Math 225 (Q1) Homework Assignment 6.

1. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$. Solve the differential equation $\underline{x}' = A\underline{x}$, that is,

$$x_1' = 2x_1 + 3x_2$$
$$x_2' = -x_1 - 2x_2,$$

where $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are functions of t and $x_1' = \frac{dx_1}{dt}$, $x_2' = \frac{dx_2}{dt}$ are the derivative of $x_1(t)$ and $x_2(t)$ with respect to time t.

2. Solve the initial value problem

$$\begin{cases} x_1' = 4x_1 + x_3 \\ x_2' = -2x_1 + x_2 \\ x_3' = -2x_1 + x_3 \end{cases}$$

where $x_1(0) = -1$, $x_2(0) = 1$ and $x_3(0) = 0$.

3. Orthogonally diagonalize the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, that is, show that A is symmetric and find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

4. Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

(a) Find a matrix P that orthogonally diagonally diagonalizes A.

(b) Determine the diagonal matrix $D = P^T AP$.

(c) Find the spectral decomposition of A.

5. Consider the matrix $A = \begin{pmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{pmatrix}$

(a) Verify that A satisfies its characteristic equation, as guaranteed by the Caley-Hamilton theorem.

(b) Find an expression for A^4 in terms of A^2, A and I and use that expression to evaluate A^4.

(c) Find an expression for A^{-1} in terms of A^2, A and I.