Math 411: Honours Complex Variables
List of Theorems

Theorem 1.1 ($\mathbb{C}$ is a Field)**.** *The complex numbers are a field. Specifically, we have:*

- $(0, 0)$ *is the identity element of addition;*
- $-(x, y) = (-x, -y)$ *for* $x, y \in \mathbb{R}$*;*
- $(1, 0)$ *is the identity element of multiplication;*
- $(x, y)^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ *for* $x, y \in \mathbb{R}$ *with* $(x, y) \neq (0, 0)$*.*

Theorem 2.1 (Cauchy–Riemann Equations)**.** *Let* $D \subset \mathbb{C}$ *be open, and let* $z_0 \in D$*. Let* $f\colon D \to \mathbb{C}$ *and denote* $u := \operatorname{Re} f$, $v := \operatorname{Im} f$*. Then the following are equivalent:*

(i) f *is complex differentiable at* z_0*;*

(ii) f *is totally differentiable at* z_0 *(in the sense of multivariable calculus), and the* Cauchy–Riemann differential equations

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$$

hold.

Corollary 2.1.1. Let $D \subset \mathbb{C}$ be open and connected, and let $f\colon D \to \mathbb{C}$ be complex differentiable. Then f is constant on D if and only if $f' \equiv 0$.

Theorem 3.1 (Radius of Convergence)**.** *Let* $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *be a complex power series. Then there exists a unique* $R \in [0, \infty]$ *with the following properties:*

- $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *converges absolutely for each* $z \in B_R(z_0)$*;*
- *for each* $r \in [0, R)$*, the series* $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *converges uniformly on* $B_r[z_0] := \{z \in \mathbb{C} : |z - z_0| \leq r\}$*;*
- $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *diverges for each* $z \notin B_R[z_0]$*.*

Moreover, R can be computed via the Cauchy–Hadamard formula:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

It is called the radius of convergence for $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Theorem 3.2 (Term-by-Term Differentiation)**.** *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series with radius of convergence R. Then*

$$f\colon B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is complex differentiable at each point $z \in B_R(z_0)$ with

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}.$$

Corollary 3.2.1 (Higher Derivatives of Power Series)**.** Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series with radius of convergence R. Then

$$f\colon B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is infinitely often complex differentiable on $B_R(z_0)$ with

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}.$$

for $z \in B_R(z_0)$ and $k \in \mathbb{N}$. In particular, when $z = z_0$ we see that

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

holds for each $n \in \mathbb{N}_0$.

Corollary 3.2.2 (Integration of Power Series)**.** Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series with radius of convergence R. Then

$$F\colon B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$$

is complex differentiable on $B_R(z_0)$ with

$$F'(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in B_R(z_0)$.

Theorem 4.1 (Antiderivative Theorem)**.** *Let $D \subset \mathbb{C}$ be open and connected and let $f\colon D \to \mathbb{C}$ be continuous. Then the following are equivalent:*

(i) *f has an antiderivative;*

(ii) *$\int_\gamma f(\zeta)\, d\zeta = 0$ for any closed, piecewise smooth curve γ in D;*

(iii) *for any piecewise smooth curve γ in D, the value of $\int_\gamma f$ depends only on the inital point and the endpoint of γ.*

Theorem 5.1 (Goursat's Lemma)**.** *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $\Delta \subset D$ be a triangle. Then we have*

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = 0.$$

Theorem 5.2. *Let $D \subset \mathbb{C}$ be open and star shaped with center z_0, and let $f\colon D \to \mathbb{C}$ be continuous such that*

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = 0$$

for each triangle $\Delta \subset D$ with z_0 as a vertex. Then f has an antiderivative.

Corollary 5.2.1. Let $D \subset \mathbb{C}$ be open and star shaped, and let $f\colon D \to \mathbb{C}$ be holomorphic. Then f has an antiderivative.

Corollary 5.2.2. Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ be holomorphic. Then, for each $z_0 \in D$, there exists an open neighbourhood $U \subset D$ of z_0 such that $f|_U$ has an antiderivative.

Corollary 5.2.3 (Cauchy's Integral Theorem for Star-Shaped Domains)**.** Let $D \subset \mathbb{C}$ be open and star shaped, and let $f : D \to \mathbb{C}$ be holomorphic. Then $\int_\gamma f(\zeta)\, d\zeta = 0$ holds for each closed curve γ in D.

Theorem 5.3 (Cauchy's Integral Formula for Circles)**.** *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for all $z \in B_r(z_0)$.

Corollary 5.3.1 (Mean Value Equation)**.** Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \, dt.$$

Theorem 5.4 (Higher Derivatives of Holomorphic Functions)**.** *Let $D \subset \mathbb{C}$ be open, let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$, and let $f : D \to \mathbb{C}$ be continuous such that*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

holds for all $z \in B_r(z_0)$. Then f is infinitely often complex differentiable on $B_r(z_0)$ and satisfies

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \tag{$*$}$$

holds for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.

Corollary 5.4.1 (Generalized Cauchy Integral Formula)**.** Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. Then f is infinitely often complex differentiable on D. Moreover, for any $z_0 \in D$ and $r > 0$ such that $B_r[z_0] \subset D$, the *generalized Cauchy integral formula* holds, i.e.

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.

Theorem 5.5 (Characterizations of Holomorphic Functions)**.** *Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be continuous. Then the following are equivalent:*

(i) *f is holomorphic;*

(ii) *the Morera condition holds, i.e.* $\int_{\partial\Delta} f(\zeta)\, d\zeta = 0$ *for each triangle* $\Delta \subset D$*;*

(iii) *for each* $z_0 \in D$ *and* $r > 0$ *with* $B_r[z_0] \subset D$*, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in B_r(z_0)$*;*

(iv) *for each* $z_0 \in D$*, there exists* $r > 0$ *with* $B_r[z_0] \subset D$ *and*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in B_r(z_0)$*;*

(v) *f is infinitely often complex differentiable on D;*

(vi) *for each* $z_0 \in D$*, there exists an open neighbourhood* $U \subset D$ *of* z_0 *such that* f *has an antiderivative on* U*.*

Theorem 5.6 (Liouville's Theorem)**.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a bounded entire function. Then* f *is constant.*

Corollary 5.6.1 (Fundamental Theorem of Algebra)**.** Let p be a non-constant polynomial with complex coefficients. Then p has a zero.

Theorem 6.1 (Uniform Convergence Preserves Continuity)**.** *Let* $D \subset \mathbb{C}$ *be open, and let* $(f_n)_{n=1}^{\infty}$ *be a sequence of continuous,* $\mathbb{C}$*-valued functions on* D *converging uniformly on* D *to* $f \colon D \to \mathbb{C}$*. Then* f *is continuous.*

Theorem 6.2 (Weierstraß Theorem)**.** *Let* $D \subset \mathbb{C}$ *be open, let* $f_1, f_2, \ldots : D \to \mathbb{C}$ *be holomorphic such that* $(f_n)_{n=1}^{\infty}$ *converges to* $f \colon D \to \mathbb{C}$ *compactly. Then* f *is holomorphic, and* $(f_n^{(k)})_{n=1}^{\infty}$ *converges compactly to* $f^{(k)}$ *for each* $k \in \mathbb{N}$*.*

Theorem 6.3 (Power Series for Holomorphic Functions)**.** *Let* $D \subset \mathbb{C}$ *be open. Then the following are equivalent for* $f \colon D \to \mathbb{C}$*:*

(i) f *is holomorphic;*

(ii) *for each* $z_0 \in D$*, there exists* $r > 0$ *with* $B_r(z_0) \subset D$ *and* $a_0, a_1, a_2, \ldots \in \mathbb{C}$ *such that* $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ *for all* $z \in B_r(z_0)$*;*

(iii) *for each* $z_0 \in D$ *and* $r > 0$ *with* $B_r(z_0) \subset D$*, we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

for all $z \in B_r(z_0)$*.*

Theorem 7.1 (Identity Theorem)**.** *Let* $D \subset \mathbb{C}$ *be open and connected, and let* $f, g \colon D \to \mathbb{C}$ *be holomorphic. Then the following are equivalent:*

(i) $f = g$*;*

(ii) *the set* $\{z \in D : f(z) = g(z)\}$ *has a cluster point in* D*;*

(iii) *there exists* $z_0 \in D$ *such that* $f^{(n)}(z_0) = g^{(n)}(z_0)$ *for all* $n \in \mathbb{N}_0$*.*

Theorem 7.2 (Open Mapping Theorem)**.** *Let* $D \subset \mathbb{C}$ *be open and connected, and let* $f \colon D \to \mathbb{C}$ *be holomorphic and not constant. Then* $f(D) \subset \mathbb{C}$ *is open and connected.*

Theorem 7.3 (Maximum Modulus Principle)**.** *Let* $D \subset \mathbb{C}$ *be open and connected, and let* $f \colon D \to \mathbb{C}$ *be holomorphic such that the function*

$$|f| \colon D \to \mathbb{C}, \quad z \mapsto |f(z)|$$

attains a local maximum on D*. Then* f *is constant.*

Corollary 7.3.1. Let $D \subset \mathbb{C}$ be open and connected, and let $f \colon D \to \mathbb{C}$ be holomorphic such that $|f|$ attains a local minimum on D. Then f is constant or f has a zero.

Corollary 7.3.2 (Maximum Modulus Principle for Bounded Domains)**.** Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f \colon \overline{D} \to \mathbb{C}$ be continuous such that $f|_D$ is holomorphic. Then $|f|$ attains its maximum over $\overline{D}$ on ∂D.

Theorem 7.4 (Schwarz's Lemma). *Let $f\colon \mathbb{D} \to \overline{\mathbb{D}}$ be holomorphic such that $f(0) = 0$. Then one has*

$$|f(z)| \leq |z| \quad \text{for } z \in \mathbb{D} \qquad \text{and} \qquad |f'(0)| \leq 1.$$

Moreover, if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for $z \in \mathbb{D}$.

Corollary 7.4.1. Let $f\colon \mathbb{D} \to \mathbb{D}$ be biholomorphic such that $f(0) = 0$. Then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for $z \in \mathbb{D}$.

Theorem 7.5 (Biholomorphisms of $\mathbb{D}$). *Let $f : \mathbb{D} \to \mathbb{D}$ be biholomorphic. Then there exist $w \in \mathbb{D}$ and $c \in \partial\mathbb{D}$ with $f(z) = c\phi_w(z)$ for $z \in \mathbb{D}$.*

Theorem 7.6 (Riemann's Removability Condition). *Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity for f. Then the following are equivalent:*

(i) *z_0 is removable;*

(ii) *there is a continuous function $g\colon D \cup \{z_0\} \to \mathbb{C}$ such that $g|_D = f$;*

(iii) *there exists $\epsilon > 0$ with $B_\epsilon(z_0) \setminus \{z_0\} \subset D$ such that f is bounded on $B_\epsilon(z_0) \setminus \{z_0\}$.*

Theorem 8.1 (Poles). *Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity of f. Then z_0 is a pole of f $\iff$ there exist a unique $k \in \mathbb{N}$ and a holomorphic function $g\colon D \cup \{z_0\} \to \mathbb{C}$ such that $g(z_0) \neq 0$ and*

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

for $z \in D$.

Theorem 8.2 (Casorati–Weierstraß Theorem). *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity of f. Then z_0 is essential $\iff$ $\overline{f(B_\epsilon(z_0) \cap D)} = \mathbb{C}$ for each $\epsilon > 0$.*

Theorem 9.1 (Cauchy's Integral Theorem for Annuli). *Let $z_0 \in \mathbb{C}$, let $r, \rho, \mathrm{P}, R \in [0, \infty]$ be such that $r < \rho < \mathrm{P} < R$, and let $f\colon A_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then we have*

$$\int_{\partial B_{\mathrm{P}}(z_0)} f(\zeta)\, d\zeta = \int_{\partial B_\rho(z_0)} f(\zeta)\, d\zeta.$$

Theorem 9.2 (Laurent Decomposition)**.** *Let $z_0 \in \mathbb{C}$, let $r, R \in [0, \infty]$ be such that $r < R$, and let $f\colon A_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then there exists a holomorphic function*

$$g\colon B_R(z_0) \to \mathbb{C} \qquad \textit{and} \qquad h\colon \mathbb{C} \setminus B_r[z_0] \to \mathbb{C}$$

with $f = g + h$ on $A_{r,R}(z_0)$. Moreover, h can be chosen such that $\lim_{|z|\to\infty} h(z) = 0$, in which case g and h are uniquely determined.

Theorem 9.3 (Laurent Coefficients)**.** *Let $z_0 \in \mathbb{C}$, let $r, R \in [0, \infty]$ be such that $r < R$, and let $f : A_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then f has a representation*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $z \in A_{r,R}(z_0)$ as a Laurent series, which converges uniformly and absolutely on compact subsets of $A_{r,R}(z_0)$. Moreover, for every $n \in \mathbb{Z}$ and $\rho \in (r, R)$, the coefficients a_n are uniquely determined as

$$a_n = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

Corollary 9.3.1. Let $z_0 \in \mathbb{C}$, let $r > 0$, and let $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with Laurent representation $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. Then the singularity z_0 of f is

(i) removable if and only if $a_n = 0$ for $n < 0$;

(ii) a pole of order $k \in \mathbb{N}$ if and only if $a_{-k} \neq 0$ and $a_n = 0$ for all $n < -k$;

(iii) essential if and only if $a_n \neq 0$ for infinitely many $n < 0$.

Proposition 10.1. *Let γ be a closed curve in $\mathbb{C}$, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then $\nu(\gamma, z) \in \mathbb{Z}$.*

Proposition 10.2 (Winding Numbers Are Locally Constant)**.** *Let γ be a closed curve in $\mathbb{C}$. Then:*

(i) *the map*

$$\mathbb{C} \setminus \{\gamma\} \to \mathbb{C}, \quad z \mapsto \nu(\gamma, z)$$

is locally constant;

(ii) *there exists* $R > 0$ *such that* $\mathbb{C} \setminus B_R[0] \subset \operatorname{ext}\gamma$.

Theorem 11.1 (Cauchy's Integral Formula)**.** *Let* $D \subset \mathbb{C}$ *be open, let* $f : D \to \mathbb{C}$ *be holomorphic, and let* γ *be a closed curve in* D *that is homologous to zero. Then, for* $n \in \mathbb{N}_0$ *and* $z \in D \setminus \{\gamma\}$*, we have*

$$\nu(\gamma, z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

Theorem 11.2 (Cauchy's Integral Theorem)**.** *Let* $D \subset \mathbb{C}$ *be open, let* $f : D \to \mathbb{C}$ *be holomorphic, and let* γ *be a closed curve in* D *that is homologous to zero. Then* $\int_\gamma f(\zeta)\, d\zeta = 0$.

Corollary 11.2.1. Let D be an open, connected subset of $\mathbb{C}$. Then D is simply connected $\iff$ every holomorphic function on D has an antiderivative.

Corollary 11.2.2 (Holomorphic Logarithms)**.** A simply connected domain admits holomorphic logarithms.

Corollary 11.2.3 (Holomorphic Roots)**.** A simply connected domain admits holomorphic roots.

Theorem 12.1 (Residue Theorem)**.** *Let* $D \subset \mathbb{C}$ *be open and simply connected,* $z_1, \ldots, z_n \in D$ *be such that* $z_j \neq z_k$ *for* $j \neq k$*,* $f \colon D \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}$ *be holomorphic, and* γ *be a closed curve in* $D \setminus \{z_1, \ldots, z_n\}$*. Then we have*

$$\int_\gamma f(\zeta)\, d\zeta = 2\pi i \sum_{j=1}^{n} \nu(\gamma, z_j) \operatorname{res}(f, z_j).$$

Corollary 12.1.1. Let $D \subset \mathbb{C}$ be open and simply connected, $f : D \to \mathbb{C}$ be holomorphic, and γ be a closed curve in D. Then we have

$$\nu(\gamma, z)\, f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for $z \in D \setminus \{\gamma\}$.

Proposition 12.1 (Rational Trigonometric Polynomials)**.** *Let* p *and* q *be polynomials of two real variables such that* $q(x, y) \neq 0$ *for all* $(x, y) \in \mathbb{R}^2$ *with* $x^2 + y^2 = 1$*. Then we have*

$$\int_0^{2\pi} \frac{p(\cos t, \sin t)}{q(\cos t, \sin t)} \, dt = 2\pi i \sum_{z \in \mathbb{D}} \operatorname{res}(f, z),$$

where

$$f(z) = \frac{1}{iz} \cdot \frac{p\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}{q\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}.$$

Proposition 12.2 (Rational Functions)**.** *Let p and q be polynomials of one real variable with $\deg q \geq \deg p + 2$ and such that $q(x) \neq 0$ for $x \in \mathbb{R}$. Then we have*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{z \in \mathbb{H}} \operatorname{res}\left(\frac{p}{q}, z\right),$$

where

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Theorem 13.1 (Meromorphic Functions Form a Field)**.** *Let $D \subset \mathbb{C}$ be open and connected. Then the meromorphic functions on D, where we define $(f+g)(z) = \lim_{w \to z}[f(w) + g(w)]$ and $(fg)(z) = \lim_{w \to z}[f(w)g(w)]$, form a field.*

Theorem 13.2 (Argument Principle)**.** *Let $D \subset \mathbb{C}$ be open and simply connected, let f be meromorphic on D, and let γ be a closed curve in $D \setminus (\mathbf{P}(f) \cup \mathbf{Z}(f))$. Then we have*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{z \in \mathbf{Z}(f)} \nu(\gamma, z) \operatorname{ord}(f, z) - \sum_{z \in \mathbf{P}(f)} \nu(\gamma, z) \operatorname{ord}(f, z).$$

Theorem 13.3 (Bifurcation Theorem)**.** *Let $D \subset \mathbb{C}$ be open, let $f \colon D \to \mathbb{C}$ be holomorphic, and suppose that, at $z_0 \in D$, the function f attains w_0 with multiplicity $k \in \mathbb{N}$. Then there exist neighbourhoods $V \subset D$ of z_0 and $W \subset f(V)$ of w_0 such that, for each $w \in W \setminus \{w_0\}$, there exist distinct $z_1, \ldots, z_k \in V$ with $f(z_1) = \cdots = f(z_k) = w$, where f attains w at each z_j with multiplicity one.*

Theorem 13.4 (Hurwitz's Theorem)**.** *Let $D \subset \mathbb{C}$ be open and connected, let $f, f_1, f_2, \ldots \colon D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^\infty$ converges to f compactly on D, and suppose that $\mathbf{Z}(f_n) = \varnothing$ for $n \in \mathbb{N}$. Then $f \equiv 0$ or $\mathbf{Z}(f) = \varnothing$.*

Corollary 13.4.1. Let $D \subset \mathbb{C}$ be open and connected, let $f, f_1, f_2, \ldots : D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^\infty$ converges to f compactly on D, and suppose that f_n is injective for $n \in \mathbb{N}$. Then f is constant or injective.

Theorem 13.5 (Rouché's Theorem)**.** *Let $D \subset \mathbb{C}$ be open and simply connected, and let $f, g \colon D \to \mathbb{C}$ be holomorphic. Suppose that γ is a closed curve in D such that $\operatorname{int}\gamma = \{z \in D \setminus \{\gamma\} : \nu(\gamma, z) = 1\}$ and that*

$$|f(\zeta) - g(\zeta)| < |f(\zeta)|$$

for $\zeta \in \{\gamma\}$. Then f and g have the same number of zeros in $\operatorname{int}\gamma$ (counting multiplicity).

Corollary 13.5.1 (Fundamental Theorem of Algebra)**.** Let p be a polynomial with $n := \deg p \geq 1$. Then p has n zeros (counting multiplicity).

Proposition 14.1 (Harmonic Components)**.** *Let $D \subset \mathbb{C}$ be open, and let $f \colon D \to \mathbb{C}$ be holomorphic. Then $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic.*

Theorem 14.1 (Harmonic Conjugates)**.** *Let $D \subset \mathbb{C}$ be open and suppose that there exists $(x_0, y_0) \in D$ with the following property: for each $(x, y) \in D$, we have*

- *$(x, t) \in D$ for each t between y and y_0 and*
- *$(s, y_0) \in D$ for each s between x and x_0.*

Then every harmonic function on D has a harmonic conjugate.

Corollary 14.1.1. Let $D \subset \mathbb{C}$ be open, and let $u \colon D \to \mathbb{R}$ be harmonic. Then, for each $z_0 \in D$, there is a neighbourhood $U \subset D$ of z_0 such that $u|_U$ has a harmonic conjugate.

Corollary 14.1.2. Let $D \subset \mathbb{C}$ be open, and let $u \colon D \to \mathbb{R}$ be harmonic. Then u is infinitely often partially differentiable.

Corollary 14.1.3. Let $D \subset \mathbb{C}$ be open and connected, and let $u \colon D \to \mathbb{R}$ be harmonic. Then the following are equivalent:

(i) $u \equiv 0$;

(ii) there exists a nonempty open set $U \subset D$ with $u|_U \equiv 0$.

Corollary 14.1.4. Let $D \subset \mathbb{C}$ be open, let $u \colon D \to \mathbb{R}$ be harmonic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

Corollary 14.1.5. Let $D \subset \mathbb{C}$ be open and connected, and let $u\colon D \to \mathbb{R}$ be harmonic with a local maximum or minimum on D. Then u is constant.

Corollary 14.1.6. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $u : \overline{D} \to \mathbb{R}$ be continuous such that $u|_D$ is harmonic. Then u attains its maximum and minimum over $\overline{D}$ on ∂D.

Theorem 14.2 (Poisson's Integral Formula)**.** *Let $r > 0$, and let $u\colon B_r[0] \to \mathbb{R}$ be continuous such that $u|_{B_r(0)}$ is harmonic. Then*

$$u(z) = \int_0^{2\pi} u(re^{i\theta}) P_r(re^{i\theta}, z)\, d\theta$$

holds for all $z \in B_r(0)$.

Theorem 14.3. *Let $r > 0$, and let $f\colon \partial B_r(0) \to \mathbb{R}$ be continuous. Define*

$$g\colon B_r[0] \to \mathbb{C}, \quad z \mapsto \begin{cases} f(z), & z \in \partial B_r(0), \\ \int_0^{2\pi} f(re^{i\theta}) P_r(re^{i\theta}, z)\, d\theta, & z \in B_r(0). \end{cases}$$

Then g is continuous and harmonic on $B_r(0)$.

Theorem 14.4. *Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ have the mean value property such that $|f|$ attains a local maximum at $z_0 \in D$. Then f is constant on a neighbourhood of z_0.*

Corollary 14.4.1. Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{R}$ be continuous and have the mean value property, and suppose that f has a local maximum or minimum at $z_0 \in D$. Then f is constant on a neighbourhood of z_0.

Corollary 14.4.2. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f\colon \overline{D} \to \mathbb{R}$ be continuous such that $f|_D$ has the mean value property. Then f attains its maximum and minimum on ∂D.

Corollary 14.4.3 (Equivalence of Harmonic and Mean-Value Properties)**.** Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{R}$ be continuous. Then the following are equivalent:

(i) f is harmonic;

(ii) f has the mean value property.

Theorem 17.1 (Conformality at Nondegenerate Points)**.** *Let $D_1, D_2 \subset \mathbb{C}$ be open, and let $f\colon D_1 \to D_2$ be holomorphic. Then f is angle preserving at $z_0 \in D_1$ whenever $f'(z_0) \neq 0$.*

Corollary 17.1.1 (Conformality of Biholomorphic Maps)**.** Let $D_1, D_2 \subset \mathbb{C}$ be open and connected, and let $f : D_1 \to D_2$ be biholomorphic. Then f is angle preserving at every point of D_1.

Theorem 17.2 (Holomorphic Inverses)**.** *Let $D_1, D_2 \subset \mathbb{C}$ be open and connected, and let $f\colon D_1 \to D_2$ be holomorphic and bijective. Then f is biholomorphic and $\mathbf{Z}(f') = \varnothing$.*

Corollary 17.2.1. Let $D \subset \mathbb{C}$ be open and connected, and let $f\colon D \to \mathbb{C}$ be holomorphic and injective. Then $\mathbf{Z}(f') = \varnothing$.

Theorem 17.3 (Riemann Mapping Theorem)**.** *Let $D \subsetneq \mathbb{C}$ be open and connected and admit holomorphic square roots, and let $z_0 \in D$. Then there is a unique biholomorphic function $f\colon D \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) > 0$.*

Theorem 17.4 (Simply Connected Domains)**.** *The following are equivalent for an open and connected set $D \subset \mathbb{C}$:*

(i) *D is simply connected;*

(ii) *D admits holomorphic logarithms;*

(iii) *D admits holomorphic roots;*

(iv) *D admits holomorphic square roots;*

(v) *D is all of $\mathbb{C}$ or biholomorphically equivalent to $\mathbb{D}$;*

(vi) *every holomorphic function $f\colon D \to \mathbb{C}$ has an antiderivative;*

(vii) *$\int_\gamma f(\zeta)\,d\zeta = 0$ for each holomorphic function $f\colon D \to \mathbb{C}$ and each closed curve γ in D;*

(viii) *for every holomorphic function $f\colon D \to \mathbb{C}$, we have*

$$\nu(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

for each closed curve γ in D and all $z \in D \setminus \{\gamma\}$;

(ix) *every harmonic function $u\colon D \to \mathbb{R}$ has a harmonic conjugate.*