Math 225 (Q1) Homework Assignment 7.

1. Let $Q(x) = x_1^2 - 6x_1x_2 + 9x_2^2$, where $x = (x_1, x_2,)$.

   (a) Find a 2×2 real, symmetric matrix A such that $Q(x) = \underline{x}^T A \underline{x}$, where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

   (b) Make a change of variable $\underline{x} = P\underline{y}$, where $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, such that $Q(x) = \underline{y}^T D \underline{y}$ has no cross terms in terms of y_1, y_2.

   (c) Classify the quadratic form Q.

2. Show that if A is a $n \times n$ positive definite matrix, then there exists a positive definite matrix B such that $A = B^T B$. Hint: Write A as PDP^T, with $P^T = P^{-1}$, that is, P is an orthogonal matrix. Let C be a diagonal matrix such that $C^T C = C^2 = D$. Define $B = PCP^T$.

3. Let $Q(x) = 3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$, where $x = (x_1, x_2, x_3)$.

   (a) Let $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find a 3×3 real, symmetric matrix A such that $Q(x_1, x_2, x_3) = \underline{x}^T A \underline{x}$.

   (b) Find a change of variable $\underline{x} = P\underline{y}$, where $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, such that $\underline{x}^T A \underline{x} = \underline{y}^T D \underline{y}$. D is a diagonal matrix. Thus, in terms of y_1, y_2, y_3, Q has no cross product terms.

   (c) Find the maximum value of $Q(x)$ subject to the constraint $\underline{x}^T \underline{x} = 1$.

   (d) Find a unit vector $\underline{u} \in \mathbf{R}^3$ where this maximum is attained.

4. Let $A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}$. Find

(a) a basis for $\mathrm{Nul}(A)$,

(b) a basis for $\mathrm{Col}(A)$,

(c) a basis for $\mathrm{Row}(A)$.

(d) $\mathrm{rank}(A)$, the rank of A, and

(e) $\mathrm{nullity}(A)$, the nullity of A.

5. Let A be a $m \times n$ matrix and B be a $n \times p$ matrix.

(a) Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$. (Hint. Show that $\mathrm{Col}(AB) \subset \mathrm{Col}(A)$).

(b) Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$. (Hint. Use part (a) to study $\mathrm{rank}((AB)^T)$.