Math 373: Mathematical Programming and Optimization I

Fall, 2023 Assignment 2

October 6, due October 26

1. Let P be a convex polyhedron. Complete the proof of the following theorem.

   Theorem 1: *A point $\boldsymbol{x} \in P$ is an extreme point of P if and only if the set $P \setminus \{\boldsymbol{x}\}$ (the set obtained by removing $\boldsymbol{x}$ from P) is convex.*

   Proof: Let $\boldsymbol{x} \in P$. Suppose $P \setminus \{\boldsymbol{x}\}$ is convex. Then it contains every convex combination of points $\boldsymbol{y}, \boldsymbol{z} \in P \setminus \{\boldsymbol{x}\}$. Since $\boldsymbol{x}$ does not belong to $P \setminus \{\boldsymbol{x}\}$ it cannot be expressed as a convex combination of points $\boldsymbol{y}$ and $\boldsymbol{z}$ in $P \setminus \{\boldsymbol{x}\}$. That is, $\boldsymbol{x}$ is an extreme point of P. 4

   Suppose $\boldsymbol{x} \in P$ is an extreme point of P. If $P \setminus \{\boldsymbol{x}\}$ were not convex, there would exist points $\boldsymbol{y}, \boldsymbol{z} \in P \setminus \{\boldsymbol{x}\} \subset P$ and $t \in (0, 1)$ such that $t\boldsymbol{y} + (1-t)\boldsymbol{z} \notin P \setminus \{\boldsymbol{x}\}$. But since P is convex, we know that $t\boldsymbol{y} + (1-t)\boldsymbol{z} \in P$. Thus $t\boldsymbol{y} + (1-t)\boldsymbol{z} = \boldsymbol{x}$, contradicting the definition of an extreme point. Thus $P \setminus \{\boldsymbol{x}\}$ must be convex. 5

2. Let $\boldsymbol{x}$ be an element of the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$. Prove that a vector $\boldsymbol{d} \in \mathbb{R}^n$ is a feasible direction at $\boldsymbol{x}$ if and only if $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$ and $d_i \geq 0$ for every i such that $x_i = 0$. 5

   If $\boldsymbol{x}$ is a feasible direction, then $\boldsymbol{x} + t\boldsymbol{d} \in P$ for some positive scalar t. That is, $\boldsymbol{A}(\boldsymbol{x} + t\boldsymbol{d}) = \boldsymbol{b}$ and $\boldsymbol{x} + t\boldsymbol{d} \geq \boldsymbol{0}$. Then $t\boldsymbol{A}\boldsymbol{d} = \boldsymbol{A}(\boldsymbol{x} + t\boldsymbol{d}) - \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0}$, so that $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$. Moreover, for each zero component x_i, the condition $x_i + td_i \geq 0$ reduces to $d_i \geq 0$.

   Conversely, if there exists a direction such that $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$ and $d_i \geq 0$ for every i such that $x_i = 0$, then $\boldsymbol{A}(\boldsymbol{x} + t\boldsymbol{d}) = \boldsymbol{A}\boldsymbol{x} + t\boldsymbol{A}\boldsymbol{d} = \boldsymbol{b} + \boldsymbol{0} = \boldsymbol{b}$ for every real t. We are also given that $x_i + td_i = td_i \geq 0$ for every component i such that $x_i = 0$. For those i for which $x_i > 0$, then unless $d_i = 0$ (in which case $x_i + td_i > 0$ for every t) let us enforce $t \leq x_i/|d_i| > 0$, so that $x_i + td_i \geq |td_i| + td_i \geq 0$. On choosing $t^* = \min\limits_{\substack{i \\ x_i > 0, d_i \neq 0}} x_i/|d_i| > 0$, we thus see that $\boldsymbol{x} + t^*\boldsymbol{d} \geq \boldsymbol{0}$. Hence $\boldsymbol{d}$ is a feasible direction at $\boldsymbol{x}$.

