Math 225 (Q1) Homework Assignment 4.

1. Let $\underline{u_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\underline{u_2} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ and $\underline{u_3} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

(a) Show that $\{\underline{u_1}, \underline{u_2}, \underline{u_3}\}$ is an orthogonal set.

(b) Using part (a), express $\underline{x} = \begin{pmatrix} 8 \\ -2 \\ 3 \end{pmatrix}$ as a linear combination of $\underline{u_1}$, $\underline{u_2}$ and $\underline{u_3}$.

2. Let $S \subset \mathbf{R^n}$ be a non–empty subset of $\mathbf{R^n}$. Define

$$S^{\perp} = \{\underline{x} \in \mathbf{R^n} \ : \ \underline{x} \cdot \underline{y} = 0, \quad \text{for all } \underline{y} \in S\}.$$

Show that $S^{\perp}$ is a subspace of $\mathbf{R^n}$, that is, $S^{\perp}$ is non-empty and for all vectors $\underline{u}, \underline{v} \in S^{\perp}$ and for all scalars $\alpha, \beta \in \mathbf{R}$, we have $\alpha\underline{u} + \beta\underline{v} \in S^{\perp}$. $S^{\perp}$ is called the "orthogonal complement" of S.

3. Let $A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix}$. Using the Gram–Schmidt process, find an orthonormal basis for the column space of A. (Recall that the column space space of A, Col(A), is the span of the column vectors of A).

4. Let A be a $m \times n$ matrix, where m and n may not be the same,

(a) Show that if the columns of A are linearly dependent, then there exists a non–zero vector $\underline{x} \in \mathbf{R}^n$ such that $A\underline{x} = \underline{0}$.

(b) Show that if $A^T A$ is invertible, then the columns of A are linearly independent. Hint: Use Question 4 in Assignment 1.

5.

(a) A square matrix U is said to be an "orthogonal" matrix if U is invertible and $U^{-1} = U^T$. Let U, V be two $n \times n$ orthogonal matrices. Show that UV is also an orthogonal matrix.

(b) Suppose $A = PRP^{-1}$, where A, P, R are $n \times n$ matrices, P is an orthogonal matrix and R is an upper triangular matrix. This is called a "Schur" factorization/decomposition of A. (Note that this says in particular that the matrix A is similar to the matrix R). Show that if A is symmetric, then R is also symmetric and hence R is a diagonal matrix.