Review of Math 373

1 Linear Programming

Objective function: Linear function (cost) to be minimized.

Constraints: Linear inequalities or equalities on decision variables.

Feasible solution: Decision variables that satisfy the constraints.

Feasible set: Set of all feasible solutions.

Optimal value: Desired minimum.

Optimal solution: A feasible solution that achieves the optimal value.

Optimal set: Set of all optimal solutions.

Cost vector: Vector of cost weights for each decision variable.

Standard form:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

1. Elimination of free variables.
2. Elimination of inequality constraints.

2 Geometry

Polyhedron:

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}.$$

Polyhedron in standard form:

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}.$$

Hyperplane:

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}^\intercal \boldsymbol{x} = b\}.$$

Half space:

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}^\intercal \boldsymbol{x} \geq b\}.$$

Convex function: $f(t\boldsymbol{p} + (1-t)\boldsymbol{q}) \leq tf(\boldsymbol{p}) + (1-t)f(\boldsymbol{q})$ for all $t \in [0,1]$, $\boldsymbol{p}, \boldsymbol{q}$.

Convex combination: $\sum_{i=1}^k t_i \boldsymbol{x}_i$, where the nonnegative weights t_i sum up to 1.

Convex hull: the set of all convex combinations.

Convex set: contains all convex combinations of points $\boldsymbol{p}, \boldsymbol{q}$ in the set.

Absolute values: Minimize $\sum_{i=1}^n c_i |x_i|$, where all $c_i \geq 0$:

$$|x_i| \to x_i^+ + x_i^-, \qquad x_i \to x_i^+ - x_i^-.$$

Active constraint: holds with equality.

Extreme point: not a convex combination of two distinct points of the polyhedron.

Vertex: lies on separating hyperplane.

Basic solution: all equality constraints active, for a total of n linearly independent active constraints.

Extreme point=vertex=basic feasible solution.

Adjacent basic solutions in $\mathbb{R}^n$: share $n-1$ linearly independent active constraints.

Edge: line segment joining two adjacent basic feasible solutions.

Basic solutions in standard-form (feasible if $\boldsymbol{x}_B \geq \mathbf{0}$):

i) $\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_m}$ are linearly independent;

ii) if $i \neq j_1, \ldots, j_m$, then $x_i = 0$.

$$\boldsymbol{B} = [\,\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_m}\,], \qquad \boldsymbol{x}_B = \begin{bmatrix} x_{j_1} \\ \vdots \\ x_{j_m} \end{bmatrix}$$

$$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}.$$

Degenerate: more than n constraints are active at $\boldsymbol{x}$.

Degenerate polyhedra in standard-form: some zero basic variables.

Contains a line.

Theorem 2.7: *Suppose that the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i^\intercal \boldsymbol{x} \geq b_i, i = 1, \ldots, k\}$ is nonempty. Then the following are equivalent:*

(i) *P does not contain a line.*

(ii) *P has at least one extreme point.*

(iii) *The set $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$ contains n linearly independent vectors.*

Theorem 2.9: *Consider the linear programming problem of minimizing $\mathbf{c}^\intercal \boldsymbol{x}$ over a polyhedron P. Suppose that P has at least one extreme point. Then either the optimal cost equals $-\infty$ or P has an optimal extreme point.*

Remark: Nonempty polyhedra that are bounded *or* in standard form always have an extreme point!

3 The Simplex Method

Basic components of jth simplex direction (feasible if nondegenerate):

$$\boldsymbol{d}_B = -\boldsymbol{B}^{-1}\boldsymbol{A}_j.$$

Reduced cost:

$$\bar{c}_j = c_j + \boldsymbol{c}_B^\intercal \boldsymbol{d}_B = c_j - \boldsymbol{c}_B^\intercal \boldsymbol{B}^{-1}\boldsymbol{A}_j.$$

Reduced cost of basic variable: 0.

Optimality: if $\bar{\boldsymbol{c}} \geq \boldsymbol{0}$, $\boldsymbol{x}$ is optimal. If $\boldsymbol{x}$ is optimal and nondegenerate, then $\bar{\boldsymbol{c}} \geq \boldsymbol{0}$.

Phase II: full tableau constructed from negative cost, reduced costs $\bar{\boldsymbol{c}}$, $\boldsymbol{x}_B$, and $\boldsymbol{B}^{-1}\boldsymbol{A}$.

Entering variable: negative reduced cost $\bar{c}_j$.

Exiting variable: minimize x_{j_i}/u_i, where $\boldsymbol{u} = -\boldsymbol{d}_B = \boldsymbol{B}^{-1}\boldsymbol{A}_j$.

Bland's rule: if there is more than than one choice for the entering or exiting variable, always choose the one with the lowest subscript.

Phase I: look for an initial basic feasible solution by considering the auxillary problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} y_i \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \\ & \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned}$$

Optimization: omit from the tableau any artificial column that is a positive multiple of another column.

4 Duality

Primal vs. Dual:

$$
\begin{array}{lll}
\text{minimize} & \boldsymbol{c}^\intercal \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{a}_i^\intercal \boldsymbol{x} \le b_i, & i \in \mathcal{M}_1, \\
& \boldsymbol{a}_i^\intercal \boldsymbol{x} \ge b_i, & i \in \mathcal{M}_2, \\
& \boldsymbol{a}_i^\intercal \boldsymbol{x} = b_i, & i \in \mathcal{M}_3, \\
& \boldsymbol{x}_j \le 0, & j \in \mathcal{N}_1, \\
& \boldsymbol{x}_j \ge 0, & j \in \mathcal{N}_2, \\
& \boldsymbol{x}_j \text{ free}, & j \in \mathcal{N}_3,
\end{array}
\qquad
\begin{array}{lll}
\text{maximize} & \boldsymbol{b}^\intercal \boldsymbol{p} & \\
\text{subject to} & p_i \le 0, & i \in \mathcal{M}_1, \\
& p_i \ge 0, & i \in \mathcal{M}_2, \\
& p_i \text{ free}, & i \in \mathcal{M}_3, \\
& \boldsymbol{A}_j^\intercal \boldsymbol{p} \ge c_j, & j \in \mathcal{N}_1, \\
& \boldsymbol{A}_j^\intercal \boldsymbol{p} \le c_j, & j \in \mathcal{N}_2, \\
& \boldsymbol{A}_j^\intercal \boldsymbol{p} = c_j, & j \in \mathcal{N}_3.
\end{array}
$$

	minimize	maximize	
constaints	$\le b_i$ $\ge b_i$ $= b_i$	≤ 0 ≥ 0 free	variables
variables	≤ 0 ≥ 0 free	$\ge c_j$ $\le c_j$ $= c_j$	constraints

Primal \ Dual	Finite optimum	Unbounded	Infeasible
Finite optimum	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Theorem 4.3 (Weak duality): *If $\boldsymbol{x}$ is a feasible solution to the primal problem and $\boldsymbol{p}$ is a feasible solution to the dual problem, then*

$$\boldsymbol{p}^\intercal \boldsymbol{b} \le \boldsymbol{c}^\intercal \boldsymbol{x}.$$

Corollary 4.3.2: Let $\boldsymbol{x}$ and $\boldsymbol{p}$ be feasible solutions to the primal and dual problems, respectively, and suppose that $\boldsymbol{p}^\intercal \boldsymbol{b} = \boldsymbol{c}^\intercal \boldsymbol{x}$. Then $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal solutions.

Theorem 4.4 (Strong duality): *If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.*

Theorem 4.5 (Complementary slackness): *Let $\boldsymbol{x}$ and $\boldsymbol{p}$ be feasible solutions to the primal and dual problem, respectively. Then $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal solutions if and only if*

$$p_i(\boldsymbol{a}_i^\intercal \boldsymbol{x} - b_i) = 0, \quad \textit{for all } i = 1, \ldots, m$$

and

$$(c_j - \boldsymbol{A}_j^\intercal \boldsymbol{p}) x_j = 0, \quad \textit{for all } j = 1, \ldots, n.$$

Theorem 4.6 (Farkas' lemma): *Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{b} \in \mathbb{R}^m$. Exactly one of the following alternatives holds:*

(i) $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *has a solution* $\boldsymbol{x} \geq \boldsymbol{0}$*;*

(ii) $\boldsymbol{A}^\mathsf{T}\boldsymbol{p} \geq \boldsymbol{0}$ *has a solution* $\boldsymbol{p}$ *with* $\boldsymbol{p}^\mathsf{T}\boldsymbol{b} < 0$.

5 Sensitivity Analysis

Look for conditions under which $\boldsymbol{B}$ remains optimal:

1. feasibility: $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$;
2. optimality: $\bar{\boldsymbol{c}}^\mathsf{T} = \boldsymbol{c}^\mathsf{T} - \boldsymbol{c}_B^\mathsf{T}\boldsymbol{B}^{-1}\boldsymbol{A} \geq \boldsymbol{0}$.

Dual simplex method ($\bar{\boldsymbol{c}} \geq \boldsymbol{0}$):

Entering variable: lowest subscript with negative pivot-row entry v_j that minimizes $\bar{c}_j/-v_j$.

Exiting variable: lowest subscript with negative basic variable.