

# Math 373: Mathematical Programming and Optimization I

Fall, 2025 Assignment 1

September 24, solutions available on October 7

1. Suppose that  $(x_1, x_2, x_3)$  is a feasible solution to the linear programming problem

$$\begin{aligned} &\text{minimize} && 4x_1 + 2x_2 + x_3 \\ &\text{subject to} && x_1 - x_2 \geq 3, \\ &&& 2x_1 + x_2 + x_3 \geq 4, \\ &&& x_1, x_2, x_3 \geq 0. \end{aligned}$$

Let  $y_1$  and  $y_2$  be non-negative numbers.

- (a) Show that

$$x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3y_2 \geq 3y_1 + 4y_2.$$

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On summing the first constraint multiplied by  $y_1$  with the second constraint multiplied by  $y_2$ , we obtain

$$y_1(x_1 - x_2) + y_2(2x_1 + x_2 + x_3) \geq 3y_1 + 4y_2.$$

The desired result then follows on rearranging to isolate  $x_1$ ,  $x_2$ , and  $x_3$ .

- (b) Find constraints on  $y_1$  and  $y_2$  so that

$$4x_1 + 2x_2 + x_3 \geq x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3y_2$$

at every feasible solution  $(x_1, x_2, x_3)$ .

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We require that  $y_1 + 2y_2 \leq 4$ ,  $-y_1 + y_2 \leq 2$ , and  $y_2 \leq 1$ .

- (c) Use parts (a) and (b) to find a lower bound to the optimal cost in terms of only the variables  $y_1$  and  $y_2$ .

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By transitivity, we find

$$4x_1 + 2x_2 + x_3 \geq x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3y_2 \geq 3y_1 + 4y_2.$$

- (d) Formulate the linear programming problem in the variables  $(y_1, y_2)$  that determines the **largest possible value** for the lower bound to the optimal cost found in part (c).

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$$\begin{aligned} &\text{maximize} && 3y_1 + 4y_2 \\ &\text{subject to} && y_1 + 2y_2 \leq 4, \\ &&& -y_1 + y_2 \leq 2, \\ &&& y_2 \leq 1, \\ &&& y_1, y_2 \geq 0. \end{aligned}$$

This is known as the *dual problem* to the original linear programming problem.

2. Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function on an interval  $I$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex increasing (but not necessarily differentiable) function on  $\mathbb{R}$ . Show that  $f \circ g$  is convex on  $I$ .

Let  $\mathbf{p}, \mathbf{q} \in I$ . Since  $g$  is convex on  $\mathbb{R}$  we know that

$$g(t\mathbf{p} + (1-t)\mathbf{q}) \leq tg(\mathbf{p}) + (1-t)g(\mathbf{q}).$$

Since  $f$  is increasing and convex on  $\mathbb{R}$ ,

$$f(g(t\mathbf{p} + (1-t)\mathbf{q})) \leq f(tg(\mathbf{p}) + (1-t)g(\mathbf{q})) \leq tf(g(\mathbf{p})) + (1-t)f(g(\mathbf{q}))$$

for all  $t \in [0, 1]$ . Hence  $f \circ g$  is convex on  $\mathbb{R}$ . 3

3. Let  $P$  be a convex polyhedron. Complete the proof of the following theorem.

**Theorem 1:** *A point  $\mathbf{x} \in P$  is an extreme point of  $P$  if and only if the set  $P \setminus \{\mathbf{x}\}$  (the set obtained by removing  $\mathbf{x}$  from  $P$ ) is convex.*

Proof: Let  $\mathbf{x} \in P$ . Suppose  $P \setminus \{\mathbf{x}\}$  is convex. Then it contains every convex combination of points  $\mathbf{y}, \mathbf{z} \in P \setminus \{\mathbf{x}\}$ . Since  $\mathbf{x}$  does not belong to  $P \setminus \{\mathbf{x}\}$  it cannot be expressed as a convex combination of points  $\mathbf{y}$  and  $\mathbf{z}$  in  $P \setminus \{\mathbf{x}\}$ . That is,  $\mathbf{x}$  is an extreme point of  $P$ . 4

Suppose  $\mathbf{x} \in P$  is an extreme point of  $P$ . If  $P \setminus \{\mathbf{x}\}$  were not convex, there would exist points  $\mathbf{y}, \mathbf{z} \in P \setminus \{\mathbf{x}\} \subset P$  and  $t \in (0, 1)$  such that  $t\mathbf{y} + (1-t)\mathbf{z} \notin P \setminus \{\mathbf{x}\}$ . But since  $P$  is convex, we know that  $t\mathbf{y} + (1-t)\mathbf{z} \in P$ . Thus  $t\mathbf{y} + (1-t)\mathbf{z} = \mathbf{x}$ , contradicting the definition of an extreme point. Thus  $P \setminus \{\mathbf{x}\}$  must be convex. 5