Math 225 (Q1) Solution to Homework Assignment 8

1. Using distributive law (A7), we have

$$(1+1)(\underline{u}+\underline{v}) = 1(\underline{u}+\underline{v}) + 1(\underline{u}+\underline{v}) = \underline{u}+\underline{v}+\underline{u}+\underline{v}.$$

On the other hand, using distributive law (A8), we have

$$(1+1)(\underline{u}+\underline{v}) = (1+1)\underline{u} + (1+1)\underline{v} = \underline{u}+\underline{u}+\underline{v}+\underline{v}.$$

Thus, $\underline{u}+\underline{v}+\underline{u}+\underline{v} = \underline{u}+\underline{u}+\underline{v}+\underline{v}$. Next, we add $(-\underline{u})$ on the left and add $(-\underline{v})$ on the right to both sides, we get

$$(-\underline{u})+\underline{u}+\underline{v}+\underline{u}+\underline{v}+(-\underline{v}) = (-\underline{u})+\underline{u}+\underline{u}+\underline{v}+\underline{v}+(-\underline{v})$$

which simplifies to $\underline{v}+\underline{u} = \underline{u}+\underline{v}$.

2. Let $[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underline{c}$. Then $\underline{x} = c_1\underline{b_1} + c_2\underline{b_2} + c_3\underline{b_3} = [\underline{b_1}\ \underline{b_2}\ \underline{b_3}]\underline{c}$. That is, $\begin{pmatrix} 8 \\ -9 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 2 \\ -1 & 4 & -2 \\ -3 & 9 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. Solving for $\underline{c}$ we get $\underline{c} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$.

3. Clearly S spans V since every vector in V is a linear combination of vectors in S. To show that $S = \{\underline{v_1}, \cdots, \underline{v_n}\}$ is a basis of V, it remains to show that S is linearly independent. Let $\alpha_1, \cdots, \alpha_n \in \mathbf{R}$ be such that $\alpha_1\underline{v_1} + \cdots + \alpha_n\underline{v_n} = \underline{0}$. We want to show that the only way this can happen is when $\alpha_1 = \cdots = \alpha_n = 0$. Now, since $\underline{0} = 0\cdot\underline{v_1} + \cdots + 0\cdot\underline{v_n}$ and $\underline{0} = \alpha_1\cdot\underline{v_1} + \cdots + \alpha_n\cdot\underline{v_n}$ are two ways to represent the zero vector $\underline{0}$ as a linear combination of vectors in S, therefore, by uniqueness, we must have $0 = \alpha_1$, $0 = \alpha_2$, etc.

4. Let $p_1(t) = 1, p_2(t) = 2t, p_3(t) = -2+4t^2, p_4(t) = -12t+8t^3$. We need to show $\mathcal{B} = \{p_1(t), p_2(t), p_3(t), p_4(t)\}$ is a basis of the the vector space of all polynomials in t of degree less than or equal to 3.

First we will show that the vectors in $\mathcal{B}$ are linearly independent. Suppose there exist $c_1, c_2, c_3, c_4 \in \mathbf{R}$ such that $c_1p_1(t) + c_2p_2(t) + c_3p_3(t) + c_4p_4(t)$ is the zero polynomial. This means that $c_1(1) + c_2(2t) + c_3(-2+4t^2) + c_4(-12t+8t^3) = 0$ for all t. We want to show that $c_i = 0$ for all $1 \le i \le 4$. Now, collecting terms in ascending powers of t, we have, $(c_1 - 2c_3) + (2c_2 - 12c_4)t + (4c_3)t^2 + (8c_4)t^3 = 0$. This shows that $c_1 - 2c_3 = 0$, $2c_2 - 12c_4 = 0$, $4c_3 = 0$ and $8c_4 = 0$. Solving for c_1, c_2, c_3, c_4, we get, $c_1 = c_2 = c_3 = c_4 = 0$, as desired.

Next, we will show that the vectors in $\mathcal{B}$ span $\mathbf{P}^3$. For that, we can either recall that $\dim(\mathbf{P}^3) = 4$ and use Basis Theorem or we can proceed directly. Let $q(t) \in \mathbf{P}^3$. Then $q(t) = q_0 + q_1t + q_2t^2 + q_3t^3$ for some $q_0, q_1, q_2, q_3 \in \mathbf{R}$. We want to show that $q(t)$ is a linear combination of vectors in $\mathcal{B}$. That is, there exist $c_1, c_2, c_3, c_4 \in \mathbf{R}$ such that $c_1p_1(t) + c_2p_2(t) + c_3p_3(t) + c_4p_4(t) = q(t)$ for all t. Expanding it out and collecting like powers of t, we have, $(c_1 - 2c_3) + (2c_2 - 12c_4)t + (4c_3)t^2 + (8c_4)t^3 = q_0 + q_1t + q_2t^2 + q_3t^3$. This means that $c_1 - 2c_3 = q_0$, $2c_2 - 12c_4 = q_1$, $4c_3 = q_2$ and $8c_4 = q_3$. Write it as a matrix equation $A\underline{c} = \underline{q}$, where $A = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$, $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$. Since $\det(A) = (1)(2)(4)(8) = 64$ is non–zero, A is invertible. Thus, $\underline{c} = A^{-1}\underline{q}$ and $q(t)$ is a linear combination of $p_1(t)$, $p_2(t)$, $p_3(t)$ and $p_4(t)$.

Since the basis $\mathcal{B}$ of $\mathbf{P}^3$ has 4 elements, $\dim(\mathbf{P}^3) = 4$.

5. We know that $\operatorname{rank}(A^T) = \operatorname{rank}(A)$, $\operatorname{rank}(AB) \le \operatorname{rank}(A)$ and $\operatorname{rank}(AB) \le \operatorname{rank}(B)$.

(a) Clearly, $\operatorname{rank}(PA) \le \operatorname{rank}(A)$. On the other hand,

$$\operatorname{rank}(A) = \operatorname{rank}(IA) = \operatorname{rank}((P^{-1}P)A) = \operatorname{rank}(P^{-1}(PA)) \le \operatorname{rank}(PA).$$

(b) $\operatorname{rank}(AQ) = \operatorname{rank}((AQ)^T) = \operatorname{rank}(Q^TA^T) = \operatorname{rank}(A^T) = \operatorname{rank}(A)$. Note that, we used part (a) and the fact that $P = Q^T$ is invertible, provided Q is invertible (in fact, $P^{-1} = (Q^T)^{-1} = (Q^{-1})^T$).