: *Let $S \subset \mathbb{R}^d$ be bounded. The following are equivalent:*

(i) *S is Jordan measurable;*

(ii) *for every $\epsilon > 0$, there exist elementary sets E and F such that $E \subset S \subset F$ and $m(F \setminus E) < \epsilon$;*

(iii) *for every $\epsilon > 0$, there exists an elementary set E such that $m^{*J}(E \triangle S) < \epsilon$.*

Remark: Many common geometrical objects are Jordan measurable:

- A compact *convex polytope* in $\mathbb{R}^d$ formed by intersecting finitely many *closed half spaces* $\{x \in \mathbb{R}^d : a{\cdot}x \geq b\}$, where $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$;

- The region under the graph of a continuous non-negative function;

- The *open Euclidean ball* $B_r(c) \doteq \{x \in \mathbb{R}^d : |x - c| < r\}$;

- The *closed Euclidean ball* $B_r[c] \doteq \bar{B}_r(c)$;

- The linear transformation of a Jordan measurable set.

Remark: Jordan measure inherits many of the properties of elementary measure.

Lemma 1.3: *Let $S, T \subset \mathbb{R}^d$ be Jordan measurable sets. Then*

(i) *$S \cup T$, $S \cap T$, $S \setminus T$, and $S \triangle T$ are Jordan measurable;* Boolean closure

(ii) *$m(S) \geq 0$;* non-negativity

(iii) *If S and T are disjoint then $m(S \cup T) = m(S) + m(T)$;* finite disjoint additivity

(iv) *If $S \subset T$, then $m(S) \leq m(T)$;* monotonicity

(v) *$m(S \cup T) \leq m(S) + m(T)$;* finite subadditivity

(vi) *For any $x \in \mathbb{R}^d$, $m(S + x) \exists = m(S)$.* translational invariance

Problem 1.7:

(a) Show that a set S and its topological closure $\bar{S}$ have the same Jordan outer measure.

(b) A set bounded S and its interior S° have the same Jordan inner measure.

(c) Show that a bounded set S is Jordan measurable iff its topological boundary ∂S has Jordan outer measure zero. Use this result to prove that $[0, 1] \cap \mathbb{Q}$ and $[0, 1] \setminus \mathbb{Q}$ are not Jordan measurable.

1.C Lebesgue measure

Definition: Let $S \subset \mathbb{R}^d$. The *Lebesgue outer measure* $m^*(S)$ of S is

$$m^*(S) \doteq \inf_{\substack{\bigcup_{k=1}^{\infty} B_k \supset S \\ B_k \text{ boxes}}} \sum_{k=1}^{\infty} |B_k|.$$

Remark: The Lebesgue outer measure of a set is the infimal "cost" required to cover it by a **countable** union of boxes.

Remark: Since we can always convert a finite union of boxes to an infinite union by adding an infinite number of empty boxes, we see that $m^*(S) \le m^{*J}(S)$.

Remark: Observe that the Lebesgue outer measure is defined for unbounded sets.

- Let $S = \{x_0, x_1, \ldots\} \subset \mathbb{R}^d$ be a countable set. The Lebesgue outer measure of S is easily seen to be zero: one simply covers S by the degenerate boxes $\{x_0\}$, $\{x_1\}$, …, each having measure zero. Alternatively given $\epsilon > 0$, one can cover each x_k by a hypercube of sidelength $\epsilon/2^k$, leading to a total cost $\sum_{k=0}^{\infty}(\frac{\epsilon}{2^k})^d = \epsilon^d/(1-2^{-d})$; on taking the infimum, the Lebesgue outer measure of S is then seen to be zero. In contrast, the Jordan outer measure of a countable set can be large: $m^{*J}(\mathbb{Q} \cap [0,R]) = R$ and $m^{*J}(\mathbb{Q}) = \infty$.

Theorem 1.2 (Properties of Lebesgue outer measure):

(i) $m^*(\emptyset) = 0$; nullity

(ii) $S \subset T \subset \mathbb{R}^d \Rightarrow m^*(S) \le m^*(T)$; monotonicity

(iii) $m^*\left(\bigcup_{k=1}^{\infty} S_k\right) \le \sum_{k=1}^{\infty} m^*(S_k)$, *where* $S_k \subset \mathbb{R}^d$. countable subadditivity

Proof:

(ii) Any countable union of boxes that covers T must also cover S. Then

$$m^*(S) \doteq \inf_{\substack{\bigcup_{k=1}^{\infty} B_k \supset S \\ B_k \text{ boxes}}} \sum_{k=1}^{\infty} |B_k| \le \inf_{\substack{\bigcup_{k=1}^{\infty} B_k \supset T \\ B_k \text{ boxes}}} \sum_{k=1}^{\infty} |B_k| = m^*(T)$$

since the infimum on the left-hand side of the inequality is over a collection of unions at least as large as the collection on the right-hand side.

(iii) Given $\epsilon > 0$, for each $k \in \mathbb{N}$, there exists a countable union $\bigcup_{n=1}^{\infty} B_{k,n}$ of boxes $B_{k,n}$ containing S_k such that

$$\sum_{n=1}^{\infty} |B_{k,n}| < m^*(S_k) + \frac{\epsilon}{2^k}.$$

Since

$$\bigcup_{k=1}^{\infty} S_k \subset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} B_{k,n},$$

we see that

$$m^*\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |B_{k,n}| < \sum_{k=1}^{\infty} \left(m^*(S_k) + \frac{\epsilon}{2^k}\right) = \sum_{k=1}^{\infty} m^*(S_k) + \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \sum_{k=1}^{\infty} m^*(S_k) + \epsilon.$$

Since this must hold for all $\epsilon > 0$, the only possibility is that

$$m^*\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} m^*(S_k).$$

Definition: Let X and Y be two nonempty subsets of $\mathbb{R}^d$. The *distance* between X and Y is

$$\operatorname{dist}(X, Y) \doteq \inf\{|x - y| : x \in X, y \in Y\}.$$

Definition: The *diameter* of a nonempty set S is $\sup\{|x - y| : x, y \in S\}$.

Definition: Two sets $S, T \subset \mathbb{R}^d$ are *separated* if $\operatorname{dist}(S, T) > 0$.

Problem 1.8: Prove that compact disjoint sets in $\mathbb{R}^d$ are separated.

Lemma 1.4 (Finite additivity for separated sets): *Let S and T be separated sets. Then $m^*(S \cup T) = m^*(S) + m^*(T)$.*

Proof: From subadditivity, we know that $m^*(S \cup T) \leq m^*(S) + m^*(T)$, so it suffices to prove $m^*(S) + m^*(T) \leq m^*(S \cup T)$. This is trivial if $S \cup T$ has infinite Lebesgue outer measure. Otherwise $S \cup T$ has finite Lebesgue outer measure and, by monotonicity, so do S and T.

Given $\epsilon > 0$, without loss of generality we can cover $S \cup T$ by a countable collection of boxes $B_1, B_2, \ldots$ with diameter less than $\operatorname{dist}(S, T) > 0$ such that $\sum_{k=1}^{\infty} |B_k| < m^*(S \cup T) + \epsilon$. By construction, each box intersects at most one of S and T. We can therefore split this collection into two countable subcollections $B'_1, B'_2, \ldots$ and $B''_1, B''_2, \ldots$ that cover S and T, respectively. Then

$$m^*(S) \le \sum_{k=1}^{\infty} |B'_k|$$

and

$$m^*(T) \le \sum_{k=1}^{\infty} |B''_k|.$$

On summing, we find that

$$m^*(S) + m^*(T) \le \sum_{k=1}^{\infty} |B'_k| + \sum_{k=1}^{\infty} |B''_k| = \sum_{k=1}^{\infty} |B_k|.$$

Thus

$$m^*(S) + m^*(T) < m^*(S \cup T) + \epsilon.$$

We have thus obtained the desired result

$$m^*(S) + m^*(T) \le m^*(S \cup T).$$

Lemma 1.5 (Outer measure of elementary set): *Let E be an elementary set. Then the Lebesgue outer measure $m^*(E)$ and the elementary measure $m(E)$ are equal.*

Proof: Since $m^*(E) \le m^{*J}(E) = m(E)$, we only need to establish that $m(E) \le m^*(E)$. We first show this when E is closed. Given $\epsilon > 0$, there exists a countable collection $B_1, B_2, \ldots$ of boxes that cover E such that

$$\sum_{k=1}^{\infty} |B_k| < m^*(E) + \epsilon.$$

Each box B_k can be enclosed within an open box B'_k such that $|B'_k| < |B_k| + \epsilon/2^k$. We have thus constructed an open cover $\cup_{k=1}^{\infty} B'_k$ of the closed set E, with

$$\sum_{k=1}^{\infty} |B'_k| < \sum_{k=1}^{\infty} \left(|B_k| + \frac{\epsilon}{2^k} \right) = \sum_{k=1}^{\infty} |B_k| + \epsilon < m^*(E) + 2\epsilon.$$

By the Heine–Borel theorem, E is compact and therefore has a finite subcover $\cup_{k=1}^{n} B'_k$ for some $n \in \mathbb{N}$. The finite subadditivity of elementary measure then yields

$$m(E) \le \sum_{k=1}^{n} |B'_k| \le \sum_{k=1}^{\infty} |B'_k| < m^*(E) + 2\epsilon.$$

Then $m(E) \le m^*(E)$ and hence $m(E) = m^*(E)$ for any closed elementary set E.

If E is not closed, express E as a finite union $\cup_{k=1}^{n} B_k$ of disjoint boxes. Given $\epsilon > 0$, for every $k \in 1, \ldots, n$ there is a closed sub-box B'_k of B_k such that $|B_k| - \epsilon/n \le |B'_k|$. Then

$$m(E) - \epsilon = \sum_{k=1}^{n} \left(|B_k| - \frac{\epsilon}{n} \right) \le \sum_{k=1}^{n} |B'_k| = m\left(\bigcup_{k=1}^{n} B'_k \right) = m^*\left(\bigcup_{k=1}^{n} B'_k \right) \le m^*(E),$$

using our result for the closed elementary set $\cup_{k=1}^{n} B'_k \subset E$ and the monotonicity of the Lebesgue outer measure. Thus $m(E) \le m^*(E)$.

Remark: We have seen that the Lebesgue outer measure of a countable set is zero. Since the elementary measure of $[0,1]$ is 1, the above lemma then establishes that the Lebesgue outer measure of $[0,1]$ is also 1. This proves that the real numbers are uncountable.

Remark: Since the Lebesgue outer measure $m^*(E)$ of an elementary set E is $m(E)$, we observe from monotonicity that for any bounded $S \subset \mathbb{R}^d$,

$$m_{*J}(S) \doteq \sup_{\substack{E \subset S \\ E \text{ elementary}}} m(E) = \sup_{\substack{E \subset S \\ E \text{ elementary}}} m^*(E) \le \sup_{\substack{E \subset S \\ E \text{ elementary}}} m^*(S) = m^*(S).$$

Thus

$$m_{*J}(S) \le m^*(S) \le m^{*J}(S).$$

Remark: Not every bounded open or compact set is Jordan measurable. For example, enumerate the countable set $\mathbb{Q} \cap [0,1]$ as $q_1, q_2, \ldots$. Given $\epsilon > 0$, the Lebesgue outer measure of the open union

$$U_\epsilon \doteq \bigcup_{k=1}^{\infty} \left(q_k - \frac{\epsilon}{2^k}, q_k + \frac{\epsilon}{2^k} \right)$$

is

$$m^*(U_\epsilon) \le \sum_{k=1}^{\infty} \frac{2\epsilon}{2^k} = 2\epsilon.$$

In contrast, since $[0,1] \subset \bar{U}_\epsilon$ (U_ϵ is dense in $[0,1]$), we see from Problem 1.7(a) that

$$1 = m^{*J}([0,1]) \le m^{*J}(\bar{U}_\epsilon) = m^{*J}(U_\epsilon).$$

For $\epsilon = 1/3$ (say), we see that the Lebesgue and Jordan outer measures of the open set $U_{1/3}$ disagree. Moreover, from the previous remark, we see that $m_{*J}(U_{1/3}) \le m^*(U_{1/3}) \le 2/3 < 1 \le m^{*J}(U_{1/3})$; that is, $U_{1/3}$ is not Jordan measurable. Likewise, the complement of $U_{1/3}$ in $[-1,2]$ is compact but not Jordan measurable.

Definition: Two boxes are *almost disjoint* if their interiors are disjoint.

- $[0,1]$ and $[1,2]$ are almost disjoint.

Remark: Since a box has the same elementary measure as its interior, the finite additivity property

$$m\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{k=1}^{n} |B_k|$$

also holds for almost disjoint boxes B_k.

Lemma 1.6 (Countable unions of almost disjoint boxes): *Let S be a countable union of almost disjoint boxes $B_1, B_2, \ldots$. Then*

$$m^*(S) = \sum_{k=1}^{\infty} |B_k|.$$

Proof: Using countable subadditivity and Lemma 1.5, we find

$$m^*(S) \le \sum_{k=1}^{\infty} m^*(B_k) = \sum_{k=1}^{\infty} |B_k|.$$

Moreover, for each $n \in \mathbb{N}$, the elementary set $\bigcup_{k=1}^{n} B_k$ is contained in S. Using Lemma 1.5 again and monotonicity, we find that

$$\sum_{k=1}^{n} |B_k| = m\left(\bigcup_{k=1}^{n} B_k\right) = m^*\left(\bigcup_{k=1}^{n} B_k\right) \le m^*(S).$$

On letting $n \to \infty$, we obtain

$$\sum_{k=1}^{\infty} |B_k| \le m^*(S),$$

which establishes the desired result.

Remark: We thus see that $\mathbb{R}^d$ has an infinite Lebesgue outer measure.

Problem 1.9: Prove that the Lebesgue outer measure of a bounded union $\bigcup_{k=1}^{\infty} B_k$ of almost disjoint boxes B_k is equal to its Jordan inner measure.

Definition: Let $n \in \mathbb{Z}$. A *closed dyadic cube* in $\mathbb{R}^d$ with sidelength 2^{-n} has the form

$$\left[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\right] \times \ldots \times \left[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\right]$$

for integers $i_1, i_2, \ldots, i_d$.

Remark: The set of closed dyadic cubes of a fixed sidelength 2^{-n} are almost disjoint and cover all of $\mathbb{R}^d$.

Remark: Each closed dyadic cube in $\mathbb{R}^d$ of sidelength 2^{-n} has 2^d children of sidelength 2^{-n-1}.

Definition: Given a collection Q of closed dyadic cubes, we say that a cube is *maximal* if it is not contained in any other cubes in Q.

Lemma 1.7: *Every open subset of* $\mathbb{R}^d$ *can be expressed as a countable union of almost disjoint closed cubes.*

Proof: Recall that around every point x of an open set S there is an open ball entirely contained within S. Since each open ball contains a closed dyadic cube that includes x, an open set is the (countable) union of a collection Q of dyadic cubes contained within it. Let us restrict $n \geq 0$ to enforce a maximum cube sidelength of 1. Note that every cube in Q is contained in exactly one maximal cube and that any two such maximal cubes are almost disjoint. We thus see that S can be expressed as a countable union of (almost disjoint) maximal cubes.

Remark: From Prob 1.9, we thus see that the Lebesgue outer measure of a bounded open set is equal to its Jordan inner measure and corresponds to the total measure of any countable partitioning of the set into almost disjoint boxes.

Q. Can we express the Lebesgue outer measure of an arbitrary set in terms of the Lebesgue outer measure of open sets?

Lemma 1.8 (Outer regularity): *Let* $S \subset R^d$ *be an arbitrary set. Then*

$$m^*(S) = \inf_{\substack{U \supset S \\ U \text{open}}} m^*(U).$$

Proof: It follows immediately from monotonicity that

$$m^*(S) \leq \inf_{\substack{U \supset S \\ U \text{ open}}} m^*(U).$$

The other direction

$$\inf_{\substack{U \supset S \\ U \text{ open}}} m^*(U) \leq m^*(S)$$

holds trivially if $m^*(S)$ is infinite. If $m^*(S)$ is finite, given $\epsilon > 0$, there exists a countable collection $B_1, B_2, \ldots$ of boxes that cover S, with

$$\sum_{k=1}^{\infty} |B_k| < m^*(S) + \epsilon.$$

Each box B_k can be enclosed within an open box B'_k such that $|B'_k| < |B_k| + \epsilon/2^k$, with

$$\sum_{k=1}^{\infty} |B'_k| < \sum_{k=1}^{\infty} \left(|B_k| + \frac{\epsilon}{2^k} \right) = \sum_{k=1}^{\infty} |B_k| + \epsilon < m^*(S) + 2\epsilon.$$

From countable subadditivity, we see that

$$m^*\left(\bigcup_{k=1}^{\infty} B'_k \right) \le \sum_{k=1}^{\infty} |B'_k| < m^*(S) + 2\epsilon,$$

from which we deduce that

$$\inf_{\substack{U \supset S \\ U \text{ open}}} m^*(U) < m^*(S) + 2\epsilon.$$

The desired result then follows. $\square$

Remark: Outer regularity motivates the following definition.

Definition: A set $S \subset \mathbb{R}^d$ is said to be *Lebesgue measurable* if for every $\epsilon > 0$, there exists an open set $U \subset R^d$ containing S such that $m^*(U \setminus S) < \epsilon$.

Definition: If S is Lebesgue measurable, we refer to $m(S) \doteq m^*(S)$ as the *Lebesgue measure* of S. To emphasize the dimension d, we sometimes write $m(S)$ as $m^d(S)$.

Remark: A Lebesgue measurable set is one that can be efficiently contained within an open set (with respect to Lebesgue outer measure).

Remark: Let $S \subset U$. We will eventually show that the Lebesgue outer measure is not finitely additive for arbitrary disjoint subsets of $\mathbb{R}^d$: it is possible for $m^*(U) = m^*((U \setminus S) \cup S)$ to be less than $m^*(U \setminus S) + m^*(S)$. It is therefore **incorrect** to assume that $m^*(U \setminus S)$ equals $m^*(U) - m^*(S)$.

Problem 1.10: Show that a Jordan measurable subset of $\mathbb{R}^d$ is Lebesgue measurable.

Let S be Jordan measurable and $\epsilon > 0$. By the definition of Jordan outer measure, there exists a union $\bigcup_{k=1}^{n} B_k \supset S$ of *disjoint* boxes B_k such that

$$\sum_{k=1}^{n} |B_k| = m\left(\bigcup_{k=1}^{n} B_k \right) < m^{*J}(S) + \frac{\epsilon}{2}.$$

Slightly enlarge each box B_k to an open box $B'_k \supset B_k$ such that $|B'_k| < |B_k| + \epsilon/(2n)$. Let $U_\epsilon \doteq \bigcup_{k=1}^{n} B'_k \supset S$. By finite subadditivity,

$$m(U_\epsilon) \le \sum_{k=1}^{n} |B'_k| < \sum_{k=1}^{n} |B_k| + \frac{\epsilon}{2} < m^{*J}(S) + \epsilon.$$

By the definition of Jordan inner measure there exists an elementary set V_ϵ such that

$$m^*(S) - \epsilon < m(V_\epsilon) \le m_{*J}(S) = m^*(S) = m^{*J}(S) \le m(U_\epsilon) < m^*(S) + \epsilon.$$

Since S is bounded, we know that $m^*(S)$ is finite. Hence $m^*(U_\epsilon \setminus S) \le m(U_\epsilon \setminus V_\epsilon) = m(U_\epsilon) - m(V_\epsilon) < 2\epsilon$. That is, S is Lebesgue measurable.

Lemma 1.9 (Lebesgue measurability of compact sets): *Every compact subset of* $\mathbb{R}^d$ *is Lebesgue measurable.*

Proof: Let S be a compact set. Given $\epsilon > 0$, Lemma 1.8 guarantees the existence of an open set U containing S with $m^*(U) < m^*(S) + \epsilon$. The set $U \setminus S = U \cap S^c$ is open and so by Lemma 1.7 is the countable union $\bigcup_{k=1}^{\infty} Q_k$ of almost disjoint closed cubes Q_k. By Lemma 1.6, $m^*(U \setminus S) = \sum_{k=1}^{\infty} |Q_k|$. For every $n \in \mathbb{N}$, the finite union $\bigcup_{k=1}^{n} Q_k$ of closed cubes is itself closed and disjoint from the compact set S. Since compact disjoint sets in $\mathbb{R}^d$ are separated, it follows from Lemma 1.4 and monotonicity that

$$m^*(S) + m^*\left(\bigcup_{k=1}^{n} Q_k\right) = m^*\left(S \cup \bigcup_{k=1}^{n} Q_k\right) \leq m^*(U) < m^*(S) + \epsilon.$$

Thus $\sum_{k=1}^{n} |Q_k| = m^*(\bigcup_{k=1}^{n} Q_k) < \epsilon$ for each $n \in \mathbb{N}$. Hence $m^*(U \setminus S) = \sum_{k=1}^{\infty} |Q_k| \leq \epsilon$; that is, S is Lebesgue measurable.

Definition: A *null set* is a set of Lebesgue outer measure zero.

Lemma 1.10 (Lebesgue measurable sets): *The following sets are Lebesgue measurable:*

(i) *an open set;*

(ii) *a closed set;*

(iii) *a null set;*

(iv) *the empty set;*

(v) *the complement of a Lebesgue measurable subset of* $\mathbb{R}^d$*;*

(vi) *a countable union of Lebesgue measurable sets;*

(vii) *a countable intersection of Lebesgue measurable sets.*

Proof:

Claims (i) and (iv) follow directly from the definition of Lebesgue measurability.

(iii) If S is a null set, we know from outer regularity that $\inf\limits_{\substack{U \supset S \\ U \text{ open}}} m^*(U) = 0$. Thus given $\epsilon > 0$, there exists an open set $U \supset S$ such that $m^*(U \setminus S) \leq m^*(U) < \epsilon$, which means that S is Lebesgue measurable.

(vi) Given $\epsilon > 0$, let $S_1, S_2, \ldots$ be a sequence of Lebesgue measurable sets. For each $k \in \mathbb{N}$, S_k is contained in an open set U_k such that $m^*(U_k \setminus S_k) < \epsilon/2^k$. The Lebesgue measurability of $\bigcup_{k=1}^{\infty} S_k$ follows from the openness of $\bigcup_{k=1}^{\infty} U_k$ and countable subadditivity:

$$m^*\left(\bigcup_{k=1}^{\infty} U_k \setminus \bigcup_{k=1}^{\infty} S_k\right) \leq m^*\left(\bigcup_{k=1}^{\infty} (U_k \setminus S_k)\right) \leq \sum_{k=1}^{\infty} m^*(U_k \setminus S_k) < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

(ii) Given a closed set $S \subset \mathbb{R}^d$, we can express it as a countable union of compact sets: $S = \bigcup_{k=1}^{\infty} (\overline{B_k(0)} \cap S)$. The result then follows from Lemma 1.9 and (vi).

(v) If S is Lebesgue measurable then for each $n \in \mathbb{N}$ there exists an open set U_n containing S such that $m^*(U_n \setminus S) < 1/n$. Let $F = \bigcup_{n=1}^{\infty} U_n^c \subset S^c$. For each $n \in \mathbb{N}$, we note that $S^c \setminus F \subset S^c \setminus U_n^c$ and hence, using monotonicity, $m^*(S^c \setminus F) \leq m^*(S^c \setminus U_n^c) < 1/n$, noting that $S^c \setminus U_n^c = U_n \setminus S$. Thus $m^*(S^c \setminus F) = 0$, so that $S^c = F \cup (S^c \setminus F)$ is a countable union of the closed sets U_n^c and a set of Lebesgue outer measure zero. By (ii), (iii), and (vi), S^c is Lebesgue measurable.

(vii) Apply DeMorgan's laws to (vi), using (v) twice.

Remark: It follows that an elementary set E is Lebesgue measurable, with $m(E)$ equal to the elementary measure. This justifies the reuse of the symbol $m(E)$ for Lebesgue measure and shows that it generalizes the concept of elementary measure.

