Math 225 (Q1) Solution to Homework Assignment 4

1.

(a) Since

$$\underline{u_1}\cdot\underline{u_2} = \begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\1\end{pmatrix} = (1)(-1)+(0)(40+(1)(1) = -1+1 = 0$$

$$\underline{u_1}\cdot\underline{u_3} = (1)(2)+(0)(1)+(1)(-2) = 2-2 = 0$$

$$\underline{u_2}\cdot\underline{u_3} = (-1)(2)+(4)(1)+(1)(-2) = -2+4-2 = 0,$$

therefore $\underline{u_1}$, $\underline{u_2}$ and $\underline{u_3}$ are pairwise orthogonal and so $\{\underline{u_1},\underline{u_2},\underline{u_3}\}$ is an orthogonal set.

(b)

$$\underline{x} = \frac{\underline{x}\cdot\underline{u_1}}{\underline{u_1}\cdot\underline{u_1}}\underline{u_1} + \frac{\underline{x}\cdot\underline{u_2}}{\underline{u_2}\cdot\underline{u_2}}\underline{u_2} + \frac{\underline{x}\cdot\underline{u_3}}{\underline{u_3}\cdot\underline{u_3}}\underline{u_3} = \frac{11}{2}\underline{u_1} - \frac{13}{18}\underline{u_2} + \frac{8}{9}\underline{u_3}.$$

2. Since $\underline{0}\cdot\underline{y} = 0$ for all $\underline{y}\in S$, therefore $\underline{0}\in S^\perp$. Thus, $S^\perp$ is non–empty. Next, let $\underline{u},\underline{v}\in S^\perp$. Then $\underline{u}\cdot\underline{y} = 0$ and $\underline{v}\cdot\underline{y} = 0$, for all $\underline{y}\in S$. For any scalars a and b and for any vector $\underline{y}\in S$, we have,

$$(a\underline{u}+b\underline{v})\cdot\underline{y} = a(\underline{u}\cdot\underline{y}) + b(\underline{v}\cdot\underline{y}) = a(0)+b(0) = 0.$$

Thus, $a\underline{u}+b\underline{v}\in S^\perp$. Hence, $S^\perp$ is a subspace of $\mathbf{R}^n$.

3. Let $\underline{a_1} = \begin{pmatrix}1\\-1\\-1\\1\\1\end{pmatrix}$, $\underline{a_2} = \begin{pmatrix}2\\1\\4\\-4\\2\end{pmatrix}$ and $\underline{a_3} = \begin{pmatrix}5\\-4\\-3\\7\\1\end{pmatrix}$ be the column vectors of the matrix A. Let $W = \text{Span}\{\underline{a_1},\underline{a_2},\underline{a_3}\} = \text{Col}(A)$. According to the Gram–Schmidt process, define

$$\underline{u_1} = \underline{a_1},$$

$$\underline{u_2} = \underline{a_2} - \frac{\underline{a_2}\cdot\underline{u_1}}{||\underline{u_1}||^2}\underline{u_1} = \underline{a_2}+\underline{u_1} = \begin{pmatrix}3\\0\\3\\-3\\3\end{pmatrix}$$

$$\underline{u_3} = \underline{a_3} - \left[\frac{\underline{a_3}\cdot\underline{u_1}}{||\underline{u_1}||^2}\underline{u_1} + \frac{\underline{a_3}\cdot\underline{u_2}}{||\underline{u_2}||^2}\underline{u_2}\right] = \underline{a_3} - [4\underline{u_1} - \frac{1}{3}\underline{u_2}] = \begin{pmatrix}2\\0\\2\\2\\-2\end{pmatrix}.$$

Then $\{\underline{u_1}, \underline{u_2}, \underline{u_3}\}$ is an orthogonal basis of $\text{Col}(A)$. Next, we normalize and let

$$\underline{v_1} = \frac{1}{||\underline{u_1}||}\underline{u_1} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \quad \underline{v_2} = \frac{1}{||\underline{u_2}||}\underline{u_2} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \underline{v_3} = \frac{1}{||\underline{u_3}||}\underline{u_3} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

Then $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ is an orthonormal basis of $\text{Col}(A)$.

4.

(a) Let $\underline{a_1}, \cdots, \underline{a_n}$ be the column vectors of A. Since these vectors are linearly dependent, there exist scalars $x_1, \cdots, x_n$ (not all of them are zero) such that $x_1\underline{a_1} + \cdots + x_n\underline{a_n} = \underline{0}$. Since the left hand side is $A\underline{x}$, where $\underline{x} = (x_1, \cdots, x_n)^T$, therefore we have $A\underline{x} = \underline{0}$, where $\underline{x} \neq \underline{0}$. Note that, this also implies $\text{Nul}(A) \neq \{\underline{0}\}$.

(b) Suppose $\underline{a_1}, \cdots, \underline{a_n}$ are not linearly independent. Then by part (a), there exists $\underline{x} \in \mathbf{R}^n$ such that $\underline{x} \neq \underline{0}$ and $A\underline{x} = \underline{0}$. Thus, $A^TA\underline{x} = A^T\underline{0} = \underline{0}$. It follows that $\underline{x} = (A^TA)^{-1}\underline{0} = \underline{0}$ which is a contradiction. Hence the columns of A are linearly independent. Alternately, we can argue as follows. Since A^TA is invertible, therefore $\text{Nul}(A^TA) = \{\underline{0}\}$. Since $\text{Nul}(A^TA) = \text{Nul}(A)$, therefore $\text{Nul}(A) = \{\underline{0}\}$. Using part (a), we see that the columns of A cannot be linearly dependent and hence they are linearly independent.

5.

(a) Since $(UV)^{-1} = V^{-1}U^{-1} = V^TU^T = (UV)^T$, therefore UV is an orthogonal matrix.

(b) $A = PRP^{-1}$ implies $P^{-1}AP = P^{-1}PRP^{-1}P = R$ so that $R = P^TAP$, since $P^{-1} = P^T$. Now, $R^T = (P^TAP)^T = P^TA^T(P^T)^T = P^TAP = R$, since $A^T = A$. Thus, R is a symmetric matrix. Finally since R is an upper triangular matrix, therefore R^T is a lower triangular matrix. Hence $R = R^T$ is both upper and lower triangular and consequently it must be a diagonal matrix.