Math 373: Mathematical Programming and Optimization I

Fall, 2023 Assignment 1

September 18, due October 3

1. Suppose that (x_1, x_2, x_3) is a feasible solution to the linear programming problem

$$\begin{array}{lrl} \text{minimize} & 4x_1 + 2x_2 + x_3 & \\ \text{subject to} & x_1 - x_2 & \geq 3, \\ & 2x_1 + x_2 + x_3 & \geq 4, \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

Let y_1 and y_2 be non-negative numbers.

(a) Show that

$$x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3y_2 \geq 3y_1 + 4y_2.$$

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On summing the first constraint multiplied by y_1 with the second constraint multiplied by y_2, we obtain

$$y_1(x_1 - x_2) + y_2(2x_1 + x_2 + x_3) \geq 3y_1 + 4y_2.$$

The desired result then follows on rearranging to isolate x_1, x_2, and x_3.

(b) Find constraints on y_1 and y_2 so that

$$4x_1 + 2x_2 + x_3 \geq x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3y_2$$

at every feasible solution (x_1, x_2, x_3). 1

We require that $y_1 + 2y_2 \leq 4$, $-y_1 + y_2 \leq 2$, and $y_2 \leq 1$.

(c) Use parts (a) and (b) to find a lower bound to the optimal cost in terms of only the variables y_1 and y_2. 1

By transitivity, we find

$$4x_1 + 2x_2 + x_3 \geq x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) + x_3y_2 \geq 3y_1 + 4y_2.$$

(d) Formulate the linear programming problem in the variables (y_1, y_2) that determines the **largest possible value** for the lower bound to the optimal cost found in part (c). 1

$$\begin{array}{lrl} \text{maximize} & 3y_1 + 4y_2 & \\ \text{subject to} & y_1 + 2y_2 & \leq 4, \\ & -y_1 + y_2 & \leq 2, \\ & y_2 & \leq 1, \\ & y_1, y_2 & \geq 0. \end{array}$$

This is known as the *dual problem* to the original linear programming problem.

2. Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex increasing (but not necessarily differentiable) function on $\mathbb{R}$. Show that $f \circ g$ is convex on I.

Let $\boldsymbol{p}, \boldsymbol{q} \in I$. Since g is convex on $\mathbb{R}$ we know that

$$g(t\boldsymbol{p} + (1-t)\boldsymbol{q}) \leq tg(\boldsymbol{p}) + (1-t)g(\boldsymbol{q}).$$

Since f is increasing and convex on $[g(\boldsymbol{p}), g(\boldsymbol{q})]$:

$$f(g(t\boldsymbol{p} + (1-t)\boldsymbol{q})) \leq f(tg(\boldsymbol{p}) + (1-t)g(\boldsymbol{q})) \leq tf(g(\boldsymbol{p})) + (1-t)f(g(\boldsymbol{q})).$$

Hence $f \circ g$ is convex on $\mathbb{R}$. 3

3. Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{b} \in \mathbb{R}^m$. Prove that exactly one of these alternatives holds:

   (i) $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$;

   (ii) $\boldsymbol{A}^\intercal \boldsymbol{p} = \boldsymbol{0}$ has a solution $\boldsymbol{p}$ with $\boldsymbol{p}^\intercal \boldsymbol{b} \neq 0$. 5

Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$. If $\boldsymbol{A}^\intercal \boldsymbol{p} = \boldsymbol{0}$, then

$$\boldsymbol{p}^\intercal \boldsymbol{b} = \boldsymbol{p}^\intercal \boldsymbol{Ax} = (\boldsymbol{A}^\intercal \boldsymbol{p})^\intercal \boldsymbol{x} = \boldsymbol{0}^\intercal \boldsymbol{x} = \boldsymbol{0}.$$

Thus (i)$\Rightarrow \overline{\text{(ii)}}$.

Alternatively, if $\boldsymbol{Ax} = \boldsymbol{b}$ does not have a solution $\boldsymbol{x}$, then $\boldsymbol{b} \notin \operatorname{span}\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n\}$. On decomposing $\boldsymbol{b} = \boldsymbol{s} + \boldsymbol{p}$, where $\boldsymbol{s} \in \operatorname{span}\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n\}$ and $\boldsymbol{p}^\intercal \boldsymbol{A} = \boldsymbol{0}^\intercal$, it follows that $\boldsymbol{p} \neq \boldsymbol{0}$. Then $\boldsymbol{p}^\intercal \boldsymbol{b} = \boldsymbol{p}^\intercal \boldsymbol{s} + \boldsymbol{p}^\intercal \boldsymbol{p} = 0 + |\boldsymbol{p}|^2 > 0$. That is, $\overline{\text{(i)}} \Rightarrow \text{(ii)}$.