Definition: A *Boolean algebra* is an algebraic structure that characterizes both set and logic operations: a subset Y of a set X is associated with a set of bits indexed by $x \in X$ as 1 or 0 according to whether or not $x \in Y$. A Boolean algebra is thus analogous to the *field of sets*, which is any nonempty collection of subsets of a given set closed under finite union, finite intersection, and complement operations.

Remark: Properties (iv), (v), and (vi) of Lemma 1.10 state that the set of Lebesgue measurable sets is closed under countably many Boolean operations and thus forms a *σ-algebra*, a generalization of a Boolean algebra that requires closure under countable unions.[1]

Lemma 1.11 (Characterization of measurability): *Let $S \subset \mathbb{R}^d$. The following are equivalent:*

(i) *S is Lebesgue measurable;*

(ii) *given $\epsilon > 0$, there exists an open set U_ϵ* **containing** *S with $m^*(U_\epsilon \setminus S) < \epsilon$;* *(outer open approximation)*

(iii) *given $\epsilon > 0$, there exists an open set U_ϵ with $m^*(U_\epsilon \triangle S) < \epsilon$;* *(almost open)*

(iv) *given $\epsilon > 0$, there exists a closed set F_ϵ* **contained in** *S with $m^*(S \setminus F_\epsilon) < \epsilon$;* *(inner closed approximation)*

---

[1] The σ (the Greek letter corresponding to the initial s in the German word "Summe" originally used to denote union) in *σ-algebra* denotes closure under countable union. Likewise, a *δ-algebra* is closed under countable intersections (δ is the Greek letter corresponding to the initial d in the German word "Durchschnitt" for intersection). DeMorgan's laws establish that the terms σ-algebra and δ-algebra are actually equivalent.

(v) *given* $\epsilon > 0$, *there exists a closed set* F_ϵ *with* $m^*(F_\epsilon \triangle S) < \epsilon$; *(almost closed)*

(vi) *given* $\epsilon > 0$, *there exists a Lebesgue measurable set* S_ϵ *with* $m^*(S_\epsilon \triangle S) < \epsilon$. *(almost measurable)*

Proof:

(i) ⇒ (ii) This is the definition of Lebesgue measurability.

(ii) ⇒ (iii) $U_\epsilon \supset S \Rightarrow U_\epsilon \triangle S = U_\epsilon \setminus S$.

(iii) ⇒ (vi) This follows from Lemma 1.10 (i).

(i) ⇒ (iv) By Lemma 1.10 (v), S^c is Lebesgue measurable: given $\epsilon > 0$, there exists U_ϵ such that $S^c \subset U_\epsilon$, with $m^*(U_\epsilon \setminus S^c) < \epsilon$. Hence $S \supset U_\epsilon^c$, where U_ϵ^c is closed. Since $U_\epsilon \setminus S^c = U_\epsilon \cap S = S \setminus U_\epsilon^c$, we see that $m^*(S \setminus U_\epsilon^c) < \epsilon$.

(iv) ⇒ (v) $S \supset F_\epsilon \Rightarrow F_\epsilon \triangle S = S \setminus F_\epsilon$.

(v) ⇒ (vi) This follows from Lemma 1.10 (ii).

(vi) ⇒ (i) First, notice for any sets A, B, and S that $(A \cup B) \triangle S \subset (A \triangle S) \cup B$.

Given $\epsilon > 0$, there exists a measurable set S_ϵ with $m^*(S_\epsilon \triangle S) < \epsilon$. Since S_ϵ is measurable, it can be contained in an open set U_ϵ such that $m^*(U_\epsilon \setminus S_\epsilon) < \epsilon$. Then

$$U_\epsilon \triangle S \subset (S_\epsilon \triangle S) \cup (U_\epsilon \setminus S_\epsilon);$$

monotonicity and subadditivity then lead to

$$m^*(U_\epsilon \triangle S) \le m^*(S_\epsilon \triangle S) + m^*(U_\epsilon \setminus S_\epsilon) < \epsilon + \epsilon = 2\epsilon.$$

By outer regularity, there exists an open set V_ϵ containing $S \setminus U_\epsilon$ with

$$m^*(V_\epsilon) \le m^*(S \setminus U_\epsilon) + \epsilon \le m^*(S \triangle U_\epsilon) + \epsilon < 2\epsilon + \epsilon = 3\epsilon.$$

Finally, we note that the open set $U_\epsilon \cup V_\epsilon$ contains S and that by monotonicity and subadditivity,

$$m^*((U_\epsilon \cup V_\epsilon) \setminus S) = m^*((U_\epsilon \cup V_\epsilon) \triangle S) \le m^*(U_\epsilon \triangle S) + m^*(V_\epsilon) < 2\epsilon + 3\epsilon = 5\epsilon,$$

as desired.

Remark: We thus see that every measurable set S can be expressed as the union of a closed set $F \subset S$ and a set $S \setminus F$ of arbitrarily small measure. Likewise, there exists a set of arbitrarily small measure whose union with S is open.

Remark: While the Lebesgue outer measure exists for any subset of $\mathbb{R}^d$, we have remarked that it does not satisfy even finite additivity for disjoint subsets of $\mathbb{R}^d$. However, if we restrict our attention to Lebesgue measurable subsets of $\mathbb{R}^d$, then we actually obtain countable additivity, as we had originally hoped for on page 9:

Lemma 1.12 (Properties of Lebesgue measure):

(i) *$m(\emptyset) = 0$;* nullity*;*

(ii) *If $S_1, S_2, \ldots \subset \mathbb{R}^d$ is a countable sequence of disjoint Lebesgue measurable sets, then $m(\bigcup_{k=1}^{\infty} S_k) = \sum_{k=1}^{\infty} m(S_k)$.* countable additivity

Proof:

(i) We note that $m^*(\emptyset \setminus \emptyset) = m^*(\emptyset) = 0$.

(ii) First suppose that all of the sets S_k are compact and let $n \in \mathbb{N}$. Since compact disjoint sets in $\mathbb{R}^d$ are separated, we see using Lemma 1.4 and monotonicity that

$$\sum_{k=1}^{n} m(S_k) = m\left(\bigcup_{k=1}^{n} S_k\right) \le m\left(\bigcup_{k=1}^{\infty} S_k\right),$$

on replacing m^* with m, in view of Lemma 1.10. As $n \to \infty$, we see that

$$\sum_{k=1}^{\infty} m(S_k) \le m\left(\bigcup_{k=1}^{\infty} S_k\right).$$

Also, from countable subadditivity, we have

$$m\left(\bigcup_{k=1}^{\infty} S_k\right) \le \sum_{k=1}^{\infty} m(S_k).$$

The result then follows.

If the sets S_k are all bounded but not necessarily compact, we know from Lemma 1.11 that they can each be expressed as the union of a compact set $K_k \subset S_k$ and a set of arbitrarily small measure:

$$m(S_k) = m(K_k \cup (S_k \setminus K_k)) \le m(K_k) + m(S_k \setminus K_k) < m(K_k) + \frac{\epsilon}{2^k},$$

using subadditivity; thus

$$\sum_{k=1}^{\infty} m(S_k) \le \sum_{k=1}^{\infty} m(K_k) + \epsilon.$$

But from the compact case and monotonicity, we know that

$$\sum_{k=1}^{\infty} m(K_k) = m\left(\bigcup_{k=1}^{\infty} K_k\right) \le m\left(\bigcup_{k=1}^{\infty} S_k\right),$$

from which we deduce

$$\sum_{k=1}^{\infty} m(S_k) \le m\left(\bigcup_{k=1}^{\infty} S_k\right) + \epsilon$$

for every $\epsilon > 0$. The result then follows from countable subadditivity:

$$m\left(\bigcup_{k=1}^{\infty} S_k\right) \le \sum_{k=1}^{\infty} m(S_k).$$

Finally, if some of the sets S_k are not bounded, we decompose each of them as a countable union of disjoint Lebesgue measurable sets, using the bounded annuli $A_m \doteq B_m(0) \setminus B_{m-1}(0)$ for $m \in N$:

$$S_k = \bigcup_{m=1}^{\infty} (S_k \cap A_m).$$

The bounded case then yields

$$m(S_k) = \sum_{m=1}^{\infty} m(S_k \cap A_m).$$

We can similarly decompose $\bigcup_{k=1}^{\infty} S_k$ as a countable union of the disjoint bounded measurable sets $S_k \cap A_m$ over $(k, m) \in \mathbb{N}\times\mathbb{N}$ to obtain

$$m\left(\bigcup_{k=1}^{\infty} S_k\right) = m\left(\bigcup_{k=1}^{\infty}\bigcup_{m=1}^{\infty} (S_k \cap A_m)\right) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m(S_k \cap A_m) = \sum_{k=1}^{\infty} m(S_k).$$

Problem 1.11 (Monotone convergence theorem for Lebesgue measurable sets):

(i) (*Upward monotone convergence*) Let $S_1 \subset S_2 \subset \ldots$ be a countable increasing sequence of Lebesgue measurable subsets of $\mathbb{R}^d$. Show that

$$m\left(\bigcup_{k=1}^{\infty} S_k\right) = \lim_{n\to\infty} m(S_n).$$

Hint: Express $\bigcup_{k=1}^{\infty} S_k$ in terms of the *lacunae* $L_n = S_n \setminus \bigcup_{k=1}^{n-1} S_k$.

(ii) (*Downward monotone convergence*) Let $S_1 \supset S_2 \supset \ldots$ be a countable decreasing sequence of Lebesgue measurable subsets of $\mathbb{R}^d$. If at least one of the $m(S_k)$ is finite, show that

$$m\left(\bigcap_{k=1}^{\infty} S_k\right) = \lim_{n\to\infty} m(S_n).$$

(iii) Show that one cannot drop the assumption in (ii) that at least one of the $m(S_m)$ is finite.

Definition: A sequence $\{S_n\}_{n=1}^{\infty}$ of sets in $\mathbb{R}^d$ *converges pointwise* to another set S in $\mathbb{R}^d$ if the indicator functions 1_{S_n} converge pointwise to 1_S.

Definition: Recall that

$$\limsup_{n\to\infty} x_n \doteq \lim_{n\to\infty} \sup_{k\geq n} x_k$$
$$\liminf_{n\to\infty} x_n \doteq \lim_{n\to\infty} \inf_{k\geq n} x_k.$$

In the extended d-dimensional reals $[-\infty,\infty]^d$, a sequence $\{x_n\}$ converges iff $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

Problem 1.12: Suppose $S_n \subset \mathbb{R}^d$, $n = 1, 2, \ldots$ are Lebesgue measurable sets that converge pointwise to a set S.

(i) Show that S is Lebesgue measurable. Hint: use the fact that $1_S(x) = \liminf_{n\to\infty} 1_{S_n}(x) = \limsup_{n\to\infty} 1_{S_n}(x)$ to write S in terms of countable unions and intersections of S_n.

(ii) (Dominated convergence theorem) Suppose that the S_n are all contained in another Lebesgue measurable set F of finite measure. Show that $m(S_n)$ converges to $m(S)$. Hint: use the upward and downward monotone convergence theorems.

(iii) Give a counterexample to show that the dominated convergence theorem fails if the S_n are not contained in a set of finite measure, even if we assume that their Lebesgue measures are uniformly bounded.

Problem 1.13: Let $S \subset \mathbb{R}^d$. Show that S is contained in a Lebesgue measurable set of measure $m^*(S)$.

Problem 1.14: Let $S \subset \mathbb{R}^d$. Provide a counterexample that establishes that the claim

$$m^*(S) = \sup_{\substack{U \subset S \\ U \text{ open}}} m^*(U)$$

is false.

Problem 1.15: (*Inner regularity*) Let $S \subset \mathbb{R}^d$ be Lebesgue measurable. Show that

$$m(S) = \sup_{\substack{K \subset S \\ K \text{ compact}}} m(K).$$

Problem 1.16: (*Characterization of finite measurability*)
Let $S \subset \mathbb{R}^d$. Given $\epsilon > 0$, show that the following are equivalent:

(i) S is Lebesgue measurable with finite measure;

(ii) there exists an open set U_ϵ of finite measure **containing** S with $m^*(U_\epsilon \setminus S) < \epsilon$; (outer open approximation)

(iii) there exists a bounded open set U_ϵ with $m^*(U_\epsilon \triangle S) < \epsilon$; (almost open bounded)

(iv) there exists a compact set K_ϵ **contained in** S with $m^*(S \setminus K_\epsilon) < \epsilon$; (inner compact approximation);

(v) there exists a compact set K_ϵ with $m^*(K_\epsilon \triangle S) < \epsilon$; (almost compact)

(vi) there exists a bounded Lebesgue measurable set S_ϵ with $m^*(S_\epsilon \triangle S) < \epsilon$; (almost bounded measurable)

(vii) there exists a Lebesgue measurable set S_ϵ with finite measure such that $m^*(S_\epsilon \triangle S) < \epsilon$; (almost finite measure)

(viii) there exists an elementary set E_ϵ such that $m^*(E_\epsilon \triangle S) < \epsilon$; (almost elementary)

(ix) there exists a finite union F_ϵ of closed dyadic cubes such that $m^*(F_\epsilon \triangle S) < \epsilon$. (almost dyadic)

(i) ⇒ (ii) Since S is Lebesgue measurable, there exists an open set $U_\epsilon \supset S$ with $m(U_\epsilon \setminus S) < \epsilon$. Then by additivity we see that U_ϵ has finite measure:

$$m(U_\epsilon) = m(S) + m(U_\epsilon \setminus S) < m(S) + \epsilon.$$

(ii) ⇒ (iii) Let $\epsilon > 0$. Since S is measurable, it is contained in a countable union of boxes B_k, $k = 1, 2, \ldots$, such that (using monotonicity)

$$m(S) \leq \sum_{k=1}^{\infty} |B_k| < m(S) + \epsilon.$$

Each box B_k can be enclosed within an open box B'_k such that $|B'_k| < |B_k| + \epsilon/2^k$, so that

$$m(S) \leq \sum_{k=1}^{\infty} |B'_k| < m(S) + 2\epsilon.$$

Since $m(S)$ is finite, the infinite sum converges and hence there exists $n \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} |B'_k| < \epsilon.$$

Let U_ϵ be the bounded open set $\bigcup_{k=1}^{n} B'_k$. Since $S \setminus U_\epsilon \subset \bigcup_{k=n+1}^{\infty} B'_k$ we know by monotonicity that $m(S \setminus U_\epsilon) < \epsilon$. Also, since $U_\epsilon \setminus S \subset \bigcup_{k=1}^{\infty} B'_k \setminus S$, we have

$$m(U_\epsilon \setminus S) + m(S) \leq m\left(\bigcup_{k=1}^{\infty} B'_k \setminus S\right) + m(S) = m\left(\bigcup_{k=1}^{\infty} B'_k\right) < m(S) + 2\epsilon.$$

Since $m(S)$ is finite, we obtain $m(U_\epsilon \triangle S) = m(U_\epsilon \setminus S) + m(S \setminus U_\epsilon) < 3\epsilon$.

(iii) ⇒ (vi) This follows from Lemma 1.10.

(i) ⇒ (iv) By inner regularity, there exists for each $\epsilon > 0$ a compact set $K_\epsilon \subset S$ such that

$$m(K_\epsilon) > m(S) - \epsilon.$$

Since $S = K_\epsilon \cup (S \setminus K_\epsilon)$ and each of these sets have finite Lebesgue measure, we deduce by additivity that

$$m(S \setminus K_\epsilon) = m(S) - m(K_\epsilon) < \epsilon.$$

(iv) ⇒ (v) $S \supset K_\epsilon \Rightarrow K_\epsilon \triangle S = S \setminus K_\epsilon$.

(v) ⇒ (vi) This follows from Lemma 1.10.

(vi) ⇒ (vii) By monotonicity, a bounded Lebesgue measurable set has finite measure.

(vii) ⇒ (viii) Let $\epsilon > 0$. We know from Lemma 1.11 that S is measurable. Since S_ϵ is measurable, it is contained in a countable union of boxes B_k, $k = 1, 2, \ldots$, such that (using monotonicity)

$$m(S_\epsilon) \leq \sum_{k=1}^{\infty} |B_k| < m(S_\epsilon) + \epsilon.$$

Since $m(S_\epsilon)$ is finite, the infinite sum converges and hence there exists $n \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} |B_k| < \epsilon.$$

Then $E_\epsilon = \bigcup_{k=1}^{n} B_k$ is an elementary set such that $S_\epsilon \setminus E_\epsilon \subset \bigcup_{k=n+1}^{\infty} B_k$ and hence by monotonicity, $m(S_\epsilon \setminus E_\epsilon) < \epsilon$. Since $S \setminus E_\epsilon \subset (S \setminus S_\epsilon) \cup (S_\epsilon \setminus E_\epsilon)$, we conclude that

$$m(S \setminus E_\epsilon) \le m(S \setminus S_\epsilon) + m(S_\epsilon \setminus E_\epsilon) < 2\epsilon.$$

Also, since $E_\epsilon \setminus S \subset \bigcup_{k=1}^{\infty} B_k \setminus S$,

$$m(E_\epsilon \setminus S) \le m\left(\bigcup_{k=1}^{\infty} B_k \setminus S\right) \le m\left(\bigcup_{k=1}^{\infty} B_k \setminus S_\epsilon\right) + m(S_\epsilon \setminus S) < 2\epsilon,$$

noting that $\bigcup_{k=1}^{\infty} B_k \setminus S \subset (\bigcup_{k=1}^{\infty} B_k \setminus S_\epsilon) \cup (S_\epsilon \setminus S)$. Hence

$$m(E_\epsilon \triangle S) = m(E_\epsilon \setminus S) + m(S \setminus E_\epsilon) < 4\epsilon.$$

(viii) ⇒ (ix) The elementary set E_ϵ can be expressed as the union of a set of disjoint boxes. Each box is a Cartesian product of intervals. Given $\epsilon' > 0$, each of these intervals can be slightly enlarged so that its endpoints are rational numbers of the form $i/2^n$ for integers i and n, where $2^{-n} < \epsilon'$. We thus see that E_ϵ can be contained within a finite union Q_ϵ of closed dyadic cubes such that $m(Q_\epsilon \setminus E_\epsilon) < \epsilon$. Finally, since

$$Q_\epsilon \triangle S = (E_\epsilon \cup (Q_\epsilon \setminus E_\epsilon)) \triangle S \subset (E_\epsilon \triangle S) \cup (Q_\epsilon \setminus E_\epsilon),$$

we see that

$$m^*(Q_\epsilon \triangle S) \le m^*(E_\epsilon \triangle S) + m(Q_\epsilon \setminus E_\epsilon) < 2\epsilon.$$

(ix) ⇒ (i) Since a finite union of closed dyadic cubes is closed, Lemma 1.11 guarantees that S is Lebesgue measurable. Moreover, since $m(S) \le m(F_\epsilon \triangle S) + m(F_\epsilon) \le \epsilon + m(F_\epsilon)$ and F_ϵ is bounded, we see that S has finite measure.

Problem 1.17: Let $S \subset \mathbb{R}^d$. Prove that S is Lebesgue measurable $\iff$

$$m^*(E \cap S) + m^*(E \cap S^c) = m(E)$$

for every elementary set E.

Problem 1.18: (*Carathéodory criterion*) Generalize Prob. 1.17 to the stronger result that $S \subset \mathbb{R}^d$ is Lebesgue measurable $\iff$

$$m^*(A \cap S) + m^*(A \cap S^c) = m^*(A)$$

for every **arbitrary** subset A of $\mathbb{R}^d$.
Hints: Begin with the definition of $m^*(A)$. Use subadditivity and monotonicity. Consider the cases where $m^*(A) < \infty$ and $m^*(A) = \infty$.

Remark: In many texts, the Carathéodory criterion is adopted as the definition of Lebesgue measurability since it readily generalizes to abstract measure spaces.

Definition: Let $S \subset \mathbb{R}^d$ be a bounded set contained within an elementary set E. The *Lebesgue inner measure* of S is

$$m_*(S) \doteq m(E) - m^*(E \setminus S).$$

Problem 1.19:

(i) Show that the definition of Lebesgue inner measure is well defined in that it does not depend on the choice of elementary set E.

(ii) Show that $m_*(S) \le m^*(S)$ and that equality holds iff S is Lebesgue measurable.

Definition: A G_δ set is a countable intersection of open sets.

Definition: An F_σ set is a countable union of closed sets.

Remark: Note that a G_δ set need not be open and a F_σ set need not be closed.

Problem 1.20: Show that the following are equivalent:

(i) S is Lebesgue measurable;

(ii) S is the difference of a G_δ set and a null set;

(iii) S is the union of an F_σ set and a null set.

Problem 1.21: (*translational invariance*)
If $S \subset \mathbb{R}^d$ is Lebesgue measurable, show that $S + x$ is Lebesgue measurable for any $x \in \mathbb{R}^d$, with $m(S + x) = m(S)$.

Problem 1.22: (*Linear change of variables*)
If $S \subset \mathbb{R}^d$ is Lebesgue measurable and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear transformation, show that $T(S)$ is Lebesgue measurable, with $m(T(S)) = |\det T| m(S)$.

Problem 1.23: Let $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^{d'}$.

(a) Show that

$$m^{(d+d')*}(A \times B) \le m^{d*}(A) m^{d'*}(B).$$

Let $\{A_i\}$ be a sequence of disjoint boxes such that $A \subset \bigcup_{i=1}^{\infty} A_i$, with $\sum_{i=1}^{\infty} |A_i| < m^{d*}(A) + \epsilon$. Let $\{B_j\}$ be a sequence of disjoint boxes such that $B \subset \bigcup_{j=1}^{\infty} B_j$, with $\sum_{j=1}^{\infty} |B_j| < m^{d'*}(B) + \epsilon$. Then $A \times B \subset \bigcup_{i,j} A_i \times B_j$. Each box $A_i \times B_j \subset \mathbb{R}^{d+d'}$ has measure $|A_i \times B_j| = |A_j||B_j|$. By monotonicity,

$$m^{(d+d')*}(A \times B) \le \sum_{i,j} |A_i||B_j| < (m^{d*}(A) + \epsilon)(m^{d'*}(B) + \epsilon).$$

Since this holds for all $\epsilon > 0$, the desired result follows.

(b) If A and B are both Lebesgue measurable (but not necessarily of finite measure), show that $A \times B$ is Lebesgue measurable, with

$$m^{d+d'}(A \times B) = m^d(A) m^{d'}(B).$$

We first assume that both A and B have finite Lebesgue measure. Given $\epsilon > 0$, let U, V be open sets such that $m(U \setminus A) < \epsilon$ and $m(V \setminus B) < \epsilon$. From additivity, we know that $m(U) = m(U \setminus A) + m(A) < m(A) + \epsilon$ and $m(V) = m(V \setminus B) + m(B) < m(B) + \epsilon$. Since

$$(U \times V) \setminus (A \times B) = ((U \setminus A) \times V) \cup (U \times (V \setminus B)),$$

we see from subadditivity and part(a) that

$$\begin{aligned} m^*((U \times V) \setminus (A \times B)) &\le m^*((U \setminus A) \times V) + m^*(U \times (V \setminus B)) \\ &\le m(U \setminus A) m(V) + m(U) m(V \setminus B) \\ &< \epsilon[m(V) + m(U)] \\ &< \epsilon[m(A) + m(B) + 2\epsilon]. \end{aligned}$$

As ϵ is arbitrary, we see that $A \times B$ is Lebesgue measurable. Moreover, we know that the open sets $U = \bigcup_{i=1}^{\infty} U_i$ and $V = \bigcup_{i=1}^{\infty} V_i$ can be expressed as countable unions of almost disjoint boxes U_i and V_i, respectively. Thus

$$m(U \times V) = \sum_{(i,j) \in \mathbb{N}^2} |U_i||V_j| = \sum_{i=1}^{\infty} |U_i| \sum_{j=1}^{\infty} |V_j| = m(U) m(V).$$

We then find from additivity, subadditivity, and part (a) that

$$\begin{aligned} m(A)m(B) - m(A \times B) &= [m(U) - m(U \setminus A)][m(V) - m(V \setminus B)] \\ &\quad - m(U \times V) + m((U \times V) \setminus (A \times B)) \\ &\le -m(U \setminus A) m(V) - m(U) m(V \setminus B) + m((U \setminus A) \times V) \\ &\quad + m(U \times (V \setminus B)) + m(U \setminus A) m(V \setminus B) \\ &< \epsilon^2. \end{aligned}$$

Since ϵ is arbitrary, we see that

$$m(A)m(B) - m(A \times B) \le 0.$$

On combining this inequality with part (a), we obtain the desired result.

