

# Math 373: Mathematical Programming and Optimization I

Fall, 2025 Assignment 2

October 14, solutions available October 21

1. Let  $\mathbf{x}$  be an element of the polyhedron  $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ . Prove that a vector  $\mathbf{d} \in \mathbb{R}^n$  is a feasible direction at  $\mathbf{x}$  if and only if  $\mathbf{A}\mathbf{d} = \mathbf{0}$  and  $d_i \geq 0$  for every  $i$  such that  $x_i = 0$ . 5

If  $\mathbf{d}$  is a feasible direction, then  $\mathbf{x} + t\mathbf{d} \in P$  for some positive scalar  $t$ . That is,  $\mathbf{A}(\mathbf{x} + t\mathbf{d}) = \mathbf{b}$  and  $\mathbf{x} + t\mathbf{d} \geq \mathbf{0}$ . Then  $t\mathbf{A}\mathbf{d} = \mathbf{A}(\mathbf{x} + t\mathbf{d}) - \mathbf{A}\mathbf{x} = \mathbf{b} - \mathbf{b} = \mathbf{0}$ , so that  $\mathbf{A}\mathbf{d} = \mathbf{0}$ . Moreover, for each zero component  $x_i$ , the condition  $x_i + td_i \geq 0$  reduces to  $d_i \geq 0$ .

Conversely, if there exists a direction such that  $\mathbf{A}\mathbf{d} = \mathbf{0}$  and  $d_i \geq 0$  for every  $i$  such that  $x_i = 0$ , then  $\mathbf{A}(\mathbf{x} + t\mathbf{d}) = \mathbf{A}\mathbf{x} + t\mathbf{A}\mathbf{d} = \mathbf{b} + \mathbf{0} = \mathbf{b}$  for every real  $t$ . We are also given that  $x_i + td_i = td_i \geq 0$  for every component  $i$  such that  $x_i = 0$ . For those  $i$  for which  $x_i > 0$ , then unless  $d_i = 0$  (in which case  $x_i + td_i > 0$  for every  $t$ ) let us enforce  $t \leq x_i/|d_i| > 0$ , so that  $x_i + td_i \geq |td_i| + td_i \geq 0$ . On choosing  $t^* = \min_{x_i > 0, d_i \neq 0} x_i/|d_i| > 0$ ,

we thus see that  $\mathbf{x} + t^*\mathbf{d} \geq \mathbf{0}$ . Hence  $\mathbf{d}$  is a feasible direction at  $\mathbf{x}$ .

2. Let  $\mathbf{x}$  be a basic feasible solution of a linear programming problem  $\Pi$  written in standard form, with associated basis matrix  $\mathbf{B}$  and set of nonbasic indices  $N$ . Let  $\mathbf{y}$  be any feasible solution to  $\Pi$  and consider the difference vector  $\mathbf{d} = \mathbf{y} - \mathbf{x}$ .

(a) Prove that  $d_j \geq 0$  for every  $j \in N$ . 1

For any feasible solution  $\mathbf{y}$  we have  $\mathbf{y} \geq \mathbf{0}$ . Since  $\mathbf{x}$  is a basic feasible solution, we know for each  $j \in N$  that  $x_j = 0$  and hence  $d_j = y_j - x_j \geq 0$ .

(b) If  $d_j = 0$  for every  $j \in N$ , prove that  $\mathbf{y} = \mathbf{x}$ . 1

This would imply that

$$\mathbf{0} = \mathbf{A}\mathbf{y} - \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{d} = \mathbf{B}\mathbf{d}_B + \sum_{j \in N} \mathbf{A}_j d_j = \mathbf{B}\mathbf{d}_B.$$

The linear independence of the columns of  $\mathbf{B}$  then implies that  $\mathbf{d}_B = \mathbf{0}$  and hence  $\mathbf{d} = \mathbf{0}$ , so that  $\mathbf{y} = \mathbf{x}$ .

(c) If the reduced cost  $\bar{c}_j$  of every nonbasic variable  $x_j$  is positive, use parts (a) and (b) to prove that  $\mathbf{x}$  is the unique optimal solution to  $\Pi$ . 2

Recall that  $\bar{c}_j$  is the rate of change along the  $j$ th simplex direction. That is, the change in cost on moving from  $\mathbf{x}$  to  $\mathbf{y}$  is

$$\mathbf{c}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x} = \mathbf{c}^\top \mathbf{d} = \mathbf{c}_B^\top \mathbf{d}_B + \sum_{j \in N} c_j d_j = \sum_{j \in N} (c_j - \mathbf{c}_B^\top \mathbf{B}^{-1} \mathbf{A}_j) d_j = \sum_{j \in N} \bar{c}_j d_j.$$

We know from part (a) that  $d_j \geq 0$ . Moreover, if  $\mathbf{y} \neq \mathbf{x}$ , we know from part (b) that  $d_j > 0$  for some  $j \in N$ . Given  $\bar{c}_j > 0$  for each  $j \in N$ , we see that

$$\mathbf{c}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x} = \sum_{j \in N} \bar{c}_j d_j > 0.$$

Since this holds for every feasible vector  $\mathbf{y} \neq \mathbf{x}$ , we see that  $\mathbf{x}$  is the unique optimal solution.

(d) Suppose that  $\mathbf{x}$  is a nondegenerate optimal solution to  $\Pi$ . If the reduced cost  $\bar{c}_j$  of some nonbasic variable  $x_j$  is zero at  $\mathbf{x}$ , prove that  $\Pi$  does not have a unique optimal solution. 1

Let  $\mathbf{d}'$  be the  $j$ th simplex direction. Since  $\mathbf{x}$  is nondegenerate, we know that the solution  $\mathbf{y} = \mathbf{x} + t\mathbf{d}'$  is feasible for some  $t > 0$ . From the definition of the  $j$ th simplex direction, we see that

$$\mathbf{c}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x} = t\bar{c}_j d'_j = 0.$$

That is,  $\mathbf{y}$  is a distinct feasible solution with the same optimal cost as  $\mathbf{x}$ .