3. Let $\boldsymbol{x}$ be a basic feasible solution of a linear programming problem Π written in standard form, with associated basis matrix $\boldsymbol{B}$ and set of nonbasic indices N. Let $\boldsymbol{y}$ be any feasible solution to Π and consider the difference vector $\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x}$.

   (a) Prove that $d_j \geq 0$ for every $j \in N$. 1

   For any feasible solution $\boldsymbol{y}$ we have $\boldsymbol{y} \geq \boldsymbol{0}$. Since $\boldsymbol{x}$ is a basic feasible solution, we know for each $j \in N$ that $x_j = 0$ and hence $d_j = y_j - x_j \geq 0$.

(b) If $d_j = 0$ for every $j \in N$, prove that $\boldsymbol{y} = \boldsymbol{x}$. 1

This would imply that

$$\mathbf{0} = \boldsymbol{A}\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{d} = \boldsymbol{B}\boldsymbol{d}_B + \sum_{j \in N} \boldsymbol{A}_j d_j = \boldsymbol{B}\boldsymbol{d}_B.$$

The linear independence of the columns of $\boldsymbol{B}$ then implies that $\boldsymbol{d}_B = \mathbf{0}$ and hence $\boldsymbol{d} = \mathbf{0}$, so that $\boldsymbol{y} = \boldsymbol{x}$.

(c) If the reduced cost $\bar{c}_j$ of every nonbasic variable x_j is positive, use parts (a) and (b) to prove that $\boldsymbol{x}$ is the unique optimal solution to Π. 2

Recall that $\bar{c}_j$ is the rate of change along the jth simplex direction. That is, the change in cost on moving from $\boldsymbol{x}$ to $\boldsymbol{y}$ is

$$\boldsymbol{c}^\mathsf{T}\boldsymbol{y} - \boldsymbol{c}^\mathsf{T}\boldsymbol{x} = \boldsymbol{c}^\mathsf{T}\boldsymbol{d} = \boldsymbol{c}_B^\mathsf{T}\boldsymbol{d}_B + \sum_{j \in N} c_j d_j = \sum_{j \in N} (c_j - c_B^\mathsf{T}\boldsymbol{B}^{-1}\boldsymbol{A}_j) d_j = \sum_{j \in N} \bar{c}_j d_j.$$

We know from part (a) that $d_j \geq 0$. Moreover, if $\boldsymbol{y} \neq \boldsymbol{x}$, we know from part (b) that $d_j > 0$ for some $j \in N$. Given $\bar{c}_j > 0$ for each $j \in N$, we see that

$$\boldsymbol{c}^\mathsf{T}\boldsymbol{y} - \boldsymbol{c}^\mathsf{T}\boldsymbol{x} = \sum_{j \in N} \bar{c}_j d_j > 0.$$

Since this holds for every feasible vector $\boldsymbol{y} \neq \boldsymbol{x}$, we see that $\boldsymbol{x}$ is the unique optimal solution.

(d) Suppose that $\boldsymbol{x}$ is a nondegenerate optimal solution to Π. If the reduced cost $\bar{c}_j$ of some nonbasic variable x_j is zero at $\boldsymbol{x}$, prove that Π does not have a unique optimal solution. 1

Let $\boldsymbol{d}'$ be the jth simplex direction. Since $\boldsymbol{x}$ is nondegenerate, we know that the solution $\boldsymbol{y} = \boldsymbol{x} + t\boldsymbol{d}'$ is feasible for some $t > 0$. From the definition of the jth simplex direction, we see that

$$\boldsymbol{c}^\mathsf{T}\boldsymbol{y} - \boldsymbol{c}^\mathsf{T}\boldsymbol{x} = t\bar{c}_j d'_j = 0.$$

That is, $\boldsymbol{y}$ is a distinct feasible solution with the same optimal cost as $\boldsymbol{x}$.