Finally, if A or B has infinite Lebesgue measure, apply upward monotone convergence to the result for $A \cap B_n(0)$ and $B \cap B_n(0)$, $n \in \mathbb{N}$.

Chapter 2

The Lebesgue Integral

Definition: Let $\{c_k\}_{k=1}^{\infty}$ be a sequence of elements of the set of complex numbers $\mathbb{C}$. We say that $\{c_k\}_{k=1}^{\infty}$ *converges absolutely* or is *absolutely summable* if

$$\sum_{k=1}^{\infty} |c_k| < \infty.$$

Remark: Recall that the partial sums $\sum_{k=1}^{n} c_k$ of an absolutely convergent sequence converge to a finite number. Furthermore, one can rearrange the terms of an absolutely convergent sequence without affecting its sum, defined in terms of its real and imaginary parts as

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \operatorname{Re} c_k + i \sum_{k=1}^{\infty} \operatorname{Im} c_k,$$

for complex c_k, where for real c_k,

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} c_k^{+} - \sum_{k=1}^{\infty} c_k^{-},$$

with $c_k^{+} \doteq \max(c_k, 0)$ and $c_k^{-} \doteq \max(-c_k, 0)$.

Definition: A (complex-valued) *simple function* $f : \mathbb{R}^d \to \mathbb{C}$ is a finite linear combination

$$f = \sum_{k=1}^{n} c_k 1_{S_k}.$$

of indicator functions 1_{S_k} of Lebesgue measurable sets $S_k \subset \mathbb{R}^d$ for $c_k \in \mathbb{C}$, where $n \in \mathbb{N}$ and $k = 1, \ldots, n$. If $f : \mathbb{R}^d \to [0, \infty]$ and $c_k \in [0, \infty]$, we say that f is an *unsigned simple function*.

Definition: Denote the complex vector space of simple functions over $\mathbb{R}^d$ as $\mathrm{Simp}(\mathbb{R}^d)$.

Definition: Denote the space of unsigned simple functions over $\mathbb{R}^d$ as $\mathrm{Simp}^+(\mathbb{R}^d)$.

Remark: In addition to the usual closure properties of a vector space, $\mathrm{Simp}(\mathbb{R}^d)$ is also closed under the pointwise product $f, g \mapsto fg$ and complex conjugation $f \to \bar{f}$, making $\mathrm{Simp}(\mathbb{R}^d)$ a commutative $*$-*algebra* (or *involutive algebra*). The space $\mathrm{Simp}^+(\mathbb{R}^d)$ is a $[0,\infty]$-module: it is closed under addition and under multiplication by elements of $[0,\infty]$.

Remark: Although we don't require the Lebesgue measurable sets $S_1,\ldots,S_n$ to be disjoint, we can achieve this by noting that they partition $\mathbb{R}^d$ into 2^n measurable sets, each of which is an intersection of $S_1,\ldots,S_n$ and their complements in $\mathbb{R}^d$.

Definition: Denote

$$\int_{\mathbb{R}^d} 1_S \doteq m(S).$$

Definition: If $f = \sum_{k=1}^n c_k 1_{S_k}$ is an unsigned simple function, define the *simple integral*

$$\mathrm{Simp}\int_{\mathbb{R}^d} f \doteq \sum_{k=1}^n c_k m(S_k).$$

Lemma 2.1 (Well-definedness of the simple integral): *Let* $n, n' \in \mathbb{N}$, $c_1,\ldots,c_n, c'_1,\ldots,c'_{n'} \in [0,\infty]$ *and* $S_1,\ldots,S_n,S'_1,\ldots,S'_{n'} \subset \mathbb{R}^d$ *be Lebesgue measurable sets such that*

$$\sum_{k=1}^{n} c_k 1_{S_k} = \sum_{k=1}^{n'} c'_k 1_{S'_k}.$$

Then

$$\sum_{k=1}^{n} c_k m(S_k) = \sum_{k=1}^{n'} c'_k m(S'_k).$$

Proof: The $n+n'$ sets S_k and $S'_{k'}$ partition $\mathbb{R}^d$ into $2^{n+n'}$ disjoint sets. The result then follows from the finite additivity of Lebesgue measure (for details see Tao, page 52).

Definition: A property $P(x)$ of points $x \in \mathbb{R}^d$ is said to hold *almost everywhere* (or *a.e.*) in $\mathbb{R}^d$ if the set of $x \in R^d$ on which $P(x)$ fails has Lebesgue measure zero (i.e. P is true outside of a null set).

Definition: Two functions f and g on $\mathbb{R}^d$ are said to *agree almost everywhere* if $f(x) = g(x)$ almost everywhere in $\mathbb{R}^d$.

Definition: The *support* of a function $f : \mathbb{R}^d \to \mathbb{C}$ or $f : \mathbb{R}^d \to [0, \infty]$ is the set $\{x \in \mathbb{R}^d : f(x) \neq 0\}$.

Problem 2.1: (*Properties of the simple unsigned integral*)
Let $f, g : \mathbb{R}^d \to [0, \infty]$ be simple unsigned functions. Then

(i)

$$\text{Simp} \int_{\mathbb{R}^d} (f + g) = \text{Simp} \int_{\mathbb{R}^d} f + \text{Simp} \int_{\mathbb{R}^d} g$$

and

$$\text{Simp} \int_{\mathbb{R}^d} cf = c \, \text{Simp} \int_{\mathbb{R}^d} f$$

for every $c \in [0, \infty]$; (unsigned linearity)

(ii) $\text{Simp} \int_{\mathbb{R}^d} f < \infty$ iff f is finite almost everywhere and its support has finite measure; (finiteness)

(iii) $\text{Simp} \int_{\mathbb{R}^d} f = 0$ iff $f = 0$ almost everywhere; (vanishing)

(iv) if f and g agree almost everywhere, $\text{Simp} \int_{\mathbb{R}^d} f = \text{Simp} \int_{\mathbb{R}^d} g$; (equivalence)

(v) if $f(x) \leq g(x)$ for almost every $x \in \mathbb{R}^d$, $\text{Simp} \int_{\mathbb{R}^d} f \leq \text{Simp} \int_{\mathbb{R}^d} g$; (monotonicity)

(vi) for any Lebesgue measurable set S, $\text{Simp} \int_{\mathbb{R}^d} 1_S = m(S)$. (compatibility)

Definition: A complex-valued simple function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be *absolutely integrable* if $\text{Simp} \int_{\mathbb{R}^d} |f| < \infty$.

Definition: If the real-valued simple function f is absolutely integrable, let

$$\text{Simp} \int_{\mathbb{R}^d} f \doteq \text{Simp} \int_{\mathbb{R}^d} f_+ - \text{Simp} \int_{\mathbb{R}^d} f_-$$

in terms of the unsigned simple functions $f_+ \doteq \max(f, 0)$ and $f_- \doteq \max(-f, 0)$ (which are both dominated by $|f|$).

Definition: If a complex-valued simple function $f : \mathbb{R}^d \to \mathbb{C}$ is absolutely integrable,

$$\text{Simp} \int_{\mathbb{R}^d} f \doteq \text{Simp} \int_{\mathbb{R}^d} \text{Re}\, f + i \, \text{Simp} \int_{\mathbb{R}^d} \text{Im}\, f$$

Remark: We note that a complex-valued simple function f is absolutely integrable iff its support has finite measure.

Remark: We now show that the space $\mathrm{Simp}^{\mathrm{abs}}(\mathbb{R}^d)$ of absolutely integrable simple functions, being closed under addition and scalar multiplication by complex numbers, is a complex vector space.

Problem 2.2: (*Properties of the simple integral*)
Let $f, g : \mathbb{R}^d \to \mathbb{C}$ be absolutely integrable simple complex-valued functions. Then

(i)

$$\mathrm{Simp}\int_{\mathbb{R}^d} (f+g) = \mathrm{Simp}\int_{\mathbb{R}^d} f + \mathrm{Simp}\int_{\mathbb{R}^d} g$$

and

$$\mathrm{Simp}\int_{\mathbb{R}^d} cf = c\,\mathrm{Simp}\int_{\mathbb{R}^d} f$$

for every $c \in \mathbb{C}$, along with

$$\mathrm{Simp}\int_{\mathbb{R}^d} \overline{f} = \overline{\mathrm{Simp}\int_{\mathbb{R}^d} f};$$

(*-linearity)

(ii) If f and g agree almost everywhere, $\mathrm{Simp}\int_{\mathbb{R}^d} f = \mathrm{Simp}\int_{\mathbb{R}^d} g$; (equivalence)

(iii) For any Lebesgue measurable set S, $\mathrm{Simp}\int_{\mathbb{R}^d} 1_S = m(S)$. (compatibility)

Hint: Use the decomposition

$$f + g = (f+g)_+ - (f+g)_- = (f_+ - f_-) + (g_+ - g_-).$$

Definition: An unsigned function $f : \mathbb{R}^d \to [0, \infty]$ is *unsigned Lebesgue measurable* (or *measurable*) if it is the pointwise limit of a sequence of unsigned simple functions.

Definition: Let $X \subset \mathbb{R}^d$. A set $U \subset X$ is *relatively open* (*relatively closed*) in X if there is an open (closed) set V in $\mathbb{R}^d$ such that $U = V \cap X$.

Lemma 2.2 (Characterization of measurable unsigned functions):
Let $f : \mathbb{R}^d \to [0,\infty]$ *be an unsigned function. The following are equivalent:*

(i) f *is unsigned Lebesgue measurable;*

(ii) f *is the pointwise limit of a sequence of unsigned simple functions;*

(iii) f *is the pointwise almost everywhere limit of unsigned simple functions;*

(iv) $f = \sup_n f_n$ *for an increasing sequence* f_n *of bounded unsigned simple functions that have finite-measure support;*

(v) *for every* $\lambda \in [0,\infty]$, *the set* $\{x \in \mathbb{R}^d : f(x) > \lambda\}$ *is Lebesgue measurable;*

(vi) *for every* $\lambda \in [0,\infty]$, *the set* $\{x \in \mathbb{R}^d : f(x) \geq \lambda\}$ *is Lebesgue measurable;*

(vii) *for every* $\lambda \in [0,\infty]$, *the set* $\{x \in \mathbb{R}^d : f(x) < \lambda\}$ *is Lebesgue measurable;*

(viii) *for every* $\lambda \in [0,\infty]$, *the set* $\{x \in \mathbb{R}^d : f(x) \leq \lambda\}$ *is Lebesgue measurable;*

(ix) *for every interval* $I \subset [0,\infty)$, *the set* $f^{-1}(I) \doteq \{x \in \mathbb{R}^d : f(x) \in I\}$ *is Lebesgue measurable;*

(x) *for every relatively open set* $U \subset [0,\infty)$, *the set* $f^{-1}(U) \doteq \{x \in \mathbb{R}^d : f(x) \in U\}$ *is Lebesgue measurable;*

(xi) *for every relatively closed set* $F \subset [0,\infty)$, *the set* $f^{-1}(F) \doteq \{x \in \mathbb{R}^d : f(x) \in F\}$ *is Lebesgue measurable.*

Proof:

(i) $\iff$ (ii) This is the definition of Lebesgue measurability of an unsigned function.

(ii) $\Rightarrow$ (iii) Everywhere implies almost everywhere.

(iv) $\Rightarrow$ (ii) Every monotone sequence in $[0,\infty]$ converges.

(iii) $\Rightarrow$ (v) We are given that for almost all $x \in \mathbb{R}^d$,

$$f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sup_{k\geq n} f_k(x) = \inf_{n\in\mathbb{N}} \sup_{k\geq n} f_k(x).$$

Then to within a set of measure zero, the set $\{x \in \mathbb{R}^d : f(x) > \lambda\}$ equals

$$\bigcup_{M\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \left\{ x \in \mathbb{R}^d : \sup_{k\geq n} f_k(x) > \lambda + \frac{1}{M} \right\},$$

or equivalently,

$$\bigcup_{M\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}\bigcup_{k\ge n}\left\{x\in\mathbb{R}^d : f_k(x)>\lambda+\frac{1}{M}\right\}.$$

But each set $\{x\in\mathbb{R}^d : f_k(x)>\lambda+1/M\}$ is Lebesgue measurable since each f_k is an unsigned simple function. Since countable unions and intersections of Lebesgue measurable sets are Lebesgue measurable, we arrive at (v).

(v) $\iff$ (vi)

(vii) $\iff$ (viii) To establish these two equivalences, let $\mathbb{Q}^+ \doteq \mathbb{Q}\cap[0,\infty)$. Then for $\lambda\in(0,\infty]$,

$$\{x\in\mathbb{R}^d : f(x)\ge\lambda\}=\bigcap_{q\in\mathbb{Q}^+:q<\lambda}\{x\in\mathbb{R}^d : f(x)>q\}.$$

Likewise for $\lambda\in[0,\infty)$,

$$\{x\in\mathbb{R}^d : f(x)>\lambda\}=\bigcup_{q\in\mathbb{Q}^+:q>\lambda}\{x\in\mathbb{R}^d : f(x)\ge q\}.$$

Since $\mathbb{Q}^+$ is countable, we then see that (v) and (vi) are equivalent and so are (vii) and (viii).

(v) $\iff$ (viii)

(vi) $\iff$ (vii)

(x) $\iff$ (xi) These three equivalences follow immediately upon taking complements.

(v)–(viii) $\Rightarrow$ (ix) Every interval $I\subset[0,\infty]$ can be expressed as the intersection of two semi-infinite intervals (half lines).

(x) $\Rightarrow$ (vii) Let $U=[0,\lambda)$.

(ix) $\Rightarrow$ (x) Every open set in $[0,\infty)$ is the union of countably many open intervals.

(ix) $\Rightarrow$ (iv) For every $n\in\mathbb{N}$, let $f_n(x)$ be the largest integer multiple of 2^{-n} bounded by $\min(f(x),n)$ for $x\in B_n[0]$ and zero elsewhere. At each x, we see that $f(x)$ is the supremum of the increasing sequence of functions $f_n(x)$. Note that f_n achieves each of its finite number of nonzero values c on a Lebesgue measurable set $f_n^{-1}(c)=f^{-1}(I_c)\cap B_n[0]$, where $I_c\subset[0,\infty)$ is an interval or half line. Thus each function f_n is a bounded unsigned simple function with finite-measure support.

Remark: Having established these characterizations of measurable functions, we now observe that many of the unsigned functions that arise in practical applications are measurable:

- every continuous unsigned function $f : \mathbb{R}^d \to [0, \infty]$;
- every unsigned simple function;
- the supremum, infimum, limit superior, and limit inferior of a sequence of unsigned measurable functions;
- an unsigned function that is almost everywhere equal to an unsigned measurable function;
- the composition $\phi \circ f$ of a continuous function $\phi : [0, \infty] \to [0, \infty]$ and an unsigned measurable function f;
- the sum and product of unsigned measurable functions.

Remark: If an unsigned measurable function f is bounded by M, the functions f_n constructed in Lemma 2.2 (ix) ⇒ (iv) are each bounded by M. Thus, f is a bounded unsigned measurable function with finite-measure support iff it is the *uniform* limit of a bounded sequence of simple functions.

Problem 2.3: Show that an unsigned function $f : \mathbb{R}^d \to [0, \infty]$ is a simple function iff it is measurable and takes on finitely many values.

If f is a simple function, it is a linear combination of n indicator functions, and therefore measurable. These indicator functions divide $\mathbb{R}^d$ into at most 2^n regions. Therefore, f can achieve at most 2^n values.

Conversely, if f is measurable and takes on only n different values $c_1, \ldots, c_n$, consider the Lebesgue measurable sets $S_k = f^{-1}(\{c_k\})$ for $k = 1, \ldots, n$. We can then express $f = \sum_{k=1}^n c_k 1_{S_k}$.

Remark: If f is measurable, Lemma 2.2 shows that $f^{-1}(S)$ is Lebesgue measurable for many, **but not all**, measurable sets S (for a counterexample, see Tao, Remark 1.3.10).

Definition: An almost everywhere-defined complex-valued function $f : \mathbb{R}^d \to \mathbb{C}$ is *Lebesgue measurable* (or *measurable*) if it is the pointwise almost-everywhere limit of a sequence of complex-valued simple functions.

Lemma 2.3 (Characterization of measurable complex-valued functions): *Let $f : \mathbb{R}^d \to \mathbb{C}$ be an almost-everywhere defined complex-valued function. The following are equivalent:*

(i) *f is measurable;*

(ii) *f is the pointwise almost-everywhere limit of a sequence of complex-valued simple functions;*

(iii) *the positive and negative parts of $\operatorname{Re} f$ and $\operatorname{Im} f$ are unsigned measurable functions;*

(iv) *for every open set $U \subset \mathbb{C}$, the set $f^{-1}(U)$ is measurable;*

(v) *for every closed set $F \subset \mathbb{C}$, the set $f^{-1}(F)$ is measurable.*

Remark: For defining a measurable function, it is enough just to test the open subsets of $[0, \infty]$; the preimage of relatively open subsets will then automatically be measurable since relatively open subsets of $[0, \infty]$ belong to the Borel σ-algebra $\mathcal{B}[\mathbb{R}]$ on $\mathbb{R}$ (the smallest σ-algebra that contains all open subsets of $\mathbb{R}$).

Problem 2.4: Let $f : \mathbb{R}^d \to \mathbb{C}$. Show that

(i) if f is continuous, it is measurable;

(ii) if f is almost everywhere equal to a measurable function, it is itself measurable.

(iii) if a sequence f_n of complex-valued measurable functions converges pointwise almost everywhere to f, then f is measurable.

(iv) if f is measurable, the composition $\phi \circ f$ of a continuous function $\phi : \mathbb{C} \to \mathbb{C}$ and f is measurable.

Problem 2.5: Show that the sum and product of measurable functions are measurable.

Definition: Let $f : \mathbb{R}^d \to [0,\infty]$ be an unsigned (but not necessarily measurable) function. The *lower unsigned Lebesgue integral* is

$$\underline{\int_{\mathbb{R}^d}} f \doteq \sup_{\substack{h \text{ simple} \\ 0 \le h \le f}} \operatorname{Simp} \int_{\mathbb{R}^d} h.$$

Likewise, the *upper unsigned Lebesgue integral* is

$$\overline{\int_{\mathbb{R}^d}} f \doteq \inf_{\substack{h \text{ simple} \\ h \ge f}} \operatorname{Simp} \int_{\mathbb{R}^d} h.$$

Remark: For any unsigned function $f : \mathbb{R}^d \to [0,\infty]$ observe that

$$\underline{\int_{\mathbb{R}^d}} f \le \overline{\int_{\mathbb{R}^d}} f$$

Theorem 2.1 (Properties of the lower and upper Lebesgue integrals): *Let $f, g : \mathbb{R}^d \to [0,\infty]$ be unsigned (not necessarily measurable) functions. Then*

(i) *if f is simple, $\underline{\int_{\mathbb{R}^d}} f = \overline{\int_{\mathbb{R}^d}} f = \operatorname{Simp} \int_{\mathbb{R}^d} f$;* compatibility

(ii) *if $f \le g$ pointwise almost everywhere, $\underline{\int_{\mathbb{R}^d}} f \le \underline{\int_{\mathbb{R}^d}} g$ and $\overline{\int_{\mathbb{R}^d}} f \le \overline{\int_{\mathbb{R}^d}} g$;* monotonicity

(iii) *$\underline{\int_{\mathbb{R}^d}} cf = c \underline{\int_{\mathbb{R}^d}} f$ for every $c \in [0,\infty)$;* scaling

(iv) *if f and g agree almost everywhere, $\underline{\int_{\mathbb{R}^d}} f = \underline{\int_{\mathbb{R}^d}} g$ and $\overline{\int_{\mathbb{R}^d}} f = \overline{\int_{\mathbb{R}^d}} g$;* equivalence

(v) *$\underline{\int_{\mathbb{R}^d}} (f+g) \ge \underline{\int_{\mathbb{R}^d}} f + \underline{\int_{\mathbb{R}^d}} g$;* lower superadditivity

(vi) *$\overline{\int_{\mathbb{R}^d}} (f+g) \le \overline{\int_{\mathbb{R}^d}} f + \overline{\int_{\mathbb{R}^d}} g$;* upper subadditivity

(vii) *for any measurable set $S \subset \mathbb{R}^d$,*

$$\underline{\int_{\mathbb{R}^d}} f = \underline{\int_{\mathbb{R}^d}} f 1_S + \underline{\int_{\mathbb{R}^d}} f 1_{S^c};$$

complementarity

(viii)

$$\lim_{n\to\infty} \underline{\int_{\mathbb{R}^d}} \min(f(x), n)\, dx = \underline{\int_{\mathbb{R}^d}} f;$$

vertical truncation

(ix)

$$\lim_{n\to\infty} \underline{\int_{\mathbb{R}^d}} f(x)\, 1_{B_n[0]}\, dx = \underline{\int_{\mathbb{R}^d}} f;$$

(use the monotone convergence theorem) horizontal truncation

(x) *if $f+g$ is a bounded simple function with finite measure support,*

$$\mathrm{Simp}\int_{\mathbb{R}^d} (f+g) = \underline{\int_{\mathbb{R}^d}} f + \overline{\int_{\mathbb{R}^d}} g.$$

reflection

Definition: If $f : \mathbb{R}^d \to [0,\infty]$ is measurable, we define the *unsigned Lebesgue integral* $\int_{\mathbb{R}^d} f$ to be the lower unsigned Lebesgue integral $\underline{\int_{\mathbb{R}^d}} f$.

Problem 2.6: Let $f : \mathbb{R}^d \to [0,\infty)$ be measurable, bounded, and have finite-measure support. Show that the lower and upper Lebesgue integrals agree. Hint: use the fact that a bounded unsigned measurable function is the uniform limit of a bounded sequence of simple functions.

Corollary 2.1.1 (Finite additivity of the Lebesgue integral): Let $f, g : \mathbb{R}^d \to [0,\infty]$ be measurable. Then $\int_{\mathbb{R}^d}(f+g) = \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g$.

Proof: We first prove this in the case where f and g are bounded and have bounded support. By Problem 2.6, the lower and upper integrals of f, g, and $f+g$ agree. The result then follows from lower superadditivity and upper subadditivity. The general case can then be reduced to this case by applying horizontal and vertical truncation and taking the corresponding limits.

Problem 2.7: Show for an arbitrary set $S \subset \mathbb{R}^d$ that $\displaystyle\overline{\int_{\mathbb{R}^d}} 1_S = m^*(S)$.

From outer regularity, we know that

$$m^*(S) = \inf_{\substack{U\supset S\\ U \text{ open}}} m(U) = \inf_{\substack{U\supset S\\ U \text{ open}}} \int_{\mathbb{R}^d} 1_U \geq \overline{\int_{\mathbb{R}^d}} 1_S$$

since $1_U \geq 1_S$. Furthermore, given $\epsilon > 0$, there exists a simple function $h_\epsilon \geq 1_S$ such that $h_\epsilon \geq 1$ on a measurable set $T_\epsilon \supset S$ and

$$\overline{\int_{\mathbb{R}^d}} 1_S + \epsilon > \int_{\mathbb{R}^d} h_\epsilon \geq \int_{\mathbb{R}^d} 1_{T_\epsilon} = m(T_\epsilon) \geq m^*(S).$$

Since ϵ is arbitrary, we conclude that $\displaystyle\overline{\int_{\mathbb{R}^d}} 1_S = m^*(S)$.

