

Review of Math 373

1 Linear Programming

Objective function: Linear function (cost) to be minimized.

Constraints: Linear inequalities or equalities on decision variables.

Feasible solution: Decision variables that satisfy the constraints.

Feasible set: Set of all feasible solutions.

Optimal value: Desired minimum.

Optimal solution: A feasible solution that achieves the optimal value.

Optimal set: Set of all optimal solutions.

Cost vector: Vector of cost weights for each decision variable.

Standard form:

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{Ax} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

1. Elimination of free variables.
2. Elimination of inequality constraints.

2 Geometry

Polyhedron:

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}.$$

Polyhedron in standard form:

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}.$$

Hyperplane:

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} = b\}.$$

Half space:

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} \geq b\}.$$

Convex function: $f(t\mathbf{p} + (1-t)\mathbf{q}) \leq tf(\mathbf{p}) + (1-t)f(\mathbf{q})$ for all $t \in [0, 1]$, \mathbf{p}, \mathbf{q} .

Convex combination: $\sum_{i=1}^k t_i \mathbf{x}_i$, where the nonnegative weights t_i sum up to 1.

Convex hull: the set of all convex combinations.

Convex set: contains all convex combinations of points \mathbf{p}, \mathbf{q} in the set.

Absolute values: Minimize $\sum_{i=1}^n c_i |x_i|$, where all $c_i \geq 0$:

$$|x_i| \rightarrow x_i^+ + x_i^-, \quad x_i \rightarrow x_i^+ - x_i^-.$$

Active constraint: holds with equality.

Extreme point: not a convex combination of two distinct points of the polyhedron.

Vertex: lies on separating hyperplane.

Basic solution: all equality constraints active, for a total of n linearly independent active constraints.

Extreme point=vertex=basic feasible solution.

Adjacent basic solutions in \mathbb{R}^n : share $n-1$ linearly independent active constraints.

Edge: line segment joining two adjacent basic feasible solutions.

Basic solutions in standard-form (feasible if $\mathbf{x}_B \geq \mathbf{0}$):

- i) $\mathbf{A}_{j_1}, \dots, \mathbf{A}_{j_m}$ are linearly independent;
- ii) if $i \neq j_1, \dots, j_m$, then $x_i = 0$.

$$\mathbf{B} = [\mathbf{A}_{j_1}, \dots, \mathbf{A}_{j_m}], \quad \mathbf{x}_B = \begin{bmatrix} x_{j_1} \\ \vdots \\ x_{j_m} \end{bmatrix}$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}.$$

Degenerate: more than n constraints are active at \mathbf{x} .

Degenerate basic solution in standard-form: some zero basic variables.

Contains a line.

Theorem 2.7: *Suppose that the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^\top \mathbf{x} \geq b_i, i = 1, \dots, k\}$ is nonempty. Then the following are equivalent:*

- (i) P does not contain a line.
- (ii) P has at least one extreme point.
- (iii) The set $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ contains n linearly independent vectors.

Theorem 2.9: *Consider the linear programming problem of minimizing $\mathbf{c}^\top \mathbf{x}$ over a polyhedron P . Suppose that P has at least one extreme point. Then either the optimal cost equals $-\infty$ or P has an optimal extreme point.*

Remark: Nonempty polyhedra that are bounded or in standard form always have an extreme point!

3 The Simplex Method

Basic components of j th simplex direction (feasible if nondegenerate):

$$\mathbf{d}_B = -\mathbf{B}^{-1}\mathbf{A}_j.$$

Reduced cost:

$$\bar{c}_j = c_j + \mathbf{c}_B^\top \mathbf{d}_B = c_j - \mathbf{c}_B^\top \mathbf{B}^{-1} \mathbf{A}_j.$$

Reduced cost of basic variable: 0.

Optimality: if $\bar{\mathbf{c}} \geq \mathbf{0}$, \mathbf{x} is optimal. If \mathbf{x} is optimal and nondegenerate, then $\bar{\mathbf{c}} \geq \mathbf{0}$.

Phase II: full tableau constructed from negative cost, reduced costs $\bar{\mathbf{c}}$, \mathbf{x}_B , and $\mathbf{B}^{-1}\mathbf{A}$.

Entering variable: negative reduced cost \bar{c}_j .

Exiting variable: minimize x_{j_i}/u_i , where $\mathbf{u} = -\mathbf{d}_B = \mathbf{B}^{-1}\mathbf{A}_j$.

Bland's rule: if there is more than one choice for the entering or exiting variable, always choose the one with the lowest subscript.

