Math 373: Mathematical Programming and Optimization I

Fall, 2023 Assignment 4

November 30, due December 8

1. Consider a two-player game G with the following payoff matrix pair

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 3 & 0 \\ 5 & 5 \end{pmatrix}. \tag{1}$$

Denote the strategy of the first player by $\boldsymbol{y}$ and of the second player by $\boldsymbol{z}$.

A Nash equilibrium $(\boldsymbol{y}, \boldsymbol{z})$ must satisfy

$$\boldsymbol{y}^\intercal \boldsymbol{A}\boldsymbol{z} \geq \boldsymbol{x}^\intercal \boldsymbol{A}\boldsymbol{z}, \tag{2a}$$
$$\boldsymbol{y}^\intercal \boldsymbol{B}\boldsymbol{z} \geq \boldsymbol{y}^\intercal \boldsymbol{B}\boldsymbol{x}, \tag{2b}$$

for every $\boldsymbol{x} \in \Delta_1 \doteq \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1\}$.

Analytically determine the set of Nash equilibria of the game. Write down the corresponding linear programming problems. $\boxed{9}$

We have

$$y = \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix}, \quad y_1 \geq 0, \qquad z = \begin{bmatrix} z_1 \\ 1 - z_1 \end{bmatrix}, \quad z_1 \geq 0.$$

Condition (2a) implies that y must maximize $x^\intercal Az$ over $x \in \Delta_1$:

$$y = \arg\max_{x \in \Delta_1} x^\intercal Az$$

We note that $\boldsymbol{y}$ is fully determined by

$$\begin{aligned} y_1 &= \arg\max_{x_1} \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ 1 - z_1 \end{bmatrix} \\ &= \arg\max_{x_1} \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} 3z_1 \\ 2 \end{bmatrix} \\ &= \arg\max_{x_1} (3z_1 - 2)x_1 + 2. \end{aligned}$$

So based on the sign of $3z_1 - 2$, y_1 is given by

$$\begin{cases} y_1 = 1 & z_1 > \frac{2}{3} \\ y_1 \in [0, 1] & z_1 = \frac{2}{3} \\ y_1 = 0 & z_1 < \frac{2}{3} \end{cases}. \tag{3}$$

Condition (2b) implies that z must maximize $y^T Bx$ over $x \in \Delta_1$:

$$z = \arg\max_{x \in \Delta_1} y^\intercal Bx.$$

We note that $\boldsymbol{z}$ is fully determined by

$$
\begin{aligned}
z_1 &= \arg\max_{x_1} \begin{bmatrix} y_1 & 1-y_1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ 1-x_1 \end{bmatrix} \\
&= \arg\max_{x_1} \begin{bmatrix} 5-2y_1 & 5-5y_1 \end{bmatrix} \begin{bmatrix} x_1 \\ 1-x_1 \end{bmatrix} \\
&= \arg\max_{x_1} 3y_1 x_1 + (5-5y_1).
\end{aligned}
$$

So z_1 is given by

$$
\begin{cases} z_1 = 1 & y_1 > 0 \\ z_1 \in [0,1] & y_1 = 0 \end{cases}. \tag{4}
$$

Now (3) and (4) result in the following possibilities:

$$
\begin{cases} y_1 = 1, z_1 = 1 \\ y_1 = 0, z_1 \leq \frac{2}{3} \end{cases} \quad \Rightarrow \begin{cases} y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, z = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, z = \begin{bmatrix} z_1 \\ 1-z_1 \end{bmatrix}, 0 \leq z_1 \leq \frac{2}{3}. \end{cases}
$$

The corresponding linear programming problems are, given (z_1, z_2):

$$
\begin{array}{llll}
\text{maximize} & 3z_1x_1 & + & 2x_2 \\
\text{subject to} & x_1 & + & x_2 \quad = 1, \\
& x_1, x_2 \geq 0, & &
\end{array}
$$

and, given (y_1, y_2):

$$
\begin{array}{llll}
\text{maximize} & (5-2y_1)x_1 & + & 5(1-y_1)x_2 \\
\text{subject to} & x_1 & + & x_2 \quad = 1, \\
& x_1, x_2 \geq 0. & &
\end{array}
$$

2. Consider a two-player game G' with payoff matrices

$$
\boldsymbol{A}' = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \boldsymbol{B}' = \begin{pmatrix} 6 & 0 \\ 10 & 10 \end{pmatrix}. \tag{5}
$$

Referring to Question 1, is it possible to make a statement about the set of Nash equilibria of G' by just comparing the payoff matrices in Eqs. (1) and (5)? Explain.
Hint: Do the Nash equilibria change if you add a constant to all entries of a column of $\boldsymbol{A}$? Justify your answer. $\boxed{3}$

Adding constants c_j to the jth column of A does not change the set of Nash equilibria:

$$\begin{aligned}
\arg\max_{x\,\in\Delta_1}\left(\sum_{i,j} x_i(A_{ij}+c_j)z_j\right) &= \arg\max_{x\in\Delta_1}\left(\sum_{i,j} x_i A_{ij} z_j + \sum_i x_i \sum_j c_j z_j\right) \\
&= \arg\max_{x\in\Delta_1}\left(\sum_{i,j} x_i A_{ij} z_j + \sum_j c_j z_j\right) \\
&= \arg\max_{x\in\Delta_1}\sum_{i,j} x_i A_{ij} z_j .
\end{aligned}$$

The same holds true when adding a constant to all entries of any row of B or when a payoff matrix is multiplied by a positive constant.

Now if we add -3 to the first column of A and -1 to the second column of A, we get A'. Moreover, if we multiply B by 2, we get B'. Hence, G and G' have the same Nash equilibria.