Remark: In view of the fact that Lebesgue outer measure is not necessarily additive, a consequence of Problem 2.7 is that the upper and lower Lebesgue integrals need not be additive.

Problem 2.8: If $f : \mathbb{R}^d \to [0, \infty]$ is Lebesgue measurable, show that the Lebesgue measure of $\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \le y \le f(x)\}$ exists and equals $\int_{\mathbb{R}^d} f$.

Remark: The statement in Problem 2.8 can be used as an alternate definition of the Lebesgue integral of a measurable function.

Remark: The Lebesgue integral is the unique map from measurable unsigned functions over $\mathbb{R}^d$ that is compatible with the simple integral and obeys finite additivity, along with the horizontal and vertical truncation properties.

Remark: We can extend a given function $f : [a, b] \to [0, \infty)$ to $\mathbb{R}$ by assigning it the value 0 on $\mathbb{R} \backslash [a, b]$. If f is Riemann integrable on $[a, b]$, then it is the pointwise limit of a sequence of piecewise constant functions and is therefore measurable. Since a lower sum for f is the integral of a simple function bounded above by f, it must be less than or equal to $\underline{\int_{\mathbb{R}}} f$. Likewise, an upper sum for f must be greater than or equal to $\overline{\int_{\mathbb{R}}} f$. Let L be the lower Riemann integral (the supremum of all lower sums) and U be the upper Riemann integral (the infimum of all upper sums). Then $L \le \underline{\int_{\mathbb{R}}} f \le \overline{\int_{\mathbb{R}}} f \le U$. Since $L = U = \int_a^b f$, we conclude that $\underline{\int_{\mathbb{R}}} f = \overline{\int_{\mathbb{R}}} f = \int_a^b f$. That is, the Lebesgue integral of a Riemann integrable function on $[a, b]$ exists and equals $\int_a^b f$.

Theorem 2.2 (Markov's inequality): *Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. Then for every $\lambda \in (0, \infty)$,*

$$m(\{x \in \mathbb{R}^d : f(x) \ge \lambda\}) \le \frac{1}{\lambda} \int_{\mathbb{R}^d} f.$$

Proof: In view of the pointwise inequality

$$\lambda 1_{\{x \in \mathbb{R}^d : f(x) \ge \lambda\}} \le f(x),$$

we see from the definition of the lower Lebesgue integral that

$$\lambda m(\{x \in \mathbb{R}^d : f(x) \ge \lambda\}) \le \int_{\mathbb{R}^d} f.$$

Corollary 2.2.1: Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable.

(i) If $\int_{\mathbb{R}^d} f < \infty$, then f is finite almost everywhere.

(ii) $\int_{\mathbb{R}^d} f = 0$ iff f is zero almost everywhere.

Proof:

(i) Consider the limit $\lambda \to \infty$.

(ii) Consider the limit $\lambda \to 0$.

Remark: The converse to Corollary 2.2.1 (i) is false: consider $\int_{\mathbb{R}^d} 1 = m(\mathbb{R}^d) = \infty$.

Definition: A measurable complex-valued function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be *absolutely integrable* if the L^1 *semi-norm*

$$|f|_{L^1(\mathbb{R}^d)} \doteq \int_{\mathbb{R}^d} |f| < \infty.$$

Definition: The space of all absolutely integrable functions is denoted $L^1(\mathbb{R}^d)$.

Definition: If f is real-valued and absolutely integrable, we define

$$\int_{\mathbb{R}^d} f \doteq \int_{\mathbb{R}^d} f_+ - \int_{\mathbb{R}^d} f_-,$$

where $f_+ \doteq \max(f, 0)$ and $f_- \doteq \max(-f, 0)$.

Definition: If f is complex-valued and absolutely integrable, we define

$$\int_{\mathbb{R}^d} f \doteq \int_{\mathbb{R}^d} \operatorname{Re} f + i \int_{\mathbb{R}^d} \operatorname{Im} f$$

Remark: From the pointwise triangle inequality $|f(x) + g(x)| \leq |f(x)| + |g(x)|$, we obtain the L^1 triangle inequality $|f+g|_{L^1(\mathbb{R}^d)} \leq |f|_{L^1(\mathbb{R}^d)} + |g|_{L^1(\mathbb{R}^d)}$ for every measurable function $f, g : \mathbb{R}^d \to \mathbb{C}$. Note also for $c \in \mathbb{C}$ that $|cf|_{L^1(\mathbb{R}^d)} = |c||f|_{L^1(\mathbb{R}^d)}$ and that $|f|_{L^1(\mathbb{R}^d)} = 0$ iff f vanishes almost everywhere. Together these properties make $L^1(\mathbb{R}^d)$ a complex vector space, with *semi-norm* L^1 (it is a semi-norm because of the almost everywhere qualifier).

Remark: The use of the integral $\int_{\mathbb{R}^d}|f|$ to control the distribution of f is the *first moment method*. In probability theory, one also uses higher moments such as $\int_{\mathbb{R}^d}|f|^p$ and Fourier moments $\int_{\mathbb{R}^d} e^{itf}$ to control the distribution of f.

Remark: The Lebesgue integral is a $*$-*linear operator* from $L^1(\mathbb{R}^d)$ to $\mathbb{C}$.

Problem 2.9: (*translational invariance*)
If $f \in L^1(\mathbb{R}^d)$, show for every $y \in \mathbb{R}^d$ that $\int_{\mathbb{R}^d} f(x+y)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$.

Problem 2.10: (*Linear change of variables*)
If $f \in L^1(\mathbb{R}^d)$ and T is an invertible linear transformation, show that

$$\int_{\mathbb{R}^d} f(T^{-1}x)\,dx = |\det T| \int_{\mathbb{R}^d} f.$$

Problem 2.11: If S and T are disjoint measurable subsets of $\mathbb{R}^d$ and $f : S \cup T \to \mathbb{C}$ is absolutely integrable, show that

$$\int_{S\cup T} (f 1_S) = \int_S f$$

and

$$\int_S f + \int_T f = \int_{S\cup T} f.$$

Lemma 2.4 (Triangle inequality): *Let* $f \in L^1(\mathbb{R}^d)$. *Then*

$$\left|\int_{\mathbb{R}^d} f\right| \le \int_{\mathbb{R}^d} |f|.$$

Proof: If $f = f_+ - f_-$ is real-valued then $|f| = f_+ + f_-$ and the claim follows from the triangle inequality on $\mathbb{R}$. If f is complex-valued we can express

$$\left|\int_{\mathbb{R}^d} f\right| = e^{i\theta} \int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} e^{i\theta} f$$

for some real phase θ. On taking real parts, we find

$$\left|\int_{\mathbb{R}^d} f\right| = \int_{\mathbb{R}^d} \operatorname{Re}\left(e^{i\theta} f\right) \le \int_{\mathbb{R}^d} |e^{i\theta} f| = \int_{\mathbb{R}^d} |f|.$$

Definition: A *step function* is a finite linear combination of indicator functions 1_B over boxes B.

Theorem 2.3 (Approximation of L^1 functions): *Let $f \in L^1(\mathbb{R}^d)$ and $\epsilon > 0$. There exists*

(i) *an absolutely integrable simple function g such that $|f - g|_{L^1(\mathbb{R}^d)} < \epsilon$.*

(ii) *a step function g such that $|f - g|_{L^1(\mathbb{R}^d)} < \epsilon$.*

(iii) *a continuous, compactly supported function $g \in L^1(\mathbb{R}^d)$ such that $|f - g|_{L^1(\mathbb{R}^d)} < \epsilon$.*

Proof:

(i) In the case where f is unsigned, by the definition of the lower Lebesgue integral there exists an unsigned simple function $g \le f$ such that $\int_{\mathbb{R}^d} g > \int_{\mathbb{R}^d} f - \epsilon$, which implies that $|f - g|_{L^1(\mathbb{R}^d)} < \epsilon$. This result can then immediately be generalized to the real-valued and complex-valued cases.

(ii) The case where f is a simple function can be reduced, using linearity and the triangle inequality, to the case where f is the indicator function of a set with finite measure. The claim then follows from the fact that such sets can be approximated by an elementary set. The general case then follows on applying (i) and the triangle inequality.

(iii) It suffices to focus on the case where f is the indicator function of a box B. Let B' be a slightly enlarged box that contains $\bar{B}$ within its interior, such that $|B'| < |B| + \epsilon$. Let $g(x) = \max(1 - C\,\mathrm{dist}(x, B), 0)$ where C is chosen sufficiently large such that $g(x) = 0$ outside of B'. Then g is continuous and compactly supported, with $|f - g|_{L^1(\mathbb{R}^d)} < \epsilon$.

Definition: A sequence of functions $f_n : \mathbb{R}^d \to \mathbb{C}$ converges *locally uniformly* to a function $f : \mathbb{R}^d \to C$ if f_n converges uniformly to f on every bounded subset $S \subset \mathbb{R}^d$.

- The sequence of functions $x \mapsto x/n$ on $\mathbb{R}$ for $n = 1, 2, \ldots$ converges locally uniformly (and hence pointwise) to 0 on $\mathbb{R}$, but not uniformly.

- The partial sums $\sum_{k=0}^{n} x^k/k!$ of the Taylor series of e^x converge to e^x locally uniformly on $\mathbb{R}$, but not uniformly.

- The functions
$$f_n(x) = \begin{cases} \dfrac{1}{nx} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$
converge pointwise everywhere to zero as $n \to \infty$, but not locally uniformly (due to the behaviour of f_n near $x = 0$).

Remark: Although pointwise convergence is evidently a weaker *mode of convergence* than locally uniform convergence, the following theorem establishes that one can recover locally uniform convergence if one is willing to delete as set of arbitrarily small measure.

Theorem 2.4 (*Egorov's theorem*): *Let $f_n : \mathbb{R}^d \to \mathbb{C}$ be a sequence of measurable functions that converge pointwise almost everywhere to $f : \mathbb{R}^d \to \mathbb{C}$. Given $\epsilon > 0$, there exists a Lebesgue measurable set S of measure at most ϵ such that f_n converges locally uniformly to f outside of S.*

Proof: By modifying f_n and f as needed on a null set (which can be absorbed into S), we may assume that f_n converges pointwise to f on $\mathbb{R}^d$. That is, for each $k \in \mathbb{N}$, the Lebesgue measurable set

$$S_{N,k} \doteq \{x \in \mathbb{R}^d : |f_n(x) - f(x)| \geq 1/k \text{ for some } n > N\}$$

obeys

$$\bigcap_{N=1}^{\infty} S_{N,k} = \emptyset.$$

For fixed $k \in N$, downward monotone convergence of the decreasing sequence of sets $S_{N,k} \cap B_k(0)$ yields

$$\lim_{N\to\infty} m(S_{N,k} \cap B_k(0)) = 0.$$

In particular, given $\epsilon > 0$, we can find $N_k \in \mathbb{N}$ such that

$$N > N_k \Rightarrow m(S_{N,k} \cap B_k(0)) < \frac{\epsilon}{2^k}.$$

Then by countable subadditivity, the Lebesgue measurable set

$$S \doteq \bigcup_{k=1}^{\infty} (S_{N,k} \cap B_k(0))$$

has measure less than ϵ. We thus see that for every $k \geq 1$ and $n > N_k$ that $|f_n(x) - f(x)| < 1/k$ (that is, f_n converges uniformly to f) on $x \in B_k(0) \setminus S$. Since every bounded set is contained within a ball $B_k(0)$ for some k, the desired result follows.

Remark: We have now witnessed the three heuristic principles of measure theory first articulated by Littlewood:

1. Every finite-measurable set is nearly a finite union of boxes;
2. Every absolutely integrable function is nearly continuous;
3. Every pointwise convergent sequence of functions is nearly locally uniformly convergent.

Chapter 3

Abstract Measure Spaces

Definition: Let X be a set. A *Boolean algebra* on X is a collection $\mathcal{B}$ of subsets of X such that:

(i) $\emptyset \in \mathcal{B}$; (empty set)

(ii) If $S \in \mathcal{B}$, then the complement $S^c \doteq X \setminus S$ is also an element of $\mathcal{B}$; (closure under complement)

(iii) If $S, T \in \mathcal{B}$, then $S \cup T \in \mathcal{B}$. (closure under finite union)

Definition: Given two Boolean algebras $\mathcal{B}$, $\mathcal{B}'$ on X, we say that $\mathcal{B}$ is *finer than* (*coarser than*) $\mathcal{B}'$ if $\mathcal{B} \supset \mathcal{B}'$ ($\mathcal{B} \subset \mathcal{B}'$).

- The coarsest Boolean algebra on a set X is the *trivial algebra* $\{\emptyset, X\}$.

- The finest Boolean algebra on a set X is the *discrete algebra* $\mathcal{P}(X) = \{S : S \subset X\}$.

Remark: All other Boolean algebras are intermediate between these two extremes: finer than the trivial algebra, but coarser than the discrete one.

- The *elementary Boolean algebra* on $\mathbb{R}^d$ is the collection of subsets of $\mathbb{R}^d$ that are either elementary or have an elementary complement.

- The *Jordan algebra* on $\mathbb{R}^d$ is the collection of subsets of $\mathbb{R}^d$ that are either Jordan-measurable or have a Jordan-measurable complement.

- The *Lebesgue algebra* $\mathcal{L}[\mathbb{R}^d]$ on $\mathbb{R}^d$ is the collection of Lebesgue-measurable subsets of $\mathbb{R}^d$.

Remark: The Lebesgue algebra is finer than the Jordan algebra, which is itself finer than the elementary Boolean algebra.

- The *null algebra* is the collection of sets in $\mathbb{R}^d$ that are either Lebesgue null sets or have null complements.

Remark: The null algebra is coarser than the Lebesgue algebra.

Remark: Let $\mathcal{F}$ be a collection of subsets of a set X. The intersection $\langle\mathcal{F}\rangle_{\text{bool}}$ of all Boolean algebras that contain $\mathcal{F}$ is itself a Boolean algebra. It is the coarsest Boolean algebra that contains $\mathcal{F}$; we say that $\langle\mathcal{F}\rangle_{\text{bool}}$ is *generated* by $\mathcal{F}$.

Definition: A Boolean algebra is *finite* if it contains only finitely many sets.

Definition: Suppose we express a set X as a union $\bigcup_{\alpha\in I} A_\alpha$ of disjoint sets A_α, called *atoms*, where I is an index set. This partitioning of X generates a Boolean algebra, the *atomic algebra* $\mathcal{A}(\{A_a : \alpha \in I\})$, defined as the collection of all unions $\bigcup_{\alpha\in J} A_\alpha$ such that $J \subset I$.

Remark: The trivial algebra corresponds to the trivial partition of X into a single atom, namely X itself.

Remark: The discrete algebra corresponds to the discrete partition of $X = \bigcup_{x\in X}\{x\}$ into singleton atoms.

- If we decompose the set $X = \{1, 2, 3, 4, 5\}$ into atoms $A_1 = \{1, 2\}$, $A_2 = \{3\}$, and $A_3 = \{4, 5\}$, over the index set $I = \{1, 2, 3\}$, we obtain the atomic boolean algebra $\mathcal{A}(\{A_1, A_2, A_3\}) = \{\emptyset, \{1, 2\}, \{3\}, \{4, 5\}, \{1, 2, 3\}, \{1, 2, 4, 5\}, \{3, 4, 5\}, X\}$.

Remark: Every finite Boolean algebra is an atomic algebra.

- Let n be an integer. The *dyadic algebra* $\mathcal{D}_n(\mathbb{R}^d)$ at scale 2^{-n} is the atomic algebra generated by taking unions and complements of half-open dyadic cubes

$$\left[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\right) \times \ldots \times \left[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\right)$$

for integers $i_1, i_2, \ldots, i_d$. Note that $\mathcal{D}_{n+1} \supset \mathcal{D}_n$.

Remark: The elementary, Jordan, Lebesgue, and null algebras cannot be expressed as nontrivial atomic algebras: they are not composed of indivisible atoms.

Definition: Let X be a set. A *σ-algebra* on X is a collection $\mathcal{B}$ of subsets of X such that

(i) $\emptyset \in \mathcal{B}$; (empty set)

(ii) If $S \in \mathcal{B}$, then the complement $S^c \doteq X \setminus S$ is also an element of $\mathcal{B}$; (closure under complement)

(iii) If $S_1, S_2, \ldots \in \mathcal{B}$, then $\bigcup_{k=1}^{\infty} S_k \in \mathcal{B}$. (closure under countable union)

- All atomic algebras are σ-algebras.

- The Lebesgue and null algebras are σ-algebras, but the elementary and Jordan algebras are not.

- Every σ-algebra is a Boolean algebra.

Remark: An intersection $\cap_{\alpha \in I} \mathcal{B}_\alpha$ of σ-algebras $\mathcal{B}_\alpha$ is itself a σ-algebra and is the finest σ-algebra that is coarser than each of the $\mathcal{B}_\alpha$.

Remark: Let $\mathcal{F}$ be a collection of subsets of X. The intersection $\langle \mathcal{F} \rangle$ of all σ-algebras that contain $\mathcal{F}$ is itself a σ-algebra. It is the coarsest σ-algebra that contains $\mathcal{F}$; we say that $\langle \mathcal{F} \rangle$ is *generated* by $\mathcal{F}$.

Remark: Observe that $\langle \mathcal{F} \rangle_{\mathrm{bool}} \subset \langle \mathcal{F} \rangle$, with equality holding iff $\langle \mathcal{F} \rangle_{\mathrm{bool}}$ is a σ-algebra.

- Let $\mathcal{F}$ be the collection of all boxes in $\mathbb{R}^d$. Then $\langle \mathcal{F} \rangle_{\mathrm{bool}}$ is the elementary algebra, which is not a σ-algebra.

Definition: Let X be a set and $\mathcal{B}$ be a σ-algebra. We refer to the pair $(X, \mathcal{B})$ as a *measurable space*.

Remark: In abstract measure theory, the σ-algebra $\mathcal{B}$ identifies the subsets of X that one is allowed to measure.

Definition: Let X be a metric space. The *Borel σ-algebra* $\mathcal{B}[X]$ on X is the σ-algebra generated by the collection of open subsets of X. The elements of $\mathcal{B}[X]$ are *Borel measurable*.

- The Borel σ-algebra contains all open sets, all closed sets, all G_δ sets, and all F_σ sets, along with countable unions and intersections thereof.

Remark: Since every open set in $\mathbb{R}^d$ is Lebesgue measurable, the Borel σ-algebra is coarser than the Lebesgue σ-algebra.

Remark: Let $\mathcal{F}$ be a collection, of cardinality κ, of subsets of X. Using *transfinite induction*, one can show that $\langle\mathcal{F}\rangle$ has cardinality at most $\kappa^{\aleph_0}$, where $\aleph_0$ denotes the cardinality of $\mathbb{N}$.

Remark: Since every open set in $\mathbb{R}^d$ can be expressed as a countable union of open balls (centered on a rational d-tuple, with rational radius), the cardinality of the generator of open sets is the same as the cardinality $\aleph_0$ of the rationals. Then $\mathcal{B}[\mathbb{R}^d]$ has cardinality at most $\aleph_0^{\aleph_0}$, which is the same as the cardinality $c \doteq 2^{\aleph_0}$ of the reals.

Problem 3.1: Let S be a set. Show that there is no surjective mapping $f : S \to \mathcal{P}(S)$. Hint: if there was, examine the element that maps to $T \doteq \{s \in S : s \notin f(s)\}$.

Remark: The Cantor set has Lebesgue measure zero, but cardinality c. Since any subset of a Lebesgue null set is also a null set, we see that the power set of the Cantor set has cardinality $2^c > c$. Thus, there exist Lebesgue-measurable sets that are not Borel measurable! We will construct one such set in Problem 6.2.

Remark: The Lebesgue σ-algebra on $\mathbb{R}^d$ is generated by the union of the Borel σ-algebra and the null σ-algebra.

Definition: Let $\mathcal{B}$ be a Boolean algebra on a set X. A *finitely additive measure* μ on $\mathcal{B}$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$; *nullity*

(ii) If S and T are disjoint elements of $\mathcal{B}$, $\mu(S \cup T) = \mu(S) + \mu(T)$; *finite additivity*

- The Lebesgue measure m is a finitely additive measure on the Lebesgue σ-algebra (and hence on the null, Jordan, and elementary sub-algebras).

- Lebesgue outer measure is not finitely additive on the discrete algebra.
- Jordan outer measure is not finitely additive on the Lebesgue algebra.
- Let x be an element of a set X and $\mathcal{B}$ be a Boolean algebra on X. The *Dirac measure* δ_x at x defined by $\delta_x(S) \doteq 1_S(x)$ for each $S \in \mathcal{B}$ is finitely additive.
- The *zero measure* $0 : S \to 0$ is a finitely additive measure on any Boolean algebra.

Remark: A linear combination of finitely additive measures is also a finitely additive measure.

Remark: Let X be a set and $\mathcal{B}$ be a Boolean algebra on X. The *counting measure* $\# : \mathcal{B} \to [0, \infty]$, defined as the cardinality of a finite set and infinity for an infinite set, is a finitely additive measure.

Problem 3.2: (*Properties of finitely additive measures*)
Let μ be a finitely additive measure on a Boolean algebra $\mathcal{B}$. Let S and T be $\mathcal{B}$-measurable sets. Show that

(i) If $S \subset T$, then $\mu(S) \le \mu(T)$; *monotonicity*

(ii) $\mu(S \cup T) \le \mu(S) + \mu(T)$; *finite subadditivity*

(iii) If S and T are disjoint, $\mu(S \cup T) = \mu(S) + \mu(T)$; *finite additivity*

(iv) $\mu(S \cup T) + \mu(S \cap T) = \mu(S) + \mu(T)$; *inclusion–exclusion*

(iv)
$$\mu(S \cap T) + \mu(S \cup T) = \mu(S \cap T) + \mu(S \setminus T) + \mu(T) = \mu(S) + \mu(T).$$

Problem 3.3: Let $\mathcal{B}$ be a finite Boolean algebra generated by a finite collection $A_1, A_2, \ldots, A_n$ of nonempty atoms. For every finitely additive measure μ on $\mathcal{B}$, show that there exists unique values $c_1, \ldots, c_n \in [0, \infty]$ such that

$$\mu(S) = \sum_{\substack{1 \le k \le n \\ A_k \subset S}} c_k \quad \forall S \in \mathcal{B}.$$

Equivalently, if $x_k \in A_k$ for $k \in \{1, \ldots, n\}$,

$$\mu = \sum_{k=1}^{n} c_k \delta_{x_k}.$$

Since $\mathcal{B}$ is a finite atomic Boolean algebra, we can express each $S \in \mathcal{B}$ as $S = \bigcup\limits_{\substack{1 \le k \le n \\ A_k \subset S}} A_k$.

Then $\mu(S) = \sum\limits_{\substack{1 \le k \le n \\ A_k \subset S}} c_k$, where $c_k = \mu(A_k)$.

Definition: Let $\mathcal{B}$ be a σ-algebra on a set X. A *countably additive measure* or *measure* μ on $\mathcal{B}$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$; *nullity*

(ii) if S_1, S_2, … are disjoint elements of $\mathcal{B}$, then $\mu(\bigcup_{k=1}^{\infty} S_k) = \sum_{k=1}^{\infty} \mu(S_k)$. *countable additivity*

Definition: Let X be a general space, $\mathcal{B}$ be a σ-algebra, and $\mu(S) \in [0, \infty]$ be a measure assigned to each $S \in \mathcal{B}$. We refer to the triple $(X, \mathcal{B}, \mu)$ as a *measure space*.

- The Lebesgue measure m is a countably additive measure on the Lebesgue σ-algebra (and hence on every sub-algebra, including the Borel σ-algebra).

- The Dirac measure is countably additive.

- The counting measure is countably additive.

- The restriction of a countably additive measure to a measurable subspace is again countably additive.

Problem 3.4: (*Countable combinations of measures*)
Let $(X, \mathcal{B})$ be a measurable space.

(i) If μ is a countably additive measure on $\mathcal{B}$ and $c \in [0, \infty]$, then $c\mu$ is also countably additive on $\mathcal{B}$.

