Math 373: Mathematical Programming and Optimization I

Fall, 2025 Assignment 3

October 30, solutions available November 11

1. Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{b} \in \mathbb{R}^m$. Prove that exactly one of these alternatives holds:

   (i) $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$;

   (ii) $\boldsymbol{A}^\intercal \boldsymbol{p} = \boldsymbol{0}$ has a solution $\boldsymbol{p}$ with $\boldsymbol{p}^\intercal \boldsymbol{b} \neq 0$. 5

   Suppose $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$. If $\boldsymbol{A}^\intercal \boldsymbol{p} = \boldsymbol{0}$, then

$$\boldsymbol{p}^\intercal \boldsymbol{b} = \boldsymbol{p}^\intercal \boldsymbol{A}\boldsymbol{x} = (\boldsymbol{A}^\intercal \boldsymbol{p})^\intercal \boldsymbol{x} = \boldsymbol{0}^\intercal \boldsymbol{x} = \boldsymbol{0}.$$

   Thus (i)$\Rightarrow \overline{\text{(ii)}}$.

   Alternatively, if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ does not have a solution $\boldsymbol{x}$, then $\boldsymbol{b} \notin \operatorname{span}\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n\}$. On decomposing $\boldsymbol{b} = \boldsymbol{s} + \boldsymbol{p}$, where $\boldsymbol{s} \in \operatorname{span}\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n\}$ and $\boldsymbol{p}^\intercal \boldsymbol{A} = \boldsymbol{0}^\intercal$, it follows that $\boldsymbol{p} \neq \boldsymbol{0}$. Then $\boldsymbol{p}^\intercal \boldsymbol{b} = \boldsymbol{p}^\intercal \boldsymbol{s} + \boldsymbol{p}^\intercal \boldsymbol{p} = 0 + |\boldsymbol{p}|^2 > 0$. That is, $\overline{\text{(i)}} \Rightarrow$ (ii).

2. Let $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i^\intercal \boldsymbol{x} \geq b_i, i = 1, \ldots, m\}$ be a convex polyhedron.

   Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are distinct basic feasible solutions that satisfy $\boldsymbol{a}_i^\intercal \boldsymbol{u} = \boldsymbol{a}_i^\intercal \boldsymbol{v} = b_i$ for $i = 1, \ldots, n-1$, so that $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ are linearly independent.

   Let $L_1 = \{t\boldsymbol{u} + (1-t)\boldsymbol{v} : 0 \leq t \leq 1\}$ be the segment that joins $\boldsymbol{u}$ and $\boldsymbol{v}$ and $L_2 = \{\boldsymbol{w} \in P : \boldsymbol{a}_i^\intercal \boldsymbol{w} = b_i,\ i = 1, \ldots, n-1\}$. Prove that $L_1 = L_2$.

   We first show that $L_1 \subset L_2$. For any vector $t\boldsymbol{u} + (1-t)\boldsymbol{v} \in L_1$, we have

$$a_i^\intercal \Big(t\boldsymbol{u} + (1-t)\boldsymbol{v} \Big) = t a_i^\intercal \boldsymbol{u} + (1-t) a_i^\intercal \boldsymbol{v} = t b_i + (1-t) b_i = b_i.$$

   Hence $t\boldsymbol{u} + (1-t)\boldsymbol{v} \in L_2$. That is, $L_1 \subset L_2$.

   Next, we show $L_2 \subset L_1$. The vectors $\boldsymbol{w} \in L_2$ satisfy a system of linear equations $\boldsymbol{a}_i^\intercal \boldsymbol{w} = b_i$ for $i = 1, \ldots, n-1$. Since the constraint vectors $\boldsymbol{a}_i$ are linearly independent, they span a space of dimension $n-1$; the rank-nullity theorem then implies that L_2 is a subset of the one-dimensional space $\{\boldsymbol{w} \in \mathbb{R}^n : \boldsymbol{a}_i^\intercal \boldsymbol{w} = b_i,\ i = 1, \ldots, n-1\}$, which is a line. Since $\boldsymbol{u}, \boldsymbol{v} \in L_2$, we know that this is the line through $\boldsymbol{u}$ and $\boldsymbol{v}$. Let $\boldsymbol{w} \in L_2 \subset P$. Then $\boldsymbol{w} = t\boldsymbol{u} + (1-t)\boldsymbol{v}$, for some $t \in \mathbb{R}$. If $t > 1$, then $\boldsymbol{u}$ lies between $\boldsymbol{w} \in P$ and $\boldsymbol{v} \in P$ and would therefore not be an extreme point (basic feasible solution) of P. If $t < 0$, then $\boldsymbol{v}$ would lie between $\boldsymbol{u} \in P$ and $\boldsymbol{w} \in P$, and would not be an extreme point of P. These contradictions establish that $t \in [0, 1]$, so that $L_2 \subset L_1$. It follows that $L_1 = L_2$. 5