Math 225 (Q1) Homework Assignment 9.

1. Let $\mathcal{E} = \{\underline{e_1}, \underline{e_2}, \underline{e_3}\}$ be the standard basis for $\mathbf{R}^3$ and let $\mathcal{B} = \{\underline{b_1}, \underline{b_2}, \underline{b_3}\}$ be a basis for a vector space V. Suppose $T : \mathbf{R}^3 \to V$ is a linear transformation with the property that

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_3 - x_2)\underline{b_1} - (x_1 + x_3)\underline{b_2} + (x_1 - x_2)\underline{b_3}.$$

(a) Compute $T(\underline{e_1})$, $T(\underline{e_2})$ and $T(\underline{e_3})$.

(b) Compute the coordinate vectors (relative to $\mathcal{B}$) $[T(\underline{e_1})]_{\mathcal{B}}$, $[T(\underline{e_2})]_{\mathcal{B}}$ and $[T(\underline{e_3})]_{\mathcal{B}}$.

(c) Find the matrix for T relative to the bases $\mathcal{E}$ and $\mathcal{B}$.

2. Find the change–of–coordinates matrix, $P_{\mathcal{E}\leftarrow\mathcal{B}}$, from $\mathcal{B} = \left\{\begin{pmatrix} 2 \\ -9 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \end{pmatrix}\right\}$ to the standard basis $\mathcal{E} = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ in $\mathbf{R}^2$. Also find $P_{\mathcal{B}\leftarrow\mathcal{E}}$.

3. Let $T : V \to W$ be a linear transformation, with $\dim(V) = n$ and $\dim(W) = m$.

(a) If T is one-one (injective), what is $\dim(\text{Ran}(T))$? Explain. Hint: Let $\{\underline{b_1}, \cdots, \underline{b_n}\}$ be a basis of V. If T is one-one, then $\{T(\underline{b_1}), \cdots, T(\underline{b_n})\}$ is a basis of $\text{Ran}(T)$.

(b) If T is onto (surjective), what is $\dim(\text{Ker}(T))$? Explain. Hint: Let $\{\underline{b_1}, \cdots, \underline{b_k}\}$ be a basis of $\text{Ker}(T)$. Extend it to a basis $\{\underline{b_1}, \cdots, \underline{b_k}, \underline{b_{k+1}}, \cdots, \underline{b_n}\}$ of V. If T is onto, then $\{T(b_{k+1}), \cdots, T(b_n)\}$ is a basis of W.

4. Let $C[-1, 1]$ denote the vector space of all continuous functions defined on the closed interval $[-1.1]$ with the inner product $< f, g >= \int_{-1}^{1} f(x)g(x)\, dx$. Let $f(x) = x^2 - x$ and $g(x) = x - 1$.

(a) Compute $< f, g >$, $||f||$ and $||g||$.

(b) Compute the cosine of the angle between f and g.

(c) Compute the distance between f and g.

(d) Perform the Gram-Schmidt process to f and g to obtain $\hat{f}$ and $\hat{g}$ such that $\text{Span}\{f, g\} = \text{Span}\{\hat{f}, \hat{g}\}$ and $< \hat{f}, \hat{g} >= 0$.

(e) Find the best mean square approximation of the function $h(x) = x^2$ by the functions in $W = \text{Span}\{f, g\}$.

5. Let $\mathcal{M}_{2,2}$ denote the vector space of 2×2 matrices. Define the mapping $T : \mathcal{M}_{2,2} \to \mathcal{M}_{2,2}$ by $T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & a \end{pmatrix}$.

(a) Show that T is a linear operator.

(b) Find a basis for $\mathrm{Ker}(T)$.

(c) Find a basis for $\mathrm{Ran}(T)$.