(ii) If $\mu_1, \mu_2, \dots$ are a sequence of countably additive measures on $\mathcal{B}$, then their sum $\sum_{k=1}^{\infty} \mu_n$ is also countably additive on $\mathcal{B}$.

Remark: Since countably additive measures are also finitely additive, they inherit the monotonicity, finite subadditivity, and inclusion-exclusion properties. In addition, one has further properties.

Problem 3.5: Let $(X, \mathcal{B}, \mu)$ be a measure space. Establish the following properties.

(i) If $S_1, S_2, \ldots$ are $\mathcal{B}$-measurable, then

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{\infty} \mu(S_k).$$

countable subadditivity

(ii) If $S_1 \subset S_2 \subset \ldots$ is an increasing sequence of $\mathcal{B}$-measurable sets, then

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) = \lim_{n\to\infty} \mu(S_n) = \sup_n \mu(S_n).$$

upward monotone convergence

(iii) If $S_1 \supset S_2 \supset \ldots$ is a decreasing sequence of $\mathcal{B}$-measurable sets and at least one of the $\mu(S_k)$ is finite, then

$$\mu\left(\bigcap_{k=1}^{\infty} S_k\right) = \lim_{n\to\infty} \mu(S_n) = \inf_n \mu(S_n).$$

downward monotone convergence

Problem 3.6: (*Dominated convergence*) Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose S_n, $n = 1, 2, \ldots$ are $\mathcal{B}$-measurable sets that converge to a set S.

(i) Show that S is $\mathcal{B}$-measurable.

(ii) Suppose that the S_n are all contained in another $\mathcal{B}$-measurable set F of finite measure. Show that $\mu(S_n)$ converges to $\mu(S)$.
Hint: apply downward monotone convergence to the sets $\bigcap_{n=k}^{\infty} S_n$.

(iii) Give a counterexample to show that the dominated convergence theorem fails if the S_n are not contained in a set of finite measure.

Problem 3.7: Let X be an at most countable set and $\mathcal{B}$ be the discrete σ-algebra. Show that every measure μ on $(X, \mathcal{B})$ can be uniquely represented as

$$\mu(S) = \sum_{x\in S} c_x \quad \forall S \subset X.$$

where each $c_x \in [0, \infty]$. Equivalently

$$\mu = \sum_{x\in X} c_x \delta_x.$$

Definition: A *null set* of a measure space $(X, \mathcal{B}, \mu)$ is a $\mathcal{B}$-measurable set of measure zero.

Definition: A *subnull* set is any subset of a null set.

Definition: A measure space is *complete* if every subnull set is a null set.

- The Lebesgue measure space $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$ is complete.

- The Borel measure space $(\mathbb{R}^d, \mathcal{B}[\mathbb{R}^d], m)$ is not complete.

Definition: The *completion* of a measure space $(X, \mathcal{B}, \mu)$ is its (unique) coarsest refinement $(X, \bar{\mathcal{B}}, \bar{\mu})$ that is complete, consisting of sets that differ from a $\mathcal{B}$-measurable set by a $\mathcal{B}$-subnull set.

- The completion of the Borel measure space $(\mathbb{R}^d, \mathcal{B}[\mathbb{R}^d], m)$ is the Lebesgue measure space $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$.

Remark: Recall that a function is continuous if the preimage of every open set is open. In a similar spirit, in view of Lemma 2.2, we can now generalize the notion of a Lebesgue measurable function.

Definition: Let $(X, \mathcal{B})$ be a measurable space and let $f : X \to [0, \infty]$ (or $f : X \to \mathbb{C}$) be an unsigned or complex-valued function. We say that f is *measurable* if $f^{-1}(U)$ is $\mathcal{B}$-measurable for every open subset U of $[0, \infty]$ (or $\mathbb{C}$).

Problem 3.8: (*Characterization of measurable functions*)
Let $(X, \mathcal{B})$ be a measurable space. Show that

(i) a function $f : X \to [0, \infty]$ is measurable iff the level sets $\{x \in X : f(x) > \lambda\}$ are measurable for every $\lambda \in [0, \infty)$;

(ii) an indicator function 1_S of a set $S \subset X$ is measurable iff S is measurable;

(iii) a function $f : X \to [0, \infty]$ (or $f : X \to \mathbb{C}$) is measurable iff $f^{-1}(S)$ is measurable for every Borel-measurable subset S of $[0, \infty]$ (or $\mathbb{C}$);

(iv) a function $f : X \to \mathbb{C}$ is measurable iff its real and imaginary parts are measurable;

(v) a function $f : X \to \mathbb{R}$ is measurable iff its positive and negative parts are measurable;

(vi) the pointwise limit f of a sequence of measurable functions $f_n : X \to [0, \infty]$ (or $\mathbb{C}$) is also measurable;

(vii) if $f : X \to [0, \infty]$ (or $\mathbb{C}$) is measurable and $\phi : [0, \infty] \to [0, \infty]$ (or $\mathbb{C} \to \mathbb{C}$) is continuous, then $\phi \circ f$ is measurable;

(viii) the sum or product of two measurable functions in $[0, \infty]$ (or $\mathbb{C}$) is measurable.

Remark: The following is an abstract version of Egorov's theorem:

Theorem 3.1 (*Egorov's theorem*): *Let $(X, \mathcal{B}, \mu)$ be a finite measure space ($\mu(X) < \infty$) and let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions that converge pointwise almost everywhere to $f : X \to \mathbb{C}$. Given $\epsilon > 0$, there exists a $\mathcal{B}$-measurable set S of measure at most ϵ such that f_n converges uniformly to f outside of S.*

Remark: Recall that the atomic algebra $\mathcal{A}(\{A_a : \alpha \in I\})$ is the collection of all subsets of X that can be represented as the union of one more disjoint atoms A_α.

Problem 3.9: Let $(X, \mathcal{B})$ be an *atomic measurable space*: $\mathcal{B} = \mathcal{A}(\{A_a : \alpha \in I\})$ for some partition $\bigcup_{\alpha \in I} A_\alpha$ of X into disjoint nonempty atoms. Show that a function $f : X \to [0, \infty]$ or $f : X \to \mathbb{C}$ is measurable iff it is constant on each atom:

$$f = \sum_{\alpha \in I} c_\alpha 1_{A_\alpha}$$

for some constants c_α in $[0, \infty]$ or in $\mathbb{C}$, as appropriate. Furthermore, c_α are uniquely determined by f.

"$\Leftarrow$" A linear combination of indicator functions of measurable sets is measurable.

"$\Rightarrow$" Since $f^{-1}((0, \lambda))$ and $f^{-1}((\lambda, \infty))$ are measurable sets for each constant $\lambda \in [0, \infty]$, so is $f^{-1}(\lambda) = f^{-1}([\lambda, \infty] \cap [0, \lambda])$. Let A be any nonempty atom of X and choose $x \in A$. Then since $f^{-1}(f(x))$ is a measurable set containing x, and the smallest such measurable set is A, we see that $A \subset f^{-1}(f(x))$. That is, f is constant on each atom.

Definition: Let $(X, \mathcal{B}, \mu)$ be a measure space, with $\mathcal{B}$ finite (and hence atomic). Let $\mathcal{B} = \mathcal{A}(\{A_a : \alpha \in I\})$ for some partition $\bigcup_{\alpha \in I} A_\alpha$ of X into disjoint nonempty atoms. If $f : X \to [0, \infty]$ is measurable, it has a unique representation of the form

$$f = \sum_{k=1}^{n} c_k 1_{A_k}$$

for some constants c_k in $[0, \infty]$. We then define the *simple integral*

$$\mathrm{Simp} \int_X f \, d\mu \doteq \sum_{k=1}^{n} c_k \mu(A_k).$$

Remark: The measurable function f in the above definition only takes on a finite number of values.

Remark: The precise decomposition of $\mathcal{B}$ into atoms does not affect the value of the simple integral.

Remark: Having defined the simple integral of unsigned measurable functions when only a finite number of subsets of X are measurable, we can also construct the simple integral of real-valued and complex-valued functions, as we did for Lebesgue measurable functions.

Remark: We immediately obtain the monotonicity property $f \leq g$ implies $\operatorname{Simp} \int f \, d\mu \leq \operatorname{Simp} \int g \, d\mu$, as well as linearity:

$$\operatorname{Simp} \int_X (f+g) \, d\mu = \operatorname{Simp} \int_X f \, d\mu + \operatorname{Simp} \int_X g \, d\mu$$

and

$$\operatorname{Simp} \int_X cf \, d\mu = c \operatorname{Simp} \int_X f \, d\mu$$

for measurable functions f and g, with $c \in [0, \infty]$.

Remark: Let $(X, \mathcal{B}, \mu)$ be a measure space and $(X, \mathcal{B}', \mu')$ be a *coarsening* of $(X, \mathcal{B}, \mu)$, in the sense that $\mathcal{B}$ contains $\mathcal{B}'$ and μ' agrees with μ on $\mathcal{B}'$. If $\mathcal{B}'$ is finite, and $f : X \to [0, \infty]$ is $\mathcal{B}'$-measurable, then

$$\operatorname{Simp} \int_X f \, d\mu = \operatorname{Simp} \int_X f \, d\mu'$$

This observation provides a means of extending the simple integral to general measure spaces $(X, \mathcal{B})$.

Definition: An *unsigned simple function* $f : X \to [0, \infty]$ on a measurable space $(X, \mathcal{B})$ is a measurable function that takes on finitely many values $a_1, \ldots, a_k$.

Remark: Simple functions are automatically measurable with respect to at least one finite sub-algebra $\mathcal{B}'$ of $\mathcal{B}$, namely the Boolean algebra $\mathcal{B}'$ generated by the preimages $f^{-1}(\{a_1\}), \ldots, f^{-1}(\{a_k\})$.

Definition: If $f : X \to [0,\infty]$ is an unsigned simple function on a measure space $(X, \mathcal{B}, \mu)$ that takes on values $a_1, \ldots, a_k$, we define

$$\operatorname{Simp} \int_X f \, d\mu = \operatorname{Simp} \int_X f \, d\mu|_{\mathcal{B}'},$$

where $\mu|_{\mathcal{B}'}$ is the *restriction* of μ to the finite Boolean algebra $\mathcal{B}'$ generated by the preimages $f^{-1}(\{a_1\}), \ldots, f^{-1}(\{a_k\})$.

Theorem 3.2 (Properties of the simple integral): *Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f, g : X \to [0,\infty]$ be simple functions. Then*

(i) *if $f \le g$ pointwise,* $\operatorname{Simp} \int_X f \, d\mu \le \operatorname{Simp} \int_X g \, d\mu$; monotonicity

(ii) $\operatorname{Simp} \int_X 1_S \, d\mu = \mu(S)$ *for every $\mathcal{B}$-measurable set S;* compatibility

(iii) $\operatorname{Simp} \int_X cf \, d\mu = c \operatorname{Simp} \int_X f \, d\mu$ *for every $c \in [0,\infty]$;* homogeneity

(iv) $\operatorname{Simp} \int_X (f+g) \, d\mu = \operatorname{Simp} \int_X f \, d\mu + \operatorname{Simp} \int_X g \, d\mu$; finite additivity

(v) *if $(X, \bar{\mathcal{B}}, \bar{\mu})$ is a refinement of $(X, \mathcal{B}, \mu)$,* $\operatorname{Simp} \int_X f \, d\bar{\mu} = \operatorname{Simp} \int_X f \, d\mu$; refinement

(vi) *if $f(x) = g(x)$ for μ-almost every $x \in X$,* $\operatorname{Simp} \int_X f \, d\mu = \operatorname{Simp} \int_X g \, d\mu$; equivalence

(vii) $\operatorname{Simp} \int_X f \, d\mu < \infty$ *iff f is finite μ-almost everywhere and is supported on a set of finite μ-measure;* finiteness

(viii) $\operatorname{Simp} \int_X f \, d\mu = 0$ *iff f is zero μ-almost everywhere.* vanishing

Definition: Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f : X \to [0,\infty]$ be measurable. The *unsigned integral* is

$$\int_X f \, d\mu \doteq \sup_{\substack{h \text{ simple} \\ 0 \le h \le f}} \operatorname{Simp} \int_X h \, d\mu.$$

Remark: If $X = \mathbb{R}^d$ and f is Lebesgue measurable, this definition reduces to the unsigned Lebesgue integral: $\int_X f \, dm = \int_{\mathbb{R}^d} f$.

Theorem 3.3 (Properties of the unsigned integral): *Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f, g : X \to [0,\infty]$ be measurable. Then*

(i) *If $f = g$ μ-almost everywhere, then $\int_X f \, d\mu = \int_X g \, d\mu$;* equivalence

(ii) *if $f \le g$ μ-almost everywhere, then $\int_X f \, d\mu \le \int_X g \, d\mu$;* monotonicity

(iii) *for every* $c \in [0, \infty]$,

$$\int_X cf \, d\mu = c \int_X f \, d\mu;$$

homogeneity

(iv) $\int_X (f + g) \geq \int_X f + \int_X g$; superadditivity

(v) *if* f *is simple, then* $\int_X f \, d\mu = \mathrm{Simp} \int_X f \, d\mu$ compatibility

(vi) *for every* $\lambda \in (0, \infty)$, Markov's inequality

$$\mu(\{x \in X : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_X f \, d\mu;$$

(vii) *if* $\int_X f \, d\mu < \infty$, *then* f *is finite for* μ*-almost every* x; finiteness

(viii) *if* $\int_X f \, d\mu = 0$, *then* f *is zero for* μ*-almost every* x; vanishing

(ix)

$$\lim_{n \to \infty} \int_X \min(f, n) \, d\mu = \int_X f \, d\mu;$$

vertical truncation

(x) *if* $S_1 \subset S_2 \subset \ldots$ *is an increasing sequence of* $\mathcal{B}$*-measurable sets,*

$$\lim_{n \to \infty} \int_X f \, 1_{S_n} \, d\mu = \int_X f \, 1_{\bigcup_{n=1}^{\infty} S_n} \, d\mu;$$

horizontal truncation

(xi) *If* Y *is a measurable subset of* X, *then* $\int_X f \, 1_Y \, d\mu = \int_Y f|_Y d\mu|_Y$, *where* $f|_Y$ *and* $\mu|_Y$ *denote the restriction of* f *and* μ *to* Y. restriction

Theorem 3.4 (Finite additivity of the unsigned integral): *Let* $(X, \mathcal{B}, \mu)$ *be a measure space and let* $f, g : X \to [0, \infty]$ *be measurable. Then*

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$

Proof: From superadditivity, we only need to establish

$$\int_X (f + g) \, d\mu \leq \int_X f \, d\mu + \int_X g \, d\mu.$$

If $\mu(X) < \infty$ and f and g are bounded, given $\epsilon > 0$, let f_ϵ and f^ϵ be the simple functions obtained by rounding f down and up, respectively, to the nearest integer multiple of ϵ. Then for all $x \in X$,

$$f_\epsilon(x) \leq f(x) \leq f^\epsilon(x)$$

and

$$f^\epsilon(x) - f_\epsilon(x) \le \epsilon.$$

We similarly define g_ϵ and g^ϵ, so that

$$f + g \le f^\epsilon + g^\epsilon \le f_\epsilon + g_\epsilon + 2\epsilon.$$

Hence

$$\begin{aligned}
\int_X (f+g)\,d\mu &\le \operatorname{Simp}\int_X (f_\epsilon + g_\epsilon + 2\epsilon)\,d\mu = \operatorname{Simp}\int_X f_\epsilon\,d\mu + \operatorname{Simp}\int_X g_\epsilon\,d\mu + 2\epsilon\mu(X) \\
&\le \int_X f\,d\mu + \int_X g\,d\mu + 2\epsilon\mu(X).
\end{aligned}$$

The desired result follows on letting $\epsilon \to 0$.

If $\mu(X) < \infty$ but f and g are not necessarily bounded, one can use vertical truncations to reduce the problem to the above case and then take the limit.

If $\mu(X) = \infty$ but $\int_X f\,d\mu$ and $\int_X g\,d\mu$ are both finite, we can conclude from Markov's inequality that $S_n \doteq \{x \in X : f(x) > 1/n\} \cup \{x \in X : g(x) > 1/n\}$ has finite measure for each $n \in \mathbb{N}$. The above finite-measure case establishes that

$$\int_X (f+g)\,1_{S_n}\,d\mu = \int_{S_n} (f+g)\,d\mu \le \int_{S_n} f\,d\mu + \int_{S_n} g\,d\mu = \int_X f\,1_{S_n}\,d\mu + \int_X g\,1_{S_n}\,d\mu.$$

Since S_n are an increasing sequence of measurable sets, we can then apply horizontal truncation:

$$\lim_{n\to\infty}\int_X (f+g)\,1_{S_n}\,d\mu = \int_X (f+g)\,1_{\bigcup_{n=1}^\infty S_n}\,d\mu = \int_X (f+g)\,d\mu,$$

noting that $\bigcup_{n=1}^\infty S_n$ is the support of $f+g$. The desired inequality then follows from another application of horizontal truncation.

Otherwise, if $\mu(X) = \infty$ and one of $\int_X f\,d\mu$ or $\int_X g\,d\mu$ is infinite, then by superadditivity so is $\int_X (f+g)\,d\mu$, from which the desired result follows.

Problem 3.10: (*Linearity in* μ) Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f : X \to [0,\infty]$ be measurable. Show that

(i) $\int_X f\,d(c\mu) = c\int_X f\,d\mu$ for every $c \in [0,\infty]$.

(ii) if $\mu_1, \mu_2, \ldots$ is a sequence of measures on $\mathcal{B}$,

$$\int_X f\,d\left(\sum_{k=1}^\infty \mu_k\right) = \sum_{k=1}^\infty \int_X f\,d\mu_k$$

Problem 3.11 (Sums as integrals): Let X be an arbitrary set, with the discrete σ-algebra, and $\#$ be the counting measure. Show that every unsigned function $f : X \to [0, \infty]$ is measurable, with

$$\int_X f \, d\# = \sum_{x \in X} f(x).$$

Definition: Let $(X, \mathcal{B}, \mu)$ be a measure space. A measurable function $f : X \to \mathbb{C}$ is said to be *absolutely integrable* if $|f|_{L^1(X,\mathcal{B},\mu)} \doteq \int_X |f| \, d\mu < \infty$.

Definition: The space of absolutely integrable functions on $(X, \mathcal{B}, \mu)$ is denoted by $L^1(X, \mathcal{B}, \mu)$ or simply $L^1(\mu)$.

Definition: If f is real-valued and absolutely integrable, we define

$$\int_X f \, d\mu \doteq \int_X f_+ \, d\mu - \int_X f_- \, d\mu,$$

where $f_+ \doteq \max(f, 0)$ and $f_- \doteq \max(-f, 0)$.

Definition: If f is complex-valued and absolutely integrable, we define

$$\int_X f \, d\mu \doteq \int_X \operatorname{Re} f \, d\mu + i \int_X \operatorname{Im} f \, d\mu.$$

Theorem 3.5: *Let $(X, \mathcal{B}, \mu)$ be a measure space and $f, g \in L^1(X, \mathcal{B}, \mu)$. Then*

(i) *$L^1(X, \mathcal{B}, \mu)$ is a complex vector space;*

(ii) *the map $f \mapsto \int_X f \, d\mu$ is a complex-linear map from $L^1(X, \mathcal{B}, \mu)$ to $\mathbb{C}$;*

(iii) $|f + g|_{L^1(\mu)} \leq |f|_{L^1(\mu)} + |g|_{L^1(\mu)}$

(iv) *$|cf|_{L^1(\mu)} = |c| |f|_{L^1(\mu)}$ for every $c \in \mathbb{C}$;*

(v) *if $f = g$ μ-almost everywhere in X, we have $\int_X f \, d\mu = \int_X g \, d\mu$;*

(vi) *if $(X, \bar{\mathcal{B}}, \bar{\mu})$ is a refinement of $(X, \mathcal{B}, \mu)$ then $f \in L^1(X, \bar{\mathcal{B}}, \bar{\mu})$ and $\int_X f \, d\bar{\mu} = \int_X f \, d\mu$;*

(vii) *$|f|_{L^1(\mu)} = 0$ iff f is zero μ-almost everywhere;*

(viii) *if $Y \subset X$ is $\mathcal{B}$-measurable, then $f_Y \in L^1(Y, \mathcal{B}|_Y, \mu|_Y)$ and $\int_X f \, 1_Y \, d\mu = \int_Y f|_Y d\mu|_Y$.*

Q. Under what conditions can we interchange integrals and limits? That is, given a sequence of measurable functions f_n that converges pointwise μ-almost everywhere to a function f, under what conditions does

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu?$$

A. There are a number of possible conditions, which we will discuss one at a time.

Theorem 3.6 (Uniform convergence on finite spaces): *Suppose that* $(X, \mathcal{B}, \mu)$ *is a finite measure space* ($\mu(X) < \infty$) *and* $f_n : X \to [0, \infty]$ *(or* $\mathbb{C}$*) is a sequence of measurable (or absolutely integrable) functions that converges uniformly to a limit* f*. Then* $\int_X f_n \, d\mu$ *converges to* $\int_X f \, d\mu$*.*

Remark: If we relax the finite measure or uniformity conditions, it is easy to construct examples in which the interchange of limit processes is invalid:

- In $(\mathbb{R}, \mathcal{L}[\mathbb{R}], m)$, consider that $f_n \doteq 1_{[n,n+1]}$ converges pointwise to 0 but $\int_{\mathbb{R}} f_n = 1$ does not converge to $\int_{\mathbb{R}} f = 0$. We say that the "mass" of the functions f_n "escapes to horizontal infinity."

- In $(\mathbb{R}, \mathcal{L}[\mathbb{R}], m)$, consider that $f_n \doteq \frac{1}{n} 1_{[0,n]}$ converges **uniformly** to 0 but $\int_{\mathbb{R}} f_n = 1$ does not converge to $\int_{\mathbb{R}} f = 0$. We say that the mass of the functions f_n "escapes to width infinity."

- In $(\mathbb{R}, \mathcal{L}[\mathbb{R}], m)$, consider that $f_n \doteq n 1_{[\frac{1}{n}, \frac{2}{n}]}$ converges pointwise to 0 but $\int_{\mathbb{R}} f_n = 1$ does not converge to $\int_{\mathbb{R}} f = 0$. We say that the mass of the functions f_n "escapes to vertical infinity."

Remark: One way to prevent these three avenues of escape to infinity is to enforce monotonicity; this prevents each function f_n from "abandoning" the location where the mass of its predecessors was concentrated.

Theorem 3.7 (Monotone convergence theorem): *Let* $(X, \mathcal{B}, \mu)$ *be a measure space and* $f_1 \leq f_2 \leq \dots$ *be an increasing sequence of unsigned measurable functions on* X*. Then*

$$\int_X \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Proof: Let $f \doteq \lim_{n\to\infty} f_n$, which by Problem 3.8 (vi) is measurable. Let $g \leq f$ be any simple unsigned function. By applying vertical truncation, we can assume that g is finite everywhere, so that $g = \sum_{k=1}^m c_k 1_{A_k}$ for some numbers $c_1, \ldots, c_m \in [0, \infty)$ and disjoint $\mathcal{B}$-measurable sets $A_1, \ldots, A_m$. Thus

$$\int_X g \, d\mu = \sum_{k=1}^m c_k \mu(A_k).$$

For each $x \in A_k$ we know

$$\sup_n f_n(x) = f(x) \geq g(x) = c_k.$$

Let $\epsilon \in (0,1)$. Since the sets

$$A_{k,n} \doteq \{x \in A_k : f_n(x) > (1-\epsilon)c_k\}$$

increase in n to A_k and are measurable, we know from upwards monotonicity that

$$\lim_{n\to\infty} \mu(A_{k,n}) = \mu(A_k).$$

Moreover, on integrating the inequality

$$f_n > (1-\epsilon) \sum_{k=1}^m c_k 1_{A_{k,n}},$$

we find

$$\int_X f_n \, d\mu \geq (1-\epsilon) \sum_{k=1}^m c_k \mu(A_{k,n}),$$

On taking the limit as $n \to \infty$, we find

$$\lim_{n\to\infty} \int_X f_n \, d\mu \geq (1-\epsilon) \sum_{k=1}^m c_k \mu(A_k) = (1-\epsilon) \int_X g \, d\mu,$$

Since $\epsilon \in (0,1)$ is arbitrary, it follows that

$$\lim_{n\to\infty} \int_X f_n \, d\mu \geq \int_X g \, d\mu.$$

On taking the supremum over all simple functions $g \leq f$, we find

$$\lim_{n\to\infty} \int_X f_n \, d\mu \geq \int_X f \, d\mu.$$

Since the reverse inequality holds by monotonicity:

$$\lim_{n\to\infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu,$$

we arrive at the desired result.

