Math 225 (Q1) Solution to Homework Assignment 7

1.

(a) $A = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$

(b) Eigenvalues of A are: $\lambda_1 = 0$, $\lambda_2 = 10$. $\underline{v_1} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$ is a unit eigenvector corresponding to $\lambda_1 = 0$ and $\underline{v_2} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{pmatrix}$ is a unit eigenvector corresponding to $\lambda_2 = 10$. Therefore, $D = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$ and $P = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{pmatrix}$. Furthermore, $Q = \underline{y}^T D \underline{y} = 10y_2^2$

(c) Q is positively semidefinite because all eigenvalues of A are non–negative.

2. Since A is positive definite, A is symmetric. Let $A = PDP^T$ be an orthogonal diagonalization of A, where $P^{-1} = P^T$ and D is a diagonal matrix. The diagonal entries of D are the eigenvalues of A and so they are positive. Let C be a diagonal matrix whose diagonal entries are the square roots of the corresponding diagonal entries of D. Then $C^2 = D$ and $C^T = C$. Finally, Define $B = PCP^T$. Then $B^T = (PCP^T)^T = (P^T)^T C^T P^T = PCP^T = B$, so that B is a symmetric matrrix. Furthermore, since $B = PCP^{-1}$, B is similar to C and so the eigenvalues of B are the diagonal entries of C which are positive. Thus, B is positive definite. Finally, $B^T B = (PCP^T)^T(PCP^T) = PCP^T PCP^T = PCCP^T = PDP^T = A$, as desired.

3.

(a) Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$. Then $Q(x_1, x_2, x_n) = \underline{x}^T A \underline{x}$.

(b) The eigenvalues of A are: $5, 2, 0$, with corresponding unit eigenvectors: $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$, $\begin{pmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$, $\begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, resp. Let $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $Q(x) = \underline{y}^T D \underline{y} = 5y_1^2 + 2y_2^2$.

(c) The maximum value of $Q(x)$ is 5.

(d) The maximum of $Q(x)$ is attained at $x = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

4. The reduced row echelon form is A is $R = \begin{pmatrix} \boxed{1} & 0 & 6 & 5 \\ 0 & \boxed{1} & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(a) x_3 and x_4 are free variables. The equations are: $x_1 + 6x_3 + 5x_4 = 0$ and $x_2 + (5/2)x_3 + (3/2)x_4 = 0$ so that

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6x_3 - 5x_4 \\ -(5/2)x_3 - (3/2)x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, $\left\{ \begin{pmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathrm{Nul}(A)$.

(b) Using the reduced row echelon form of A, we see that columns 1 and 2 are the pivot columns. Therefore, $\left\{ \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ 8 \end{pmatrix} \right\}$ is a basis of $\mathrm{Col}(A)$.

(c) Since $\mathrm{Row}(A) = \mathrm{Row}(R)$, $\{(1, 0, 6, 5), (0, 1, 5/2, 3/2)\}$ is a basis of $\mathrm{Row}(A)$.

(d) $\mathrm{rank}(A) = 2$.

(e) $\mathrm{nullity}(A) = 2$.

5.

(a) If we can show that $\mathrm{Col}(AB)$ is a subspace of $\mathrm{Col}(A)$, then

$$\mathrm{rank}(AB) = \dim(\mathrm{Col}(AB)) \leq \dim(\mathrm{Col}(A)) = \mathrm{rank}(A).$$

Denote the columns of A by $\underline{a_1}, \cdots \underline{a_n}$. Then a linear combination $x_1\underline{a_1} + \cdots x_n\underline{a_n}$ is really $A\underline{x}$, where $\underline{x} = (x_1, \cdots, x_n)^T$. Thus $\mathrm{Col}(A)$, which is the set of all linear combinations of $\underline{a_1}, \cdots, \underline{a_n}$, is the same as the set $\{A\underline{x} \ : \ \underline{x} \in \mathbf{R}^n\}$. To show $\mathrm{Col}(AB) \subset \mathrm{Col}(A)$, let $\underline{y} \in \mathrm{Col}(AB)$. By the above, there exists $\underline{z} \in \mathbf{R}^p$ such that $\underline{y} = (AB)\underline{z} = A(B\underline{z}) = A\underline{x}$, where $\underline{x} = B\underline{z} \in \mathbf{R}^n$. Thus $\underline{y} \in \mathrm{Col}(A)$.

(b) By part (a),

$$\operatorname{rank}(AB) = \operatorname{rank}((AB)^T) = \operatorname{rank}(B^T A^T) \leq \operatorname{rank}(B^T) = \operatorname{rank}(B),$$

since $\operatorname{rank}(C^T) = \operatorname{rank}(C)$ for any matrix C (Note. Row rank is the same as column rank).