Phase I: look for an initial basic feasible solution by considering the auxillary problem

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^m y_i \\ &\text{subject to} && \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b}, \\ &&& \mathbf{x}, \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Optimization: omit from the tableau any artificial column that is a positive multiple of another column.

4 Duality

Primal vs. Dual:

$$\begin{array}{ll}
 \text{minimize} & \mathbf{c}^\top \mathbf{x} \\
 \text{subject to} & \mathbf{a}_i^\top \mathbf{x} \leq b_i, \quad i \in \mathcal{M}_1, \\
 & \mathbf{a}_i^\top \mathbf{x} \geq b_i, \quad i \in \mathcal{M}_2, \\
 & \mathbf{a}_i^\top \mathbf{x} = b_i, \quad i \in \mathcal{M}_3, \\
 & \mathbf{x}_j \leq 0, \quad j \in \mathcal{N}_1, \\
 & \mathbf{x}_j \geq 0, \quad j \in \mathcal{N}_2, \\
 & \mathbf{x}_j \text{ free}, \quad j \in \mathcal{N}_3,
 \end{array}
 \qquad
 \begin{array}{ll}
 \text{maximize} & \mathbf{b}^\top \mathbf{p} \\
 \text{subject to} & p_i \leq 0, \quad i \in \mathcal{M}_1, \\
 & p_i \geq 0, \quad i \in \mathcal{M}_2, \\
 & p_i \text{ free}, \quad i \in \mathcal{M}_3, \\
 & \mathbf{A}_j^\top \mathbf{p} \geq c_j, \quad j \in \mathcal{N}_1, \\
 & \mathbf{A}_j^\top \mathbf{p} \leq c_j, \quad j \in \mathcal{N}_2, \\
 & \mathbf{A}_j^\top \mathbf{p} = c_j, \quad j \in \mathcal{N}_3.
 \end{array}$$

	minimize	maximize	
constraints	$\leq b_i$	≤ 0	variables
	$\geq b_i$	≥ 0	
	$= b_i$	free	
variables	≤ 0	$\geq c_j$	constraints
	≥ 0	$\leq c_j$	
	free	$= c_j$	

Primal \ Dual	Finite optimum	Unbounded	Infeasible
Finite optimum	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Theorem 4.3 (Weak duality): *If \mathbf{x} is a feasible solution to the primal problem and \mathbf{p} is a feasible solution to the dual problem, then*

$$\mathbf{p}^\top \mathbf{b} \leq \mathbf{c}^\top \mathbf{x}.$$

Corollary 4.3.2: Let \mathbf{x} and \mathbf{p} be feasible solutions to the primal and dual problems, respectively, and suppose that $\mathbf{p}^\top \mathbf{b} = \mathbf{c}^\top \mathbf{x}$. Then \mathbf{x} and \mathbf{p} are optimal solutions.

Theorem 4.4 (Strong duality): *If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.*

Theorem 4.5 (Complementary slackness): *Let \mathbf{x} and \mathbf{p} be feasible solutions to the primal and dual problem, respectively. Then \mathbf{x} and \mathbf{p} are optimal solutions if and only if*

$$p_i(\mathbf{a}_i^\top \mathbf{x} - b_i) = 0, \quad \text{for all } i = 1, \dots, m$$

and

$$(c_j - \mathbf{A}_j^\top \mathbf{p})x_j = 0, \quad \text{for all } j = 1, \dots, n.$$

Theorem 4.6 (Farkas' lemma): *Let \mathbf{A} be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. Exactly one of the following alternatives holds:*

- (i) $\mathbf{Ax} = \mathbf{b}$ has a solution $\mathbf{x} \geq \mathbf{0}$;
- (ii) $\mathbf{A}^\top \mathbf{p} \geq \mathbf{0}$ has a solution \mathbf{p} with $\mathbf{p}^\top \mathbf{b} < 0$.

5 Sensitivity Analysis

Look for conditions under which \mathbf{B} remains optimal:

1. feasibility: $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$;
2. optimality: $\bar{\mathbf{c}}^\top = \mathbf{c}^\top - \mathbf{c}_B^\top \mathbf{B}^{-1} \mathbf{A} \geq \mathbf{0}$.

Dual simplex method ($\bar{\mathbf{c}} \geq \mathbf{0}$):

Entering variable: lowest subscript with negative pivot-row entry v_j that minimizes $\bar{c}_j / -v_j$.

Exiting variable: lowest subscript with negative basic variable.