Corollary 3.7.1 (Tonelli's theorem for sums and integrals): Let $(X, \mathcal{B}, \mu)$ be a measure space and $f_1, f_2, \ldots$ be a sequence of unsigned measurable functions on X. Then

$$\int_X \sum_{k=1}^{\infty} f_k \, d\mu = \sum_{k=1}^{\infty} \int_X f_k \, d\mu.$$

Proof: Apply Theorem 3.7 to the partial sums $\sum_{k=1}^{n} f_k$.

Corollary 3.7.2 (Borel–Cantelli lemma): Let $(X, \mathcal{B}, \mu)$ be a measure space and $S_1, S_2, \ldots$ be a sequence of $\mathcal{B}$-measurable sets such that $\sum_{k=1}^{\infty} \mu(S_k) < \infty$. Then for μ-almost every $x \in X$ the set $\{k \in \mathbb{N} : x \in S_k\}$ is finite.

Proof: Apply Tonelli's theorem to 1_{S_k}.

Remark: When one does not have monotonicity, *Fatou's lemma* at least provides an inequality:

Corollary 3.7.3 (Fatou's lemma): Let $(X, \mathcal{B}, \mu)$ be a measure space and $f_1, f_2, \ldots$ be a sequence of unsigned measurable functions on X. Then

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Proof: By definition,

$$\liminf_{n\to\infty} f_n = \lim_{n\to\infty} F_n,$$

where $F_n = \inf\limits_{k \ge n} f_k$. Monotonicity implies that $\int_X F_n \, d\mu \le \int_X f_k \, d\mu$ for all $k \ge n$; hence,

$$\int_X F_n \, d\mu \le \inf_{k \ge n} \int_X f_k \, d\mu.$$

Since $\{F_n\}_{n=1}^{\infty}$ is an increasing sequence of measurable functions, we know by Theorem 3.7 that

$$\int_X \lim_{n\to\infty} F_n \, d\mu = \lim_{n\to\infty} \int_X F_n \, d\mu \le \lim_{n\to\infty} \inf_{k \ge n} \int_X f_k \, d\mu = \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Remark: Fatou's lemma tells us that while the "mass" $\int_X f_n \, d\mu$ can be destroyed in taking pointwise limits, as we saw in the three escapes to infinity, it cannot be created.

Definition: Let $(X, \mathcal{B}, \mu)$ be a measure space and $(Y, \mathcal{C})$ be a measurable space. A *measurable morphism* is a function $\phi : X \to Y$ such that $\phi^{-1}(S)$ is $\mathcal{B}$-measurable for every $\mathcal{C}$-measurable set S.

Remark: Let $(X, \mathcal{B}, \mu)$ be a measure space, $(Y, \mathcal{C})$ be a measurable space, and ϕ be a measurable morphism from X to Y. The *pushforward* $\phi_*\mu(S) \doteq \mu(\phi^{-1}(S))$ is a measure on $\mathcal{C}$, so that $(Y, \mathcal{C}, \phi_*\mu)$ is a measure space.

- If $T : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear transformation, then $T_* m = \frac{1}{|\det T|} m$.

Corollary 3.7.4 (Change of variables): Let $(X, \mathcal{B}, \mu)$ be a measure space, and ϕ be a measurable morphism from X to Y. If $f : Y \to [0, \infty]$ is measurable, then

$$\int_Y f \, d\phi_*\mu = \int_X f \circ \phi \, d\mu.$$

Remark: Another important way to avoid loss of mass in taking pointwise limits is to dominate all of the functions by an absolutely integrable one.

Theorem 3.8 (Dominated convergence theorem): *Let $(X, \mathcal{B}, \mu)$ be a measure space and $f_1, f_2, \ldots$ be a sequence of complex-valued measurable functions on X that converge pointwise μ-almost everywhere on X. Suppose that there exists an unsigned absolutely integrable function $G : X \to [0, \infty]$ such that for each $n \in \mathbb{N}$, $|f_n| \leq G$ μ-almost everwhere. Then*

$$\int_X \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Proof: Without loss of generality, by considering real and imaginary parts, we may assume that the functions f_n are real and modify them on null sets so that μ-almost "everywhere" becomes "everywhere". Then $-G \leq f_n \leq G$. Let $f \doteq \lim_{n\to\infty} f_n$.

On applying Fatou's lemma to $f_n + G$ and $G - f_n$, we find

$$\int_X (f + G) \, d\mu \leq \liminf_{n\to\infty} \int_X (f_n + G) \, d\mu$$

and

$$\int_X (G - f) \, d\mu \leq \liminf_{n\to\infty} \int_X (G - f_n) \, d\mu.$$

Since G is absolutely integrable, we may subtract the finite quantity $\int_X G \, d\mu$ from both sides of these equations to obtain

$$\limsup_{n\to\infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

The desired result then follows from the fact that $\liminf_{n\to\infty} \int_X f_n \, d\mu \leq \limsup_{n\to\infty} \int_X f_n \, d\mu$.

Chapter 4

Modes of Convergence

Definition: A sequence of functions $f_n : X \to \mathbb{C}$ converges *pointwise almost everywhere* to $f : X \to C$ if for every $\epsilon > 0$ and almost every $x \in X$ there exists $N = N(\epsilon, x)$ such that $|f_n(x) - f(x)|< \epsilon$ whenever $n \geq N$.

Definition: A sequence of functions $f_n : X \to \mathbb{C}$ converges *uniformly almost everywhere* (or in L^∞ *norm*) to $f : X \to C$ if for every $\epsilon > 0$ there exists $N = N(\epsilon)$ such that $|f_n(x) - f(x)|< \epsilon$ for almost every $x \in X$ whenever $n \geq N$.

Definition: A sequence of functions $f_n : X \to \mathbb{C}$ converges *almost uniformly* to $f : X \to C$ if for every $\epsilon > 0$ there exists an exceptional set E_ϵ of measure less than ϵ such that f_n converges uniformly to f on $X \setminus E_\epsilon$.

Definition: A sequence of functions $f_n : X \to \mathbb{C}$ converges *in the* L^1 *norm* if $|f_n - f|_{L^1}$ converges to 0 as $n \to \infty$.

Definition: A sequence of functions $f_n : X \to \mathbb{C}$ converges *in measure* μ if for every $\epsilon > 0$, the measures $\mu(\{x \in X : |f_n(x) - f(x)|\geq \epsilon\})$ converge to 0 as $n \to \infty$.

- In $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$, we see that $f_n \doteq 1_{[n,n+1]}$ converges pointwise to 0 but not uniformly, in the L^∞ or L^1 norms, almost uniformly, or in measure.

- In $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$, we see that $f_n \doteq \frac{1}{n} 1_{[0,n]}$ converges uniformly to 0 (and hence pointwise, in the L^∞ norm, almost uniformly, and in measure) but not in the L^1 norm.

- In $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$, we see that $f_n \doteq n 1_{[\frac{1}{n}, \frac{2}{n}]}$ converges to 0 pointwise and almost uniformly (and hence in measure), but not uniformly or in the L^∞ or L^1 norms.

- In $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$, the *typewriter* sequence

  $$f_n = 1_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]} \qquad \text{for } k \geq 0 \text{ and } n \in [2^k, 2^{k+1}-1]$$

  converges to zero in measure and in the L^1 norm, but not pointwise almost everywhere, almost uniformly, or in the L^∞ norm.

Remark: The L^∞ norm $|f|_{L^\infty}$ of a measurable function is the infimum of all $M \in [0, \infty]$ such that $|f| \leq M$ for almost all x.

Remark: The five modes of convergence are all compatible in the sense that, outside of a set of measure zero, they never disagree about which function a sequence of functions converges to.

Remark: If a sequence of absolutely integrable functions f_n converges to f in the L^1 norm, the triangle inequality implies that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Definition: We say that a sequence $f_n : X \to \mathbb{C}$ is *dominated* if there exists an absolutely integrable function $g : X \to [0, \infty]$ such that $|f_n(x)| \leq g(x)$ for all n and almost every $x \in X$.

Problem 4.1: If a dominated sequence of measurable functions $f_n : X \to \mathbb{C}$ converges pointwise almost everywhere, use the dominated convergence theorem to show that it converges in the L^1 norm.

Definition: A sequence $f_n : X \to \mathbb{C}$ of absolutely integrable functions is said to be *uniformly integrable* if the following three statements all hold:

(i) Uniform bound on L^1 norm: $\sup_n |f_n|_{L^1} < \infty$.

(ii) No escape to vertical infinity: $\sup_n \int_{|f_n| \geq M} |f_n| \, d\mu \to 0$ as $M \to \infty$.

(iii) No escape to width infinity: $\sup_n \int_{|f_n| \leq \delta} |f_n| \, d\mu \to 0$ as $\delta \to 0$.

Remark: Given an absolutely integrable function f, we can apply the monotone convergence theorem to $|f| 1_{B_{1/n}[0]}$ and $|f| 1_{B_n[0]}$ for $n = 1, 2, \ldots$ to conclude that the constant sequence of functions f is uniformly integrable.

Theorem 4.1 (Uniformly integrable convergence in measure): *Let $f_n : X \to \mathbb{C}$ be a uniformly integrable sequence of functions, and let $f : X \to \mathbb{C}$ be another function. Then f_n converges in L^1 norm iff f_n converges to f in measure.*

Proof: See Tao, Theorem 1.5.13.

Chapter 5

Differentiation Theorems

Definition: Let $[a, b]$ be a compact interval of positive length. A function $F : [a, b] \to \mathbb{R}$ is *almost-everywhere differentiable* if the limit

$$F'(x) \doteq \lim_{\substack{y \to x \\ y \in [a,b] \setminus \{x\}}} \frac{F(y) - F(x)}{y - x}$$

exists for almost all $x \in [a, b]$.

Problem 5.1: Provide an example that illustrates that an almost-everywhere differentiable function need not be continuous everywhere.

Problem 5.2: If $F : [a, b] \to \mathbb{R}$ is almost-everywhere differentiable, show that F is continuous almost everywhere.

Problem 5.3: If $F : [a, b] \to \mathbb{R}$ is almost-everywhere differentiable, show that (the almost-everywhere defined derivative) F' is measurable.
Hint: $F'(x) = \lim\limits_{n \to \infty} n(F(x + 1/n) - F(x))$.

Definition: If F is differentiable and its *derivative* F' is continuous, we say that F is *continuously differentiable*.

Problem 5.4: Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function. Show that the indefinite integral $F(x) \doteq \int_{[-\infty,x]} f(t)\, dt$ is a continuous function on $\mathbb{R}$.

Remark: In order to extend the fundamental theorem of calculus to the Lebesgue integral, we first need to establish some preliminary results.

Lemma 5.1 (Rising sun lemma): *Let F be a real-valued continuous function on $[a, b]$ and $S = \{x \in [a, b] : F(x) < F(y) \text{ for some } y \in (x, b]\}$. Define $U = S \cap (a, b)$. Then U is open and may be written as a countable union of disjoint intervals $U = \bigcup_k (a_k, b_k)$ such that $F(a_k) = F(b_k)$, unless $a_k = a \in S$ for some k, in which case $F(a) < F(b_k)$ for that one k. Furthermore, if $x \in (a_k, b_k)$, then $F(x) < F(b_k)$.*

Remark: Imagine the graph of the function F as a hilly landscape, with the sun shining horizontally from the right (rising from the east). The set U consist of those points that are in shadow.

Proof:

Claim: If $[c,d) \subset S$, with $d \notin S$, then $F(c) < F(d)$. Otherwise, suppose $F(c) \geq F(d)$. Then F achieves its maximum on $[c,d]$ at some point $x < d$. Since $x \in S$, we know that $F(x) < F(y)$ for some $y \in (x,b]$. But $F(x) < F(y)$ implies that $y \notin [c,d]$. Hence $y \in (d,b]$ and $F(d) \leq F(x) < F(y)$, contradicting $d \notin S$. The claim thus holds.

Since F is continuous, U is open and can be expressed as a countable union of disjoint intervals (a_k, b_k).

Since each $b_k \notin S$, the claim establishes that $x \in (a_k, b_k) \Rightarrow F(x) < F(b_k)$. Since F is continuous, $F(a_k) \leq F(b_k)$.

If some $a_k = a \in S$, the claim tells us that $F(a) < F(b_k)$.

Otherwise if $a \notin S$, then $a_k \notin S$ and hence $F(a_k) \geq F(b_k)$ for all $k \in \mathbb{N}$. Thus, $F(a_k) = F(b_k)$.

Lemma 5.2 (One-sided Hardy-Littlewood maximal inequality): *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function and let $\lambda > 0$. Then*

$$m\left(\left\{x \in \mathbb{R} : \sup_{h>0} \frac{1}{h} \int_{[x,x+h]} |f(t)|\, dt \geq \lambda\right\}\right) \leq \frac{1}{\lambda} \int_{\mathbb{R}} |f(t)|\, dt.$$

Proof: Let $[a,b]$ be any compact interval and define

$$S_\lambda \doteq \left\{ x \in [a,b] : \sup_{\substack{h>0 \\ [x,x+h] \subset [a,b]}} \frac{1}{h} \int_{[x,x+h]} |f(t)|\, dt > \lambda \right\}.$$

We first establish for any $\epsilon \in (0,\lambda)$ that

$$m(S_{\lambda-\epsilon}) \leq \frac{1}{\lambda - \epsilon} \int_{\mathbb{R}} |f(t)|\, dt.$$

For $x \in [a,b]$, consider the continuous function

$$F(x) = \int_{[a,x]} |f(t)|\ dt - x(\lambda - \epsilon)$$

and note that the inequality $\frac{1}{h} \int_{[x,x+h]} |f(t)|\, dt > \lambda - \epsilon$ reduces to $F(x+h) > F(x)$.

On applying the rising sun lemma to F, we see that there exists a countable sequence of disjoint intervals $\{(a_k, b_k)\}_{k=1}^{\infty}$ such that

$$S_{\lambda-\epsilon} \subset \bigcup_{k=1}^{\infty} (a_k, b_k) \cup \{a\} \cup \{b\}.$$

By countable additivity and monotonicity, we then find that

$$m(S_{\lambda-\epsilon}) \le \sum_{k=1}^{\infty}(b_k - a_k).$$

The rising sun lemma also tells us that

$$0 \le F(b_k) - F(a_k) = \int_{[a_k,b_k]} |f(t)|\ dt - (b_k - a_k)(\lambda - \epsilon),$$

so

$$m(S_{\lambda-\epsilon}) \le \sum_{k=1}^{\infty}(b_k - a_k) \le \frac{1}{\lambda-\epsilon}\sum_{k=1}^{\infty}\int_{[a_k,b_k]} |f(t)|\ dt \le \frac{1}{\lambda-\epsilon}\int_{[a,b]} |f(t)|\ dt,$$

where we have exploited additivity and monotonicity.

Finally, since

$$\left\{ x \in [a,b] : \sup_{\substack{h>0 \\ [x,x+h]\subset[a,b]}} \frac{1}{h}\int_{[x,x+h]} |f(t)|\, dt \ge \lambda \right\} \subset S_{\lambda-\epsilon},$$

the desired result follows on letting $\epsilon \to 0$ and then applying upward monotonicity.

Theorem 5.1 (Lebesgue differentiation theorem on $\mathbb{R}$): *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function. Then*

$$\lim_{h\to 0^+} \frac{1}{h}\int_{[x,x+h]} f(t)\, dt = f(x)$$

and

$$\lim_{h\to 0^+} \frac{1}{h}\int_{[x-h,x]} f(t)\, dt = f(x)$$

for almost every $x \in \mathbb{R}$.

Proof: Let $\epsilon > 0$. By Littlewood's second principle, there exists a continuous, compactly supported function $g : \mathbb{R} \to \mathbb{C}$ such that

$$\int_{\mathbb{R}} |f(t) - g(t)|\, dt \le \epsilon.$$

Since g is continuous, we know from the fundamental theorem of calculus that

$$\lim_{h\to 0^+} \frac{1}{h}\int_{[x,x+h]} g(t)\, dt = g(x).$$

Let $n \in \mathbb{N}$. For sufficiently small $h > 0$,

$$\left| \frac{1}{h} \int_{[x,x+h]} g(t)\,dt - g(x) \right| < \frac{1}{n}.$$

From the Hardy-Littlewood maximal inequality we know that

$$m\left(\left\{ x \in \mathbb{R} : \sup_{h>0} \frac{1}{h} \int_{[x,x+h]} |f(t) - g(t)|\,dt \geq \frac{1}{n} \right\} \right) \leq n\epsilon.$$

Likewise, Markov's inequality implies that

$$m\left(\left\{ x \in \mathbb{R} : |f(x) - g(x)| \geq \frac{1}{n} \right\} \right) \leq n\epsilon.$$

By subadditivity, the measure of the union of the sets on the left-hand sides of the above two inequalities is at most $2n\epsilon$. For x outside of this union and all $h > 0$,

$$\frac{1}{h} \int_{[x,x+h]} |f(t) - g(t)|\,dt < \frac{1}{n} \qquad \text{and} \qquad |f(x) - g(x)| < \frac{1}{n}.$$

For such x, we then deduce from the triangle inequality that

$$\limsup_{h \to 0^+} \left| \frac{1}{h} \int_{[x,x+h]} f(t)\,dt - f(x) \right| < \frac{3}{n}.$$

On taking the limits as $\epsilon \to 0$ and then $n \to \infty$, we see for almost all real x that

$$\limsup_{h \to 0^+} \left| \frac{1}{h} \int_{[x,x+h]} f(t)\,dt - f(x) \right| = 0.$$

Since the corresponding limit inferior is non-negative, the first statement in the theorem then follows. The second statement follows on applying the first statement to the reflected function $x \mapsto f(-x)$.

Corollary 5.1.1: Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function, and let $F : \mathbb{R} \to \mathbb{C}$ be the indefinite integral $F(x) \doteq \int_{[-\infty,x]} f(t)\,dt$. Then F is continuous and almost everywhere differentiable, with $F'(x) = f(x)$ for almost every $x \in \mathbb{R}$.

Remark: The Lebesgue differentiation theorem has an analogue in higher dimensions.

Definition: Let $f : \mathbb{R}^d \to \mathbb{C}$ be an absolutely integrable function. A point x where

$$\lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0$$

is known as a *Lebesgue point*.

Theorem 5.2 (Lebesgue differentiation theorem on $\mathbb{R}^d$): *Let $f : \mathbb{R}^d \to \mathbb{C}$ be an absolutely integrable function. Then almost every $x \in \mathbb{R}^d$ is a Lebesgue point for f.*

Proof: See Tao, Theorem 1.6.19.

Corollary 5.2.1: Let $f : \mathbb{R}^d \to \mathbb{C}$ be an absolutely integrable function. Then

$$\lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) \, dy = f(x)$$

for almost every $x \in \mathbb{R}^d$.

Proof: Apply the triangle inequality to Theorem 5.2.

Theorem 5.3 (Monotone differentiation theorem): *Every monotone function $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere.*

Proof: See Tao, Theorem 1.6.25.

Problem 5.5 (Cantor function): Define the functions $F_0, F_1, F_2, \ldots : [0,1] \to \mathbb{R}$ recursively: let $F_0(x) \doteq x$ for $x \in [0,1]$ and for $n \in \mathbb{N}$ define

$$F_n(x) \doteq \begin{cases} \frac{1}{2}F_{n-1}(3x) & \text{if } x \in \left[0, \frac{1}{3}\right]; \\ \frac{1}{2} & \text{if } x \in \left(\frac{1}{3}, \frac{2}{3}\right); \\ \frac{1}{2} + \frac{1}{2}F_{n-1}(3x-2) & \text{if } x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

(i) Graph F_0, F_1, F_2, and F_3 on a single graph.

(ii) Using induction, show for each $n = 0, 1, \ldots$ that F_n is a continuous monotone increasing function with $F_n(0) = 0$ and $F_n(1) = 1$.

(iii) Show for each $n = 0, 1, \ldots$ and $x \in [0,1]$ that $|F_{n+1}(x) - F_n(x)| \leq 2^{-n}$. Conclude that $\{F_n\}_{n=1}^{\infty}$ converges uniformly to a limit $F : [0;1] \to \mathbb{R}$. The limit $F(x)$, known as the *Cantor function*, expresses the fraction of the "mass" of the Cantor set in $[0, x]$.

(iv) Show that the Cantor function F is continuous and monotone increasing, with $F(0) = 0$ and $F(1) = 1$.

(v) Show that if $x \in [0,1]$ lies outside the Cantor set $\mathcal{C}$, then F is constant in a neighbourhood of x, so that $F'(x) = 0$. Conclude that $\int_{[0,1]} F'(x)\,dx = 0 \neq 1 = F(1) - F(0)$ and hence the fundamental theorem of calculus fails.

(vi) Show that $F\left(\sum_{k=1}^{\infty} a_k 3^{-k}\right) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k}$ for any digits $a_1, a_2, \ldots \in \{0,2\}$.

(vii) Let $I_n = [\sum_{k=1}^{n} a_k 3^{-k}, 3^{-n} + \sum_{k=1}^{n} a_k 3^{-k}]$ for $n \geq 0$ and $a_1, \ldots, a_n \in \{0,2\}$. Show that I_n is an interval of length 3^{-n}, but $F(I_n)$ is an interval of length 2^{-n}.

(viii) Show that F is not differentiable at any element of the Cantor set $\mathcal{C}$.

Definition: The *total variation* of a function $F : \mathbb{R} \to \mathbb{R}$ on an (finite or infinite) interval I is

$$|F|_{\mathrm{TV(I)}} \doteq \sup_{\substack{x_0 < \ldots < x_n \\ x_0, \ldots, x_n \in I}} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|.$$

If $|F|_{\mathrm{TV(I)}}$ is finite, we say that F has *bounded variation on* I. If F has bounded variation on $\mathbb{R}$, we say that F has *bounded variation*.

Problem 5.6: If $F : \mathbb{R} \to \mathbb{R}$ is a monotone function, show that $|F|_{\mathrm{TV([a,b])}} = |F(b) - F(a)|$ for any interval $[a,b]$. Conclude that F has bounded variation on $\mathbb{R}$ iff it is bounded.

Problem 5.7: For any functions $F, G : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$, show that $|F + G|_{\mathrm{TV}(\mathbb{R})} \leq |F|_{\mathrm{TV}(\mathbb{R})} + |G|_{\mathrm{TV}(\mathbb{R})}$ and $|cF|_{\mathrm{TV}(\mathbb{R})} = |c| |F|_{\mathrm{TV}(\mathbb{R})}$.

Problem 5.8: If $F : \mathbb{R} \to \mathbb{R}$ is a function, show that $|F|_{\mathrm{TV([a,b])}} + |F|_{\mathrm{TV([b,c])}} = |F|_{\mathrm{TV([a,c])}}$ whenever $a \leq b \leq c$.

Theorem 5.4: *A function $F : \mathbb{R} \to \mathbb{R}$ has bounded variation iff it is the difference of two bounded monotone functions.*

Proof: "$\Leftarrow$" This follows from Prob 5.6 and 5.7.
"$\Rightarrow$" Define the *positive variation* of F to be the bounded increasing function

$$F^+(x) \doteq \sup_{x_0 < \ldots < x_n \leq x} \sum_{i=1}^{n} \max(F(x_i) - F(x_{i-1}), 0).$$

We claim that $F^+ - F$ is monotone increasing: for $b \geq a$,

$$F^+(b) \geq F^+(a) + F(b) - F(a)$$

If $F(b) - F(a) < 0$, this follows from the monotonicity of F^+. Otherwise, if $F(b) - F(a) \geq 0$, one can include a and b in any sequence $x_0 < \ldots < x_n \leq a$, which will increase

$$\sup_{x_0 < \ldots < x_n} \sum_{i=1}^{n} \max(F(x_i) - F(x_{i-1}), 0)$$

by at least $F(b) - F(a)$, thereby establishing the claim. The result then follows from the observation that $F = F_+ - (F_+ - F)$.

Corollary 5.4.1 (BV differentiation theorem): A function of bounded variation is differentiable almost everywhere.

Problem 5.9:

(i) Show that every function $F : \mathbb{R} \to \mathbb{R}$ of bounded variation is bounded and that $\lim\limits_{x\to\infty} F(x)$ and $\lim\limits_{x\to-\infty} F(x)$ exist.

(ii) Provide a counterexample of a bounded, continuous function F with bounded support that does not have bounded variation.

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is *locally of bounded variation* if it has bounded variation on every compact interval of $\mathbb{R}$.

Remark: A function that is locally of bounded variation is differentiable almost everywhere.

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz continuous* if there exists a positive constant C such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$. The smallest C with this property is known as the *Lipschitz constant* of f.

Theorem 5.5 (1D Lipschitz differentiation theorem): *Every Lipschitz continuous function is locally of bounded variation, and hence differentiable almost everywhere. Furthermore, its derivative, when it exists, is bounded by its Lipschitz constant.*

Problem 5.10: Show that every convex function $f : \mathbb{R} \to \mathbb{R}$ is continuous and almost everywhere differentiable, with derivative almost everywhere equal to an increasing function.

Recall that the convexity condition can be re-expressed in terms of the slope of a secant:

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x} \qquad \forall x \in (a, b), \quad \forall a \neq b \in \mathbb{R}.$$

Applying this criterion repeatedly, we see for all real numbers $A < a < x < y < b < B$ that

$$\frac{f(a)-f(A)}{a-A} \le \frac{f(y)-f(x)}{y-x} \le \frac{f(B)-f(b)}{B-b}.$$

On letting

$$C = \max\left(\left|\frac{f(a)-f(A)}{a-A}\right|, \left|\frac{f(B)-f(b)}{B-b}\right|\right),$$

we thus see that f is locally Lipschitz, and therefore continuous, on every compact interval $[a,b] \subset \mathbb{R}$:

$$|f(x)-f(y)| \le C|x-y| \qquad \forall x,y \in [a,b].$$

Moreover, f is locally of bounded variation and hence differentiable almost everywhere.

Let

$$m(x) = \frac{f(x)-f(a)}{x-a} \quad (x \neq a), \qquad M(x) = \frac{f(b)-f(x)}{b-x} \quad (x \neq b).$$

From convexity, we know that

$$m(x) \le m(b) = M(a) \le M(x)$$

whenever $a < x < b$. At points a and b where f is differentiable we then see that

$$f'(a) = \lim_{x\to a} m(x) \le m(b) = M(a) \le \lim_{x\to b} M(x) = f'(b).$$

Theorem 5.6 (Upper bound for fundamental theorem): *Let $F : [a,b] \to \mathbb{R}$ be increasing, so that the unsigned function $F' : [a,b] \to [0,\infty]$ exists almost everywhere and is measurable. Then*

$$\int_{[a,b]} F' \le F(b) - F(a).$$

Proof: Extend F to $\mathbb{R}$ by defining $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$, so that F is a bounded monotone function on $\mathbb{R}$ such that F' vanishes outside of $[a,b]$. As F is almost everywhere differentiable, the sequence of functions

$$f_n(x) \doteq \frac{F(x+1/n)-F(x)}{1/n}$$

converges pointwise almost everywhere to F'. Apply Fatou's lemma to conclude that

$$\begin{aligned}
\int_{[a,b]} F'(x)\,dx &\le \liminf_{n\to\infty} \int_{[a,b]} f_n\,dx = \liminf_{n\to\infty} n \int_{[a,b]} \left[F\left(x+\frac{1}{n}\right) - F(x)\right] dx \\
&= \liminf_{n\to\infty} n \left[\int_{[a+1/n,b+1/n]} F(x)\,dx - \int_{[a,b]} F(x)\,dx\right] \\
&= \liminf_{n\to\infty} n \left[\int_{[b,b+1/n]} F(x)\,dx - \int_{[a,a+1/n]} F(x)\,dx\right] \\
&\le \liminf_{n\to\infty} n \left[F(b) \int_{[b,b+1/n]} dx - F(a) \int_{[a,a+1/n]} dx\right] \\
&= F(b) - F(a).
\end{aligned}$$

Remark: In Theorem 5.6 we note that F' is absolutely integrable. This implies that every function of bounded variation has an almost-everywhere defined derivative that is absolutely integrable.

Problem 5.11: Prove that the product of two Lipschitz continuous functions is itself Lipschitz continuous.

Problem 5.12 (Integration by parts): Let $F, G : [a, b] \to \mathbb{R}$ be Lipschitz continuous functions. Show that

$$\int_{[a,b]} F'G = F(b)G(b) - F(a)G(a) - \int_{[a,b]} FG'$$

Definition: A function $F : \mathbb{R} \to \mathbb{R}$ is said to be *absolutely continuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_{k=1}^{n} |F(b_k) - F(a_k)| < \epsilon$ for every finite collection of disjoint intervals $(a_1, b_1) \ldots (a_n, b_n)$ of total length $\sum_{k=1}^{n} (b_k - a_k) < \delta$.

- The function $x \to \sqrt{x}$ is absolutely continuous, but not Lipschitz continuous, on the interval $[0, 1]$.

- The Cantor function is continuous, monotone, and uniformly continuous, but not absolutely continuous, on $[0, 1]$.

Problem 5.13: Show that

(i) every absolutely continuous function is uniformly continuous (and therefore continuous);

(ii) every absolutely continuous function is of bounded variation (and hence differentiable almost everywhere) on every compact interval $[a, b]$ (hint: first show this is true for every sufficiently small interval);

(iii) every Lipschitz continuous function is absolutely continuous;

(iv) if $f : \mathbb{R} \to \mathbb{R}$ is absolutely integrable, the indefinite integral $F(x) \doteq \int_{[-\infty,x]} f(y)\, dy$ is absolutely continuous and differentiable almost everywhere, with $F'(x) = f(x)$ for almost every x;

(v) the sum and product of two absolutely continuous functions on an interval $[a, b]$ are absolutely continuous (what happens if we replace $[a, b]$ by $\mathbb{R}$)?

Problem 5.14: (*1D Besicovitch covering lemma*)
Let $I_1, \ldots, I_n$ be a finite collection of open intervals in $\mathbb{R}$ (not necessarily disjoint). Show that there exist a subcollection $I'_1, \ldots, I'_m$ of intervals such that $\bigcup_{i=1}^n I_i = \bigcup_{j=1}^m I'_j$ and every point of $\mathbb{R}$ is contained in at most two of the intervals I'_j. Hint: First refine the collection of intervals so that no interval I_i is contained in the union of the other intervals. Then show that it is no longer possible for a point to be contained in three of the intervals.

Problem 5.15: (*Cousin's theorem*)
Given any (so-called *gauge*) function $\delta : [a,b] \to (0,\infty)$ on a compact interval $[a,b]$ of positive length, show that there exists a partition $a = t_0 < \ldots < t_n = b$ of $[a,b]$, where $n \in \mathbb{N}$, and real numbers $t_k^* \in [t_{k-1}, t_k]$ for each $k = 1, \ldots, n$ such that $t_k - t_{k-1} \le \delta(t_k^*)$. Hint: use the Heine–Borel theorem and the Besicovitch covering lemma.

Theorem 5.7 (Fundamental theorem for absolutely continuous functions): *Let $F : [a,b] \to \mathbb{R}$ be absolutely continuous. Then*

$$\int_{[a,b]} F' = F(b) - F(a).$$

Proof: Let $\epsilon > 0$. Since F is absolutely continuous, there exists $N_1 \in \mathbb{N}$ such that $\sum_{k=1}^n |F(b_k) - F(a_k)| < \epsilon$ for every finite collection of disjoint intervals $(a_1, b_1) \ldots (a_n, b_n)$ of total length $\sum_{k=1}^n (b_k - a_k) < 1/N_1$.

We know that the absolutely continuous function F is of bounded variation and therefore has an almost-everywhere defined derivative F' that is absolutely integrable.

Let $E \subset [a,b]$ be the null set consisting of points that are not Lebesgue points of F', together with the endpoints a and b and those points where F is not differentiable. By outer regularity, for each $n \in \mathbb{N}$, there exists an open set U_n containing E with measure $m(U_n) < 1/n$. Consider the decreasing sequence of open sets $V_n = \cap_{k=1}^n U_k$ and note that monotonicity implies $m(V_n) < 1/n$.

Since

$$\int_{[a,b]\setminus V_n} |F'| + \int_{V_n} |F'| = \int_{[a,b]} |F'| < \infty,$$

the monotone convergence theorem implies

$$\lim_{n\to\infty} \int_{V_n} |F'| = 0.$$

In particular, there exists $N_2 \in N$ such that $\int_{V_n} |F'| < \epsilon$ whenever $n \ge N_2$.

Let $N = \max(N_1, N_2)$ and define a gauge function $\delta : [a,b] \to (0,\infty)$:

- if $x \in E$, we choose $\delta(x)$ small enough so that $(x - \delta(x), x + \delta(x))$ is contained in the open set V_N;

- if $x \notin E$, then F is diffentiable at x and x is a Lebesgue point of F', so we can choose $\delta(x)$ small enough such that $|y-x|< \delta(x)$ implies
$$|F(y)-F(x)-(y-x)F'(x)|\leq \epsilon|y-x|$$
and
$$\left|\frac{1}{|I|}\int_I F'(y)\,dy - F'(x)\right| < \epsilon$$
whenever I is an interval of length at most $\delta(x)$ containing x.

Using this gauge, according to Cousin's theorem, we can find a partition $a = t_0 < \ldots < t_n = b$ of $[a,b]$, where $n \in \mathbb{N}$, and real numbers $t_k^* \in [t_{k-1}, t_k]$ for each $k = 1,\ldots,n$ such that $t_k - t_{k-1} \leq \delta(t_k^*)$. For those k such that $t_k^* \in E$, we note from the choice of $\delta(x)$ that the disjoint intervals (t_{k-1}, t_k) are each contained in V_N, with

$$\sum_{k:t_k^*\in E}(t_k - t_{k-1}) \leq m(V_N) < \frac{1}{N} \leq \frac{1}{N_1}.$$

Then
$$\sum_{k:t_k^*\in E}|F(t_k)-F(t_{k-1})| < \epsilon.$$

We can express this statement with the $\mathcal{O}$ notation:

$$\sum_{k:t_k^*\in E}[F(t_k)-F(t_{k-1})] = \mathcal{O}(\epsilon).$$

For those k such that $t_k^* \notin E$, we have

$$F(t_k)-F(t_k^*) = (t_k - t_k^*)F'(t_k^*) + \mathcal{O}(\epsilon(t_k - t_k^*))$$

and
$$F(t_k^*)-F(t_{k-1}) = (t_k^* - t_{k-1})F'(t_k^*) + \mathcal{O}(\epsilon(t_k^* - t_{k-1})),$$
so that
$$F(t_k)-F(t_{k-1}) = (t_k - t_{k-1})F'(t_k^*) + \mathcal{O}(\epsilon(t_k - t_{k-1})).$$
We also know that

$$\int_{[t_{k-1},t_k]} F' = (t_k - t_{k-1})F'(t_k^*) + \mathcal{O}(\epsilon(t_k - t_{k-1})).$$

On combining these two results, we find

$$F(t_k)-F(t_{k-1}) = \int_{[t_{k-1},t_k]} F' + \mathcal{O}(\epsilon(t_k - t_{k-1})).$$

Let $S \subset [a,b]$ be the union of all $[t_{k-1}, t_k]$ such that $t_k^* \notin E$. Then

$$\sum_{k : t_k^* \notin E} [F(t_k) - F(t_{k-1})] = \int_S F' + \mathcal{O}(\epsilon |b-a|).$$

Since S contains $[a,b] \setminus V_N$ and $\int_{V_n} |F'| < \epsilon$, we see that

$$\int_S F' = \int_{[a,b]} F' + \mathcal{O}(\epsilon).$$

Together, these observations give us an expression for the telescoping sum

$$F(b) - F(a) = \sum_{k=1}^{n} [F(t_k) - F(t_{k-1})] = \int_{[a,b]} F' + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon |b-a|),$$

from which the desired result follows.

Chapter 6

Outer Measures, Premeasures, and Product Measures

Definition: Given a set X, an *outer measure* is a map $\mu^* : \mathcal{P}(X) \mapsto [0,\infty]$ such that

(i) $\mu^*(\emptyset) = 0$; *nullity*

(ii) $S \subset T \subset X \Rightarrow \mu^*(S) \le \mu^*(T)$; *monotonicity*

(iii) $\mu^*\left(\bigcup_{k=1}^{\infty} S_k\right) \le \sum_{k=1}^{\infty} \mu^*(S_k)$, where $S_k \subset X$. *countable subadditivity*

- The Lebesgue outer measure is an outer measure.

- The Jordan outer measure is not actually an outer measure and should more properly be called *Jordan outer content*.

Remark: Although outer measures, requiring only countable subadditivity (rather than countable additivity), are weaker than measures, they provide a measure for all subsets of X (rather than for just a σ-algebra of measurable sets).

Remark: The concept of an open set, which was used to define the Lebesgue measure, is not available in an arbitrary space X. For this reason, we restate Lebesgue measurability in a form that can be generalized to abstract measure spaces:

Definition: Let μ^* be an outer measure on a set X. A set $S \subset X$ is said to be *Carathéodory measurable* if the *Carathéodory criterion*

$$\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^c)$$

holds for every set $A \subset X$.

Remark: In Problem 1.18, we showed that a set $S \subset \mathbb{R}^d$ is Carathéodory measurable with respect to Lebesgue outer measure if and only if it is Lebesgue measurable.

Definition: A set S is a *null set* for an outer measure μ^* if $\mu^*(S) = 0$.

Problem 6.1: Suppose S is a null set for an outer measure μ^*. Show that S is Carathéodory measurable with respect to μ^*.

Since $A \subset (A \setminus S) \cup S$, we see that $\mu^*(A) \le \mu^*(A \setminus S) + \mu^*(S) = \mu^*(A \setminus S) \le \mu^*(A)$, noting that $A \setminus S \subset A$. Also, $A \cap S \subset S \Rightarrow 0 \le \mu^*(A \cap S) \le \mu^*(S) = 0$. Hence $\mu^*(A \cap S) + \mu^*(A \cap S^c) = \mu^*(A \setminus S) = \mu^*(A)$.

Remark: If the Lebesgue outer measure m^* were finitely additive, then

$$m^*(A \cap S) + m^*(A \cap S^c) = m^*((A \cap S) \cup (A \cap S^c)) = m^*(A)$$

for every set $A \subset \mathbb{R}^d$, so that every subset S of $\mathbb{R}^d$ would be Lebesgue measurable! But we have already documented the existence of nonmeasurable sets like the Vitali set constructed on page 10. So the Lebesgue outer measure is not finitely additive for arbitrary disjoint sets. In a similar manner, one can construct a nonmeasurable subset of any set with positive Lebesgue measure.

Problem 6.2: Consider the Cantor function F.

(i) Show that $F(\mathcal{C}^c)$ is countable, where $\mathcal{C}^c$ denotes the complement of the Cantor set in $[0, 1]$.

(ii) Show that $m(F(\mathcal{C})) = 1$.

(iii) Show that $G : x \mapsto F(x) + x$ is strictly monotonic and continous on $[0, 1]$, so that it has a continuous inverse G^{-1}.

(iv) Show that $m(G(\mathcal{C})) = 1$.

(v) Let S be a non-Lebesgue measurable subset of $G(\mathcal{C})$. Is $G^{-1}(S)$ Lebesgue measurable?

(vi) Is $G^{-1}(S)$ in part (v) Borel measurable?

Problem 6.3: Let $\mathcal{B}$ be a Boolean algebra on a set X. Show that $\mathcal{B}$ is a σ-algebra iff it is closed under countable **disjoint** unions.

This follows from the fact that for any countable sequence of sets $\{S_k\}_{k=1}^{\infty}$ in $\mathcal{B}$, the disjoint *lacunae* $S_n \setminus \bigcup_{k=1}^{n-1} S_k$ are also in $\mathcal{B}$ and their union equals $\bigcup_{k=1}^{\infty} S_k$.

Theorem 6.1 (Carathéodory lemma): *Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on a set X, let $\mathcal{B}$ be the collection of all subsets of X that are Carathéodory measurable with respect to μ^* and let $\mu : \mathcal{B} \to [0, \infty]$ be the restriction of μ^* to $\mathcal{B}$. Then $\mathcal{B}$ is a σ-algebra and μ is a measure.*

Proof: The empty set lies in $\mathcal{B}$ and so does the complement of every set in $\mathcal{B}$. Let $S, T \in \mathcal{B}$ and $A \subset X$. Decompose A into four disjoint sets:

$$\begin{aligned} A_{00} &= A \setminus (S \cup T), \\ A_{10} &= S \cap (A \setminus T), \\ A_{01} &= T \cap (A \setminus S), \\ A_{11} &= A \cap S \cap T. \end{aligned}$$

The Carathéodory measurability of S establishes that

$$\mu^*(A) = \mu^*(A_{00} \cup A_{10} \cup A_{11} \cup A_{01}) = \mu^*(A_{10} \cup A_{11}) + \mu^*(A_{00} \cup A_{01})$$

and

$$\begin{aligned} \mu^*(A \cap (S \cup T)) &= \mu^*(A \cap (S \cup T) \cap S) + \mu^*(A \cap (S \cup T) \cap S^c) \\ &= \mu^*(A_{10} \cup A_{11}) + \mu^*(A_{01}). \end{aligned}$$

The Carathéodory measurability of T guarantees that

$$\mu^*(A \setminus S) = \mu^*(A_{00} \cup A_{01}) = \mu^*(A_{01}) + \mu^*(A_{00}).$$

It then follows that $S \cup T \in \mathcal{B}$:

$$\mu^*(A) = \mu^*(A_{10} \cup A_{11}) + \mu^*(A_{01}) + \mu^*(A_{00}) = \mu^*(A \cap (S \cup T)) + \mu^*(A \setminus (S \cup T)).$$

This establishes that $\mathcal{B}$ is a Boolean algebra. To show that $\mathcal{B}$ is a σ-algebra, in view of Problem 6.3, we only need to establish that it is closed under countable disjoint unions. Let $S_1, S_2, \ldots$ be a sequence of disjoint sets in $\mathcal{B}$ and let $A \subset X$. Let $U_n \doteq \bigcup_{k=1}^{n} S_k$ for $n \in \mathbb{N} \cup \{\infty\}$. For each $n \in \mathbb{N}$, we have already shown that $U_n \in \mathcal{B}$ and hence

$$\mu^*(A) = \mu^*(A \cap U_n) + \mu^*(A \setminus U_n)$$

and

$$\mu^*(A \cap U_{n+1}) = \mu^*(A \cap U_n) + \mu^*(A \cap S_{n+1}),$$

noting that $U_{n+1} = U_n \cup S_{n+1}$ and using disjointness. By induction and countable subadditivity, we then see that

$$\lim_{n\to\infty} \mu^*(A \cap U_n) = \sum_{k=1}^{\infty} \mu^*(A \cap S_k) \geq \mu^*(A \cap U_\infty). \tag{6.1}$$

Moreover, we know from monotonicity that

$$\mu^*(A \setminus U_n) \geq \mu^*(A \setminus U_\infty)$$

for each $n \in \mathbb{N}$ and hence

$$\lim_{n\to\infty} \mu^*(A \setminus U_n) \geq \mu^*(A \setminus U_\infty).$$

On combining these results, we obtain

$$\mu^*(A) \geq \mu^*(A \cap U_\infty) + \mu^*(A \setminus U_\infty).$$

It follows from subadditivity that $U_\infty \in \mathcal{B}$.

Finally, on setting $A = X$ in Eq. (6.1), we obtain

$$\lim_{n\to\infty} \mu^*(U_n) = \sum_{k=1}^{\infty} \mu^*(S_k).$$

But from monotonicity and subadditivity, we know for each $n \in \mathbb{N}$ that

$$\mu^*(U_n) \leq \mu^*(U_\infty) \leq \sum_{k=1}^{\infty} \mu^*(S_k).$$

On taking the limit as $n \to \infty$, we see that

$$\mu^*(U_\infty) = \sum_{k=1}^{\infty} \mu^*(S_k).$$

That is, the restriction μ of μ^* to $\mathcal{B}$ is in fact a measure.

Q. We have seen that the finitely additive elementary measure can be extended to the countably additive Lebesgue measure. Given a finitely additive measure μ_0 on a Boolean algebra $\mathcal{B}_0$, is it always possible to find a σ-algebra $\mathcal{B}$ refining $\mathcal{B}_0$ and a countably additive measure μ that is the extension of μ_0 to $\mathcal{B}$?

A. An obvious necessary condition is that μ_0 is countably additive on any countable sequence of sets whose union happens to belong to $\mathcal{B}_0$. We will now see that this condition is also sufficient.

Definition: A *premeasure* on a Boolean algebra $\mathcal{B}_0$ is a finitely additive measure $\mu_0 : \mathcal{B}_0 \to [0,\infty]$ such that $\mu_0(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu_0(E_k)$ whenever E_1, E_2, … are disjoint subsets of $\mathcal{B}_0$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{B}_0$.

Theorem 6.2 (Hahn–Kolmogorov): *Every premeasure $\mu_0 : \mathcal{B}_0 \to [0,\infty]$ on a Boolean algebra $\mathcal{B}_0$ in X can be extended to a countably additive measure $\mu : \mathcal{B} \to [0,\infty]$.*

Proof: In analogy with the how elementary measure (specifically the volume of boxes) was used to construct the Lebesgue outer measure, for any subset A of X we define

$$\mu^*(A) = \inf_{\substack{\bigcup_{k=1}^{\infty} E_k \supset A \\ E_k \in \mathcal{B}_0}} \sum_{k=1}^{\infty} \mu_0(E_k),$$

which, following the proof of Theorem 1.2, is seen to be an outer measure.

Let $\mathcal{B}$ be the collection of all subsets of X that are Carathéodory measurable with respect to μ^*. By Theorem 6.1, we know that $\mathcal{B}$ is a σ-algebra and the restriction μ of μ^* to $\mathcal{B}$ is a countably additive measure, so we only need to show that $\mathcal{B}$ contains $\mathcal{B}_0$ and μ extends μ_0.

Let $\epsilon > 0$. Given $A \subset X$, we can find sets $E_1, E_2, \ldots \in \mathcal{B}_0$ that cover A such that

$$\sum_{k=1}^{\infty} \mu_0(E_k) < \mu^*(A) + \epsilon.$$

Given $E \in \mathcal{B}_0$, we know from that finite additivity of μ_0 on $\mathcal{B}_0$ that

$$\mu_0(E_n \cap E) + \mu_0(E_n \setminus E) = \mu_0(E_n)$$

for each $n \in \mathbb{N}$. Since the sets $E_n \cap E$ lie in $\mathcal{B}_0$ and cover $A \cap E$,

$$\mu^*(A \cap E) \le \sum_{n=1}^{\infty} \mu_0(E_n \cap E).$$

Likewise,

$$\mu^*(A \setminus E) \le \sum_{n=1}^{\infty} \mu_0(E_n \setminus E).$$

Hence

$$\mu^*(A \cap E) + \mu^*(A \setminus E) \le \sum_{n=1}^{\infty} \mu_0(E_n) \le \mu^*(A) + \epsilon.$$

Since ϵ is arbitrary, we deduce

$$\mu^*(A \cap E) + \mu^*(A \setminus E) \le \mu^*(A).$$

The reverse inequality follows from subadditivity. Thus

$$\mu^*(A \cap E) + \mu^*(A \setminus E) = \mu^*(A).$$

That is, $E \in \mathcal{B}$ and hence $\mathcal{B} \supset \mathcal{B}_0$.

Since each element E of $\mathcal{B}_0$ covers itself, $\mu^*(E) \le \mu_0(E)$. From each covering $E_1, E_2, \ldots \in \mathcal{B}_0$ of E, construct the disjoint subsets $E \cap E_n \setminus \bigcup_{k=1}^{n-1} E_k$, whose union is exactly equal to E. Since μ_0 is a premeasure, we know from monotonicity that

$$\mu_0(E) = \sum_{n=1}^{\infty} \mu_0\left(E \cap E_n \setminus \bigcup_{k=1}^{n-1} E_k \right) \le \sum_{n=1}^{\infty} \mu_0(E_n).$$

This provides us with a lower bound for $\mu^*(E)$:

$$\mu^*(E) \ge \mu_0(E),$$

so that $\mu^*(E) = \mu_0(E)$. We thus see that the restriction μ of μ^* to $\mathcal{B}$ extends μ_0 from the Boolean algebra $\mathcal{B}_0$ to the σ-algebra $\mathcal{B}$.

Definition: A measure space $(X, \mathcal{B}, \mu)$ is σ-finite if X can be expressed as the countable union of sets of finite measure.

- $\mathbb{R}^d$ with the Lebesgue measure is σ-finite.
- $\mathbb{R}^d$ with the counting measure is not σ-finite.

Problem 6.4: Let $\mu_0 : \mathcal{B}_0 \to [0, \infty]$ be a σ-finite premeasure on a Boolean algebra $\mathcal{B}_0$ in a space X. Suppose that $\mu, \mu' : \mathcal{B} \to [0, \infty]$ are Hahn–Kolmogorov extensions of μ_0 to σ-algebras $\mathcal{B}$ and $\mathcal{B}'$ containing $\mathcal{B}_0$. Show that $\mu = \mu'$ on $\mathcal{B} \cap \mathcal{B}'$.

Definition: A *Borel measure* is any measure defined on $\mathcal{B}[\mathbb{R}^d]$.

Remark: We now introduce a powerful tool for constructing Borel measures.

Theorem 6.3 (Lebesgue-Stieltjes measure): *Let $F : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function and define*

$$F_-(x) \doteq \sup_{y<x} F(y), \qquad F_+(x) \doteq \inf_{y>x} F(y),$$

so that $F_-(x) \le F(x) \le F_+(x)$ for all $x \in \mathbb{R}$. Then there exists a unique Borel measure $\mu_F : \mathcal{B}[\mathbb{R}] \to [0, \infty]$, known as the Lebesgue-Stieltjes measure *of F, such that*

$$\begin{aligned} \mu_F([a,b]) &= F_+(b) - F_-(a), \\ \mu_F([a,b)) &= F_-(b) - F_-(a), \\ \mu_F((a,b]) &= F_+(b) - F_+(a), \\ \mu_F((a,b)) &= F_-(b) - F_+(a), \\ \mu_F([a,a]) &= F_+(a) - F_-(a), \end{aligned} \tag{6.2}$$

whenever $-\infty < a < b < \infty$.

Proof: See Tao, Theorem 1.7.9.

- When $F(x) = x$, μ_F is just the Lebesgue measure m.

- If $F : \mathbb{R} \to \mathbb{R}$ is increasing and absolutely continuous (so that, by Problem 5.13 (ii), F' exists almost everywhere and is absolutely integrable on every closed interval $[a, b]$) and S is a Borel-measurable set,
$$\mu_F(S) = \int_S F'(x)\, dx$$
is a Borel measure. For any unsigned Borel-measurable function $f : \mathbb{R} \to [0, \infty]$, the integral
$$\int_{\mathbb{R}} f\, d\mu_F = \int_{\mathbb{R}} f(x)F'(x)\, dx$$
is known as the *Lebesgue–Stieltjes integral* of f with respect to F and is often abbreviated as $\int_{\mathbb{R}} f\, dF$. In particular, when $f = 1$, we can exploit the absolute continuity of F to recover the fundamental theorem of calculus on $[a, b]$:
$$\int_{[a,b]} dF = \mu_F([a, b]) = F(b) - F(a).$$

Problem 6.5: Evaluate the Lebesgue–Stieltjes integral
$$\int_0^2 x^2 d\left(e^{x^2}\right).$$

$$\int_0^2 x^2 d\left(e^{x^2}\right) = \int_0^2 x^2 e^{x^2} 2x\, dx = \int_0^4 u e^u\, du = [ue^u]_0^r - \int_0^4 e^u\, du = 4e^4 - e^4 + 1 = 3e^4 + 1.$$

Remark: The Lebesgue–Stieltjes integral can be generalized to handle the case where F is of bounded variation, by writing F as a difference of monotone increasing functions.

Definition: A *Radon measure* on $\mathbb{R}$ is a Borel measure μ obeying:

1. (Local finiteness) $\mu(K) < \infty$ for every compact K;
2. (Outer regularity) $\mu(S) = \inf\limits_{\substack{U \supset S \\ U \text{open}}} \mu(U)$ for every Borel set S;
3. (Inner regularity) $\mu(S) = \sup\limits_{\substack{K \subset S \\ K \text{compact}}} \mu(K)$ for every Borel set S.

Problem 6.6: Show that the Lebesgue–Stieltjes measure μ_F for every monotone function $F : \mathbb{R} \to \mathbb{R}$ is a Radon measure on $\mathbb{R}$. Conversely, if μ is a Radon measure on $\mathbb{R}$, show that there exists a monotone function $F : \mathbb{R} \to \mathbb{R}$ such that $\mu = \mu_F$.

Definition: Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be measurable spaces. The σ-algebra

$$\mathcal{B}_X \times \mathcal{B}_Y \doteq \langle \{A \times B : A \in \mathcal{B}_X, B \in \mathcal{B}_Y\} \rangle ,$$

in $X \times Y$ is a *product σ-algebra*.

Remark: The σ-algebra $\mathcal{B}_X \times \mathcal{B}_Y$ is the coarsest σ-algebra on $X \times Y$ for which every product of a $\mathcal{B}_X$-measurable set and a $\mathcal{B}_Y$-measurable set is $\mathcal{B}_X \times \mathcal{B}_Y$ measurable.

Remark: If $f : X \times Y \to [0, \infty]$ is measurable with respect to $\mathcal{B}_X \times \mathcal{B}_Y$, the function $f_x : y \mapsto f(x, y)$ is $\mathcal{B}_Y$ measurable for every $x \in X$ and $f^y : x \mapsto f(x, y)$ is $\mathcal{B}_X$ measurable for every $y \in Y$.

Definition: Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be σ-finite measure spaces. A *product measure* $\mu_X \times \mu_Y$ on $\mathcal{B}_X \times \mathcal{B}_Y$ satisfies $(\mu_X \times \mu_Y)(E \times F) = \mu_X(E)\mu_Y(F)$ whenever $E \in \mathcal{B}_X$ and $F \in \mathcal{B}_Y$.

Theorem 6.4 (Existence and uniqueness of product measure): *Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be σ-finite measure spaces. Then there exists a unique product measure $\mu_X \times \mu_Y$ on $\mathcal{B}_X \times \mathcal{B}_Y$.*

Proof: Consider the Boolean algebra $\mathcal{B}_0$ consisting of all finite unions

$$S \doteq (E_1 \times F_1) \cup \ldots \cup (E_m \times F_m)$$

of Cartesian products of $\mathcal{B}_x$-measurable sets $E_1, \ldots, E_m$ and $\mathcal{B}_Y$-measurable sets $F_1, \ldots, F_m$. In analogy with the decomposition of elementary sets in $\mathbb{R}^d$ into finite unions of disjoint boxes, we can without loss of generality assume that $E_1 \times F_1, \ldots, E_m \times F_m$ are disjoint. We then introduce the finitely additive measure $\mu_0 : \mathcal{B}_0 \to [0, \infty]$:

$$\mu_0(S) \doteq \sum_{j=1}^{m} \mu_X(E_j)\mu_Y(F_j),$$

whenever S is a disjoint union of Cartesian products $E_1 \times F_1, \ldots, E_m \times F_m$ of $\mathcal{B}_x$- and $\mathcal{B}_Y$-measurable sets, independent of exactly how S is decomposed.

To establish that μ_0 is a premeasure on $\mathcal{B}_0$, we need to show for every countable union $S \in \mathcal{B}_0$ of disjoint sets $S_1, S_2, \ldots \in \mathcal{B}_0$ that $\mu_0(S) = \sum_{k=1}^{\infty} \mu_0(S_k)$. Using the finite additivity of μ_0, we can reduce the problem to the case $m = 1$; that is, $S = E{\times}F$ and $S_k = E_k{\times}F_k$ for $k = 1, 2, \ldots$. We begin by re-expressing the decomposition $S = \bigcup_{k=1}^{\infty} S_k$ into disjoint sets as the pointwise identity

$$1_E(x)1_F(y) = \sum_{k=1}^{\infty} 1_{E_k}(x)1_{F_k}(y)$$

for all $x \in X$ and $y \in Y$. For fixed $x \in X$, we can integrate both sides in y:

$$\int_Y 1_E(x)1_F(y)\, d\mu_Y(y) = \int_Y \sum_{k=1}^{\infty} 1_{E_k}(x)1_{F_k}(y)\, d\mu_Y(y).$$

Using the monotone convergence theorem to interchange the integration and summation on the right-hand side, this identity evaluates to

$$1_E(x)\mu_Y(F) = \sum_{k=1}^{\infty} 1_{E_k}(x)\mu_Y(F_k).$$

Similarly, we may now integrate in x and use monotone convergence to obtain

$$\mu_X(E)\mu_Y(F) = \sum_{k=1}^{\infty} \mu_X(E_k)\mu_Y(F_k);$$

thereby demonstrating that μ_0 is a premeasure.

Using the Hahn–Kolmogorov theorem, we can then extend μ_0 to a countably additive measure $\mu_X{\times}\mu_Y$ on a σ-algebra containing $\mathcal{B}_0$. Since $\mathcal{B}_X{\times}\mathcal{B}_Y$ is the coarsest such σ-algebra, we see that $\mu_X{\times}\mu_Y$ is a countably additive measure on $\mathcal{B}_X{\times}\mathcal{B}_Y$, with $(\mu_X{\times}\mu_Y)(E{\times}F) = \mu_X(E)\mu_Y(F)$ whenever $E \in \mathcal{B}_X$ and $F \in \mathcal{B}_Y$. The uniqueness of $\mu_X{\times}\mu_Y$ follows from Prob 6.4.

Definition: A *monotone class* in a space X is a collection $\mathcal{B}$ of subsets of X such that

1. The union $\bigcup_{k=1}^{\infty} S_k$ of every countable increasing sequence of sets $S_1 \subset S_2 \subset \ldots$ in $\mathcal{B}$ is itself in $\mathcal{B}$.
2. The intersection $\bigcap_{k=1}^{\infty} S_k$ of every countable decreasing sequence of sets $S_1 \supset S_2 \supset \ldots$ in $\mathcal{B}$ is itself in $\mathcal{B}$.

Remark: Every σ-algebra is a monotone class, but a monotone class need not be a σ-algebra.

Lemma 6.1 (Monotone class lemma): *Let $\mathcal{B}$ be a Boolean algebra on X. Then $\langle\mathcal{B}\rangle$ is the smallest monotone class containing $\mathcal{B}$.*

Proof: Let $\mathcal{I}$ be the intersection of all monotone classes containing $\mathcal{B}$. Since $\langle\mathcal{B}\rangle$ is one such class, we see that $\mathcal{I} \subset \langle\mathcal{B}\rangle$.

Consider the set $\mathcal{J} \subset \mathcal{I}$ composed of those elements of $\mathcal{I}$ whose complements are also in $\mathcal{I}$. Since $\left(\bigcup_{k=1}^{\infty} S_k\right)^c = \bigcap_{k=1}^{\infty} S_k^c$, we see that $\mathcal{J}$ is also a monotone class containing $\mathcal{B}$, so that $\mathcal{I} \subset \mathcal{J}$. Hence $\mathcal{J} = \mathcal{I}$ is closed under complements.

Likewise, consider the set $\mathcal{K} \subset \mathcal{I}$ composed of those elements of $\mathcal{I}$ for which $S \setminus T$, $T \setminus S$, $S \cap T$, and $X \setminus (S \cup T)$ also belong to $\mathcal{I}$ for every $T \in \mathcal{I}$. Since $(\bigcup_{k=1}^{\infty} S_k)\backslash T = \bigcup_{k=1}^{\infty}(S_k\backslash T)$, etc., we see that $\mathcal{K}$ is also a monotone class containing $\mathcal{B}$, so that $\mathcal{I} \subset \mathcal{K}$. Thus $\mathcal{I} = \mathcal{K}$.

We now know that $\mathcal{I}$ is closed under finite unions as well as finite intersections. Since $\mathcal{I} \supset \mathcal{B}$ and $\emptyset \in \mathcal{B}$, we see that $\mathcal{I}$ is a Boolean algebra. Given any countable sequence of elements S_k of $\mathcal{I}$, $k = 1, 2, \ldots$, consider the increasing sequence $T_n = \bigcup_{k=1}^{n} S_k$ for $n = 1, 2, \ldots$. Since $\mathcal{I}$ is a monotone class, we know that $\bigcup_{k=1}^{\infty} S_k = \bigcup_{n=1}^{\infty} T_n \in \mathcal{I}$, so $\mathcal{I}$ is a σ-algebra. Since $\mathcal{I}$ contains $\mathcal{B}$, we deduce that $\mathcal{I} \supset \langle\mathcal{B}\rangle$. Thus $\mathcal{I} = \langle\mathcal{B}\rangle$.

Theorem 6.5 (Tonelli's theorem): *Let $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$ be σ-finite measure spaces and let $f : X\times Y \to [0, \infty]$ be measurable with respect to $\mathcal{B}_X\times\mathcal{B}_Y$. Then:*

(i) the functions $x \to \int_Y f(x, y)\, d\mu_Y(y)$ and $y \to \int_X f(x, y)\, d\mu_X(x)$ are measurable with respect to $\mathcal{B}_X$ and $\mathcal{B}_Y$, respectively;

(ii)

$$\int_{X\times Y} f(x, y)\, d\mu_X\times\mu_Y(x, y) = \int_X \int_Y f(x, y)\, d\mu_Y(y)\, d\mu_X(x) = \int_Y \int_X f(x, y)\, d\mu_X(x)\, d\mu_Y(y).$$

Proof: Since the σ-finite spaces X and Y can be decomposed as increasing unions of finite-measure sets, the monotone convergence theorem can be used to reduce the problem to the case where X and Y, and by Theorem 6.4, $X\times Y$, have finite measure.

Since every unsigned measurable function is the limit of an increasing sequence of unsigned simple functions, by the monotone convergence theorem and linearity, we can further reduce the problem to the case where $f = 1_S$ for some $S \in \mathcal{B}_X\times\mathcal{B}_Y$.

Let $\mathcal{C}$ be the collection of all elements of $\mathcal{B}_X\times\mathcal{B}_Y$ for which statements (i) and (ii) hold. Repeated application of the monotone convergence theorem and downward monotone convergence (which applies in this finite-measure setting) establishes that $\mathcal{C}$ is a monotone class.

Theorem 6.4 guarantees that every product $E\times F$ with $E\in\mathcal{B}_X$ and $F\in\mathcal{B}_Y$ is in $\mathcal{C}$. By finite additivity, every finite union

$$(E_1\times F_1)\cup\ldots\cup(E_m\times F_m)$$

of such products, which can always be re-expressed as a union of disjoint products, is an element of $\mathcal{C}$. Since $\mathcal{C}$ is a monotone class, Lemma 6.1 then establishes that $\mathcal{C}$ contains the entire σ-algebra $\mathcal{B}_X\times\mathcal{B}_Y$.

Corollary 6.5.1: Let $(X,\mathcal{B}_X,\mu_X)$ and $(Y,\mathcal{B}_Y,\mu_Y)$ be σ-finite measure spaces and let $E\in\mathcal{B}_X\times\mathcal{B}_Y$ be a null set with respect to $\mu_X\times\mu_Y$. Then for μ_X-almost every $x\in X$ the set $E_x\doteq\{y\in Y:(x,y)\in E\}$ is a μ_Y-null set. Similarly, for μ_Y-almost every $y\in Y$ the set $E^y\doteq\{x\in X:(x,y)\in E\}$ is a μ_X-null set.

Proof: On applying Tonelli's theorem to the indicator function 1_E, we find

$$0=\int_X\int_Y 1_E(x,y)\,d\mu_Y(y)\,d\mu_X(x)=\int_Y\int_X 1_E(x,y)\,d\mu_X(x)\,d\mu_Y(y).$$

Thus

$$0=\int_X \mu_Y(E_x)\,d\mu_X(x)=\int_Y \mu_X(E^y)\,d\mu_Y(y),$$

from which the result immediately follows.

Remark: One limitation of the product σ-algebra $\mathcal{B}_X\times\mathcal{B}_Y$ is that it may not be complete.

- Consider the two-dimensional Cartesian product of the one-dimensional Lebesgue measure space $(\mathbb{R},\mathcal{L}[\mathbb{R}],m)$ with itself. This yields a measure m^2 on $\mathbb{R}^2$ that is incomplete. For example, let V be a nonmeasurable set like the Vitali set constructed on p. 10 and consider that $V\times\{0\}$ is not measurable in $\mathbb{R}^2$ even though $V\times\{0\}$ is a subset of the m^2-null set $\mathbb{R}\times\{0\}$. We can fix this deficiency by replacing m^2 by its completion $\overline{m^2}$.

Remark: It is easy to extend Tonelli's theorem to complete product measures.

Corollary 6.5.2 (Tonelli's theorem for complete product measures): Let $(X,\mathcal{B}_X,\mu_X)$ and $(Y,\mathcal{B}_Y,\mu_Y)$ be complete σ-finite measure spaces and let $f:X\times Y\to[0,\infty]$ be measurable with respect to $\overline{\mathcal{B}_X\times\mathcal{B}_Y}$. Then:

(i) the μ_X-almost everywhere defined function $x\to\int_Y f(x,y)\,d\mu_Y(y)$ and μ_Y-almost everywhere defined function $y\to\int_X f(x,y)\,d\mu_X(x)$ are measurable with respect to $\mathcal{B}_X$ and $\mathcal{B}_Y$, respectively;

(ii)

$$\int_{X\times Y} f(x,y)\, d\overline{\mu_X\times\mu_Y}(x,y) = \int_X \int_Y f(x,y)\, d\mu_Y(y)\, d\mu_X(x) = \int_Y \int_X f(x,y)\, d\mu_X(x)\, d\mu_Y(y).$$

Proof: Every measurable set in $\overline{\mathcal{B}_X\times\mathcal{B}_Y}$ is equal to a measurable set in $\mathcal{B}_X\times\mathcal{B}_Y$ outside of a $\mu_X\times\mu_Y$-subnull set. The result then follows from Theorem 6.5 and Corollary 6.5.1.

Corollary 6.5.3 (Fubini's theorem): Let $(X,\mathcal{B}_X,\mu_X)$ and $(Y,\mathcal{B}_Y,\mu_Y)$ be complete σ-finite measure spaces and let $f: X\times Y \to \mathbb{C}$ be absolutely integrable with respect to $\overline{\mathcal{B}_X\times\mathcal{B}_Y}$. Then:

(i) the μ_X-almost everywhere defined function $x \to \int_Y f(x,y)\, d\mu_Y(y)$ and μ_Y-almost everywhere defined function $y \to \int_X f(x,y)\, d\mu_X(x)$ are absolutely integrable with respect to μ_X and μ_Y, respectively;

(ii)

$$\int_{X\times Y} f(x,y)\, d\overline{\mu_X\times\mu_Y}(x,y) = \int_X \int_Y f(x,y)\, d\mu_Y(y)\, d\mu_X(x) = \int_Y \int_X f(x,y)\, d\mu_X(x)\, d\mu_Y(y).$$

Proof: The result follows on applying Corollary 6.5.2 to the positive and negative parts of the real and imaginary parts of f, noting from the absolute integrability of f that $\int_Y f(x,y)\, d\mu_Y(y)$ is finite for μ_X-almost all $x \in X$ and that $\int_X f(x,y)\, d\mu_X(x)$ is finite for μ_Y-almost all $y \in Y$.

Remark: If $\int_X \int_Y |f(x,y)|\, d\mu_Y(y)\, d\mu_X(x) < \infty$, Tonelli's theorem can be used to establish the absolute integrability requirement of Fubini's theorem.

Remark: The Tonelli and Fubini theorems provide us with practical tools for computing integrals.

Problem 6.7: (*Area interpretation of integral*)
Let $(X,\mathcal{B},\mu)$ be a σ-finite measure space and let $\mathbb{R}$ be equipped with the Lebesgue measure m and the Borel σ-algebra $\mathcal{B}[\mathbb{R}]$. Show that $f: X \to [0,\infty]$ is measurable iff $S \doteq \{(x,t) \in X\times\mathbb{R} : 0 \le t < f(x)\}$ is measurable in $\mathcal{B}\times\mathcal{B}[\mathbb{R}]$, with

$$(\mu\times m)(S) = \int_X f(x)\, d\mu(x).$$

For each $q \in \mathbb{Q}$, construct

$$S_q \doteq \{(x,t) \in X\times\mathbb{R} : 0 \le t \le q < f(x)\} = \{x \in X : q < f(x)\}\times[0,q],$$

which is a rectangle with measurable sides and thus belongs to $\mathcal{B}\times\mathcal{B}(\mathbb{R})$. Since $S = \bigcup_{q\in\mathbb{Q}} S_q$, this implies that $S \in \mathcal{B}\times\mathcal{B}(\mathbb{R})$. This establishes that 1_S is a measurable function, so that we can apply Tonelli's theorem:

$$(\mu\times m)(S) = \int_{X\times\mathbb{R}} 1_S \, d(\mu\times m) = \int_X \int_{\mathbb{R}} 1_{[0,f(x)]} \, dt \, d\mu(x) = \int_X f(x) \, d\mu(x).$$

Problem 6.8: (*Distribution formula*)

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $f : X \to [0,\infty]$ be measurable. Show that

$$\int_X f(x) \, d\mu = \int_{[0,\infty]} \mu(\{x \in X : f(x) \geq t\}) \, dt.$$

For each t, let $S_t \doteq \{x \in X : f(x) \geq t\}$. First note that $\mu(S_t)$ is a monotonic function in t and monotonic functions are Lebesgue measurable since $\{t : \mu(S_t) > \lambda\}$ is a (possibly infinite) interval for each $\lambda \in [0,\infty)$. Following Prob 6.7, we then see that

$$\int_{[0,\infty]} \mu(S_t) \, dt = \int_{[0,\infty]} \int_X 1_{S_t} \, d\mu(x) \, dt = \int_X \int_{[0,\infty]} 1_{[0,f(x)]} \, dt \, d\mu(x) = \int_X f(x) \, d\mu(x).$$

Bibliography

[Bogachev 2007] V. I. Bogachev, *Measure Theory*, Springer, 2007, https://link.springer.com/book/10.1007/978-3-540-34514-5.

[Cohn 2013] D. Cohn, *Measure theory*, Birkhäuser, 2nd edition, 2013, https://www.springer.com/us/book/9781461469551.

[Stein & Shakarchi 2005] E. Stein & R. Shakarchi, *Real Analysis: Measure Theory, Integration, & Hilbert Spaces*, Princeton University Press, 2005, https://www.springer.com/us/book/9781461469551.

[Tao 2011] T. Tao, *An introduction to measure theory*, American Mathematical Society Providence, RI, 2011, https://terrytao.files.wordpress.com/2011/01/measure-book1.